Acknowledgements

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How to Learn from a Math Textbook

A math textbook is different from other types of books. Here are tips on how to learn math from your math textbook.

Eliminate Distractions
Eliminate distractions so you can concentrate on the math. A quiet location helps. Soft music without words is OK and may help drown out other noises.

Slow Down
If you are reading a novel and are somewhat distracted, you can still get the story. When you are not concentrating on math, you will get very little out of it, and it will seem more difficult that it really is.

Every Word Counts
Extra words get in the way of clarity. Each page assumes you have mastered the previous pages. If lost, it is usually better to go back rather than forward. Math is logical and each step is necessary.

Understand Each Paragraph
Understand each paragraph before you go on. Reread as necessary for you to master an idea. Reading a math textbook can be difficult. It could take you half an hour to understand just one page.

Diagrams are Important
Diagrams and other illustrations are important. They are part of the written text. Stick with them until you thoroughly understand their content.

Write as you read!

> Work out sample problems. Rewrite each problem in your own words. Fill in all steps to clarify your understanding. When you go back and review, the steps are already filled in so you will continue to understand how each step was completed.
> Starting with the second example of each concept in the text, work the example. Do not worry if you cannot work the example at this time. After you have tried the example on your own, read the solution.

After reading an example, cover it up and try to work it out yourself. Continue rewriting and working the example until you can do it without the aid of the text.
Highlight formulas, definitions, cautionary notes (with an asterisk, check mark, etc.) in a way that is consistent. Do not overuse marking of the text.
Note any questions on concepts or procedures you need to have clarified.

Create a Cheat Sheet
Create a resource or “cheat” sheet by recording key points on a separate piece of paper or into a notebook.

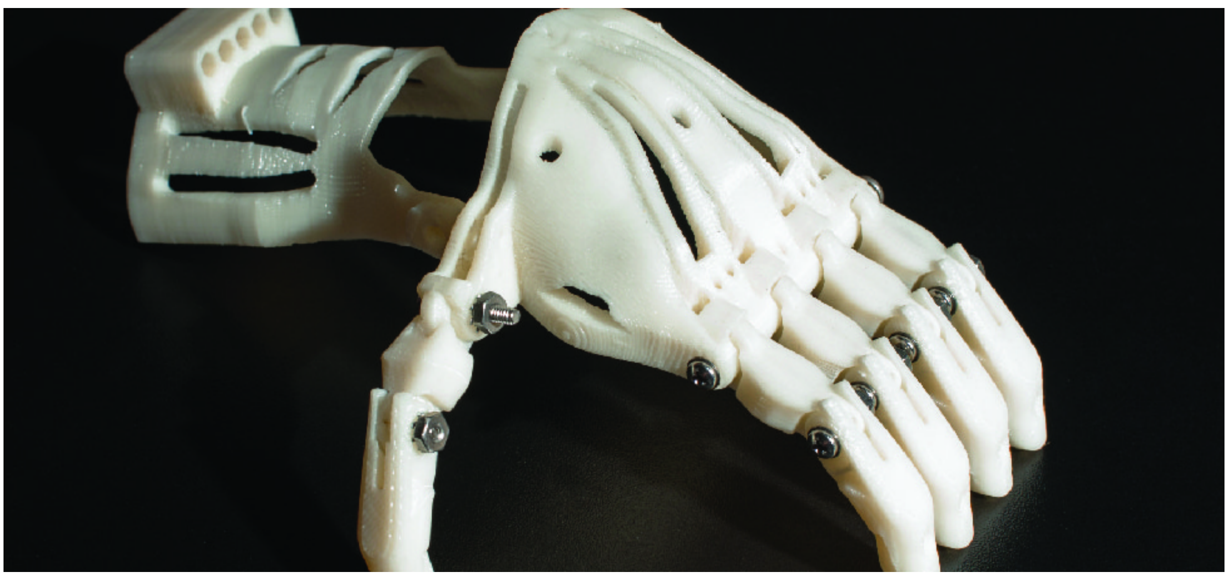
Review
Review the day’s lesson after class and before the next class.

If you do not understand the material

Go back to the previous page and reread the information to maintain a train of thought.
Read the misunderstood paragraph aloud. Sometimes this helps you capture the meaning.
Consider watching a video on the subject. Some are suggested in the textbook.
Define exactly what you do not understand and call a study buddy for help.
Prepare questions on the confusing information and contact your math tutor or math instructor for help. Ask those questions at the next class meeting or contact the tutor or instructor by email.

Introduction
class="introduction"

This hand may change someone’s life. Amazingly, it was created using a special kind of printer known as a 3D printer. (credit: U.S. Food and Drug Administration/Wikimedia Commons)



For years, doctors and engineers have worked to make artificial limbs, such as this hand for people who need them. This particular product is different, however, because it was developed using a 3D printer. As a result, it can be printed much like you print words on a sheet of paper. This makes producing the limb less expensive and faster than conventional methods.

Biomedical engineers are working to develop organs that may one day save lives. Scientists at NASA are designing ways to use 3D printers to build on the moon or Mars. Already, animals are benefitting from 3D-printed parts,

including a tortoise shell and a dog leg. Builders have even constructed entire buildings using a 3D printer.

The technology and use of 3D printers depend on the ability to understand the language of algebra. Engineers must be able to translate observations and needs in the natural world to complex mathematical commands that can provide directions to a printer. In this chapter, you will review the language of algebra and take your first steps toward working with algebraic concepts.

Use the Language of Algebra: ASE
By the end of this section, you will be able to:

- Find factors, prime factorizations, and least common multiples
- Use variables and algebraic symbols
- Simplify expressions using the order of operations
- Evaluate an expression
- Identify Properties of Addition and Multiplication
- Use the Distributive Property
- Identify and combine like terms
- Translate an English phrase to an algebraic expression

Find Factors, Prime Factorizations, and Least Common Multiples

The numbers 2, 4, 6, 8, 10, 12 are called multiples of 2. A **multiple** of 2 can be written as the product of a counting number and 2.

Multiples of 2:

2,	4,	6,	8,	10,	12, ...
$2 \cdot 1$	$2 \cdot 2$	$2 \cdot 3$	$2 \cdot 4$	$2 \cdot 5$	$2 \cdot 6$

Similarly, a multiple of 3 would be the product of a counting number and 3.

Multiples of 3:

3,	6,	9,	12,	15,	18, ...
$3 \cdot 1$	$3 \cdot 2$	$3 \cdot 3$	$3 \cdot 4$	$3 \cdot 5$	$3 \cdot 6$

We could find the multiples of any number by continuing this process.

Counting Number	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
Multiples of 2	2	4	6	8	10	12	14	16	18	20	22	24
Multiples of 3	3	6	9	12	15	18	21	24	27	30	33	36
Multiples of 4	4	8	12	16	20	24	28	32	36	40	44	48
Multiples of 5	5	10	15	20	25	30	35	40	45	50	55	60
Multiples of 6	6	12	18	24	30	36	42	48	54	60	66	72

Counting Number	1	2	3	4	5	6	7	8	9	10	11	12
Multiples of 7	7	14	21	28	35	42	49	56	63	70	77	84
Multiples of 8	8	16	24	32	40	48	56	64	72	80	88	96
Multiples of 9	9	18	27	36	45	54	63	72	81	90	99	108

Note:
Multiple of a Number
A number is a **multiple** of n if it is the product of a counting number and n.

Another way to say that 15 is a multiple of 3 is to say that 15 is **divisible** by 3. That means that when we divide 3 into 15, we get a counting number. In fact, $15 \div 3$ is 5, so 15 is $5 \cdot 3$.

Note:
Divisible by a Number
If a number m is a multiple of n, then m is **divisible** by n.

If we were to look for patterns in the multiples of the numbers 2 through 9, we would discover the following divisibility tests:

Note:
Divisibility Tests
A number is divisible by:

- 2 if the last digit is 0, 2, 4, 6, or 8.
- 3 if the sum of the digits is divisible by 3.
- 5 if the last digit is 5 or 0.
- 6 if it is divisible by both 2 and 3.
- 10 if it ends with 0.

Example:
Exercise:

Problem: Is 5,625 divisible by ⓐ 2? ⓑ 3? ⓒ 5 or 10? ⓓ 6?

Solution:

ⓐ
Is 5,625 divisible by 2?

Does it end in 0, 2, 4, 6 or 8?	No. 5,625 is not divisible by 2.

ⓑ
Is 5,625 divisible by 3?

What is the sum of the digits?	$5+6+2+5=18$
Is the sum divisible by 3?	Yes. 5,625 is divisible by 3.

ⓒ
Is 5,625 divisible by 5 or 10?

What is the last digit? It is 5.	5,625 is divisible by 5 but not by 10.

ⓓ
Is 5,625 divisible by 6?

Is it divisible by both 2 and 3?	No, 5,625 is not divisible by 2, so 5,625 is not divisible by 6.

Note:
Exercise:

Problem: Is 4,962 divisible by ⓐ 2? ⓑ 3? ⓒ 5? ⓓ 6? ⓔ 10?

Solution:

ⓐ yes ⓑ yes ⓒ no ⓓ yes
ⓔ no

Note:
Exercise:

Problem: Is 3,765 divisible by ⓐ 2? ⓑ 3? ⓒ 5? ⓓ 6? ⓔ 10?

Solution:

ⓐ no ⓑ yes ⓒ yes ⓓ no
ⓔ no

In mathematics, there are often several ways to talk about the same ideas. So far, we've seen that if *m* is a multiple of *n*, we can say that *m* is divisible by *n*. For example, since 72 is a multiple of 8, we say 72 is divisible by 8. Since 72 is a multiple of 9, we say 72 is divisible by 9. We can express this still another way.

Since $8 \cdot 9 = 72$, we say that 8 and 9 are **factors** of 72. When we write $72 = 8 \cdot 9$, we say we have factored 72.

$$\underbrace{8 \cdot 9}_{\textit{factors}} = \underbrace{72}_{\textit{product}}$$

Other ways to factor 72 are $1 \cdot 72, \quad 2 \cdot 36, \quad 3 \cdot 24, \quad 4 \cdot 18,$ and $6 \cdot 12$. The number 72 has many factors: $1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36,$ and 72.

Note:
Factors
If $a \cdot b = m$, then *a* and *b* are **factors** of *m*.

Some numbers, such as 72, have many factors. Other numbers have only two factors. A **prime number** is a counting number greater than 1 whose only factors are 1 and itself.

Note:
Prime number and Composite number
A **prime number** is a counting number greater than 1 whose only factors are 1 and the number itself.
A **composite number** is a counting number that is not prime. A composite number has factors other than 1 and the number itself.

The counting numbers from 2 to 20 are listed in the table with their factors. Make sure to agree with the "prime" or "composite" label for each!

Number	Factors	Prime or Composite?
2	1,2	Prime
3	1,3	Prime
4	1,2,4	Composite
5	1,5	Prime
6	1,2,3,6	Composite
7	1,7	Prime
8	1,2,4,8	Composite
9	1,3,9	Composite
10	1,2,5,10	Composite
11	1,11	Prime

Number	Factors	Prime or Composite?
12	1,2,3,4,6,12	Composite
13	1,13	Prime
14	1,2,7,14	Composite
15	1,3,5,15	Composite
16	1,2,4,8,16	Composite
17	1,17	Prime
18	1,2,3,6,9,18	Composite
19	1,19	Prime
20	1,2,4,5,10,20	Composite

The prime numbers less than 20 are 2, 3, 5, 7, 11, 13, 17, and 19. Notice that the only even prime number is 2.

A composite number can be written as a unique product of primes. This is called the **prime factorization** of the number. Finding the prime factorization of a composite number will be useful in many topics in this course.

Note:
Prime Factorization
The **prime factorization** of a number is the product of prime numbers that equals the number.

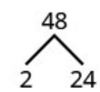
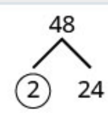
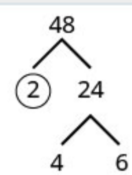
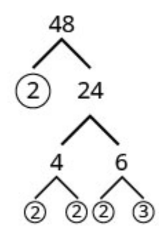
To find the prime factorization of a composite number, find any two factors of the number and use them to create two branches. If a factor is prime, that branch is complete. Circle that prime. Otherwise it is easy to lose track of the prime numbers.

If the factor is not prime, find two factors of the number and continue the process. Once all the branches have circled primes at the end, the factorization is complete. The composite number can now be written as a product of prime numbers.

Example:
How to Find the Prime Factorization of a Composite Number
Exercise:

Problem: Factor 48.

Solution:

Step 1. Find two factors whose product is the given number. Use these numbers to create two branches.	$48 = 2 \cdot 24$	
Step 2. If a factor is prime, that branch is complete. Circle the prime.	2 is prime. Circle the prime.	
Step 3. If a factor is not prime, write it as the product of two factors and continue the process.	24 is not prime. Break it into 2 more factors.	
	4 and 6 are not prime. Break them each into two factors. 2 and 3 are prime, so circle them.	
Step 4. Write the composite number as the product of all the circled primes.		$48 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$

We say $2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$ is the prime factorization of 48. We generally write the primes in ascending order. Be sure to multiply the factors to verify your answer.

If we first factored 48 in a different way, for example as $6 \cdot 8$, the result would still be the same. Finish the prime factorization and verify this for yourself.

Note:
Exercise:

Problem: Find the prime factorization of 80.

Solution:

$2 \cdot 2 \cdot 2 \cdot 2 \cdot 5$

Note:
Exercise:

Problem: Find the prime factorization of 60.

Solution:

$2 \cdot 2 \cdot 3 \cdot 5$

Note:
Find the prime factorization of a composite number.

Find two factors whose product is the given number, and use these numbers to create two branches.
If a factor is prime, that branch is complete. Circle the prime, like a leaf on the tree.
If a factor is not prime, write it as the product of two factors and continue the process.
Write the composite number as the product of all the circled primes.

One of the reasons we look at primes is to use these techniques to find the **least common multiple** of two numbers. This will be useful when we add and subtract fractions with different denominators.

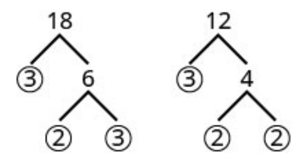
Note:
Least Common Multiple
The **least common multiple (LCM)** of two numbers is the smallest number that is a multiple of both numbers.

To find the least common multiple of two numbers we will use the Prime Factors Method. Let's find the LCM of 12 and 18 using their prime factors.

Example:
How to Find the Least Common Multiple Using the Prime Factors Method
Exercise:

Problem: Find the least common multiple (LCM) of 12 and 18 using the prime factors method.

Solution:

Step 1. Write each number as a product of primes.		
Step 2. List the primes of each number. Match primes vertically when possible.	List the primes of 12. List the primes of 18. Line up with the primes of 12 when possible. If not create a new column.	$12 = 2 \cdot 2 \cdot 3$ $18 = 2 \cdot \quad\; 3 \cdot 3$
Step 3. Bring down the number from each column.		$12 = 2 \cdot 2 \cdot 3$ $18 = 2 \cdot \quad\; 3 \cdot 3$ $\text{LCM} = 2 \cdot 2 \cdot 3 \cdot 3$
Step 4. Multiply the factors.		$\text{LCM} = 36$

Notice that the prime factors of 12 $(2 \cdot 2 \cdot 3)$ and the prime factors of 18 $(2 \cdot 3 \cdot 3)$ are included in the LCM $(2 \cdot 2 \cdot 3 \cdot 3)$. So 36 is the least common multiple of 12 and 18.

By matching up the common primes, each common prime factor is used only once. This way you are sure that 36 is the *least* common multiple.

Note:
Exercise:

Problem: Find the LCM of 9 and 12 using the Prime Factors Method.

Solution:

36

Note:
Exercise:

Problem: Find the LCM of 18 and 24 using the Prime Factors Method.

Solution:

72

Note:
Find the least common multiple using the Prime Factors Method.

Write each number as a product of primes.
List the primes of each number. Match primes vertically when possible.
Bring down the columns.
Multiply the factors.

Use Variables and Algebraic Symbols

In algebra, we use a letter of the alphabet to represent a number whose value may change. We call this a **variable** and letters commonly used for variables are x, y, a, b, c.

Note:
Variable
A **variable** is a letter that represents a number whose value may change.

A number whose value always remains the same is called a **constant**.

Note:
Constant
A **constant** is a number whose value always stays the same.

To write algebraically, we need some operation symbols as well as numbers and variables. There are several types of symbols we will be using. There are four basic arithmetic operations: addition, subtraction, multiplication, and division. We'll list the symbols used to indicate these operations below.

Note:
Operation Symbols

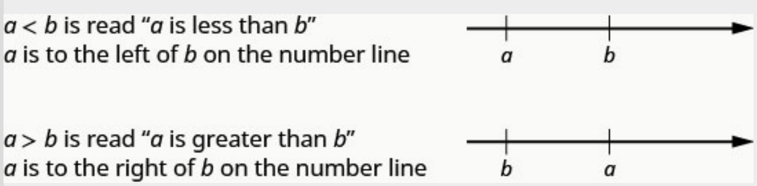
Operation	Notation	Say:	The result is…
Addition	$a+b$	a plus b	the sum of a and b
Subtraction	$a-b$	a minus b	the difference of a and b
Multiplication	$a\cdot b, ab, (a)\,(b),$ $(a)b, a\,(b)$	a times b	the product of a and b
Division	$a\div b, a/b, \frac{a}{b},\ b\overline{)a}$	a divided by b	the quotient of a and b; a is called the dividend, and b is called the divisor

When two quantities have the same value, we say they are equal and connect them with an equal sign.

Note:
Equality Symbol
$a=b$ is read "a is equal to b."
The symbol "=" is called the equal sign.

On the number line, the numbers get larger as they go from left to right. The number line can be used to explain the symbols "<" and ">".

Note:
Inequality



The expressions $a<b$ or $a>b$ can be read from left to right or right to left, though in English we usually read from left to right. In general,
Equation:

$$\begin{array}{ll} a < b & \text{is equivalent to } b > a. \text{ For example, } 7 < 11 \text{ is equivalent to } 11 > 7. \\ a > b & \text{is equivalent to } b < a. \text{ For example, } 17 > 4 \text{ is equivalent to } 4 < 17. \end{array}$$

Note:
Inequality Symbols

Inequality Symbols	**Words**
$a \neq b$	a is *not equal to b.*
$a < b$	a is *less than b.*
$a \leq b$	a is *less than or equal to b.*
$a > b$	a is *greater than b.*
$a \geq b$	a is *greater than or equal to b.*

Grouping symbols in algebra are much like the commas, colons, and other punctuation marks in English. They help identify an **expression**, which can be made up of number, a variable, or a combination of numbers and variables using operation symbols. We will introduce three types of grouping symbols now.

Note:
Grouping Symbols
Equation:

$$\begin{array}{ll} \text{Parentheses} & (\ ) \\ \text{Brackets} & [\ ] \\ \text{Braces} & \{\ \} \end{array}$$

Here are some examples of expressions that include grouping symbols. We will simplify expressions like these later in this section.
Equation:

$$8(14-8) \qquad 21-3[2+4(9-8)] \qquad 24 \div \{13-2[1(6-5)+4]\}$$

What is the difference in English between a phrase and a sentence? A phrase expresses a single thought that is incomplete by itself, but a sentence makes a complete statement. A sentence has a subject and a verb. In

algebra, we have *expressions* and *equations*.

Note:
Expression
An **expression** is a number, a variable, or a combination of numbers and variables using operation symbols.

Equation:

Expression	**Words**	**English Phrase**
$3+5$	3 plus 5	the sum of three and five
$n-1$	n minus one	the difference of n and one
$6\cdot 7$	6 times 7	the product of six and seven
$\frac{x}{y}$	x divided by y	the quotient of x and y

Notice that the English phrases do not form a complete sentence because the phrase does not have a verb.

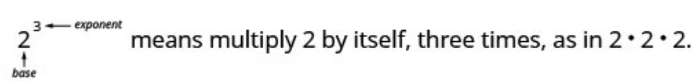
An **equation** is two expressions linked by an equal sign. When you read the words the symbols represent in an equation, you have a complete sentence in English. The equal sign gives the verb.

Note:
Equation
An **equation** is two expressions connected by an equal sign.

Equation:

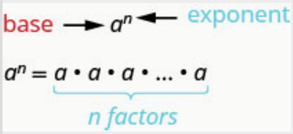
Equation	**English Sentence**
$3+5=8$	The sum of three and five is equal to eight.
$n-1=14$	n minus one equals fourteen.
$6\cdot 7=42$	The product of six and seven is equal to forty-two.
$x=53$	x is equal to fifty-three.
$y+9=2y-3$	y plus nine is equal to two y minus three.

Suppose we need to multiply 2 nine times. We could write this as $2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2$. This is tedious and it can be hard to keep track of all those 2s, so we use exponents. We write $2\cdot 2\cdot 2$ as 2^3 and $2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2$ as 2^9. In expressions such as 2^3, the 2 is called the *base* and the 3 is called the *exponent*. The exponent tells us how many times we need to multiply the base.



We read 2^3 as “two to the third power” or “two cubed.”

Note:
Exponential Notation
We say 2^3 is in *exponential notation* and $2 \cdot 2 \cdot 2$ is in *expanded notation*.
a^n means multiply *a* by itself, *n* times.



The expression a^n is read *a* to the n^{th} power.

While we read a^n as \\a to the n^{th} power", we usually read:
Equation:

$$\begin{array}{ll} a^2 & \text{"}a\text{ squared"} \\ a^3 & \text{"}a\text{ cubed"} \end{array}$$

We'll see later why a^2 and a^3 have special names.

[link] shows how we read some expressions with exponents.

Expression	In Words	
7^2	7 to the second power or	7 squared
5^3	5 to the third power or	5 cubed
9^4	9 to the fourth power	
12^5	12 to the fifth power	

Simplify Expressions Using the Order of Operations

To **simplify an expression** means to do all the math possible. For example, to simplify $4 \cdot 2 + 1$ we would first multiply $4 \cdot 2$ to get 8 and then add the 1 to get 9. A good habit to develop is to work down the page, writing each step of the process below the previous step. The example just described would look like this:
Equation:

$$\begin{array}{c} 4 \cdot 2 + 1 \\ 8 + 1 \\ 9 \end{array}$$

By not using an equal sign when you simplify an expression, you may avoid confusing expressions with equations.

Note:
Simplify an Expression
To **simplify an expression**, do all operations in the expression.

We've introduced most of the symbols and notation used in algebra, but now we need to clarify the **order of operations**. Otherwise, expressions may have different meanings, and they may result in different values.

For example, consider the expression $4 + 3 \cdot 7$. Some students simplify this getting 49, by adding $4 + 3$ and then multiplying that result by 7. Others get 25, by multiplying $3 \cdot 7$ first and then adding 4.

The same expression should give the same result. So mathematicians established some guidelines that are called the order of operations.

Note:
Use the order of operations.

Parentheses and Other Grouping Symbols
- Simplify all expressions inside the parentheses or other grouping symbols, working on the innermost parentheses first.

Exponents
- Simplify all expressions with exponents.

Multiplication and Division
- Perform all multiplication and division in order from left to right. These operations have equal priority.

Addition and Subtraction
- Perform all addition and subtraction in order from left to right. These operations have equal priority.

Students often ask, "How will I remember the order?"
Parentheses and other grouping symbols exist just so we can easily change the order operations are performed so what is inside them must be done first.

The key to the rest of the order is to remember to do the simpler operation after the more complex operation. We defined exponents in terms of multiplication so multiplication is simpler. Division is the inverse operation for multiplication so those are equally complex. Similarly, we defined multiplication in terms of addition so addition is simpler. Subtraction is the inverse operation for addition so those are equally complex.

The result is: Parentheses before Exponents, Exponents before Multiplication and Division, and Multiplication and Division before Addition and Subtraction.

A final rule is for operations of the same complexity is we do them from left to right.

If you are unsure of the order of operations for something you want to write, you can always use parenthesis to force the computation to be in the order you want.

Example:
Exercise:

Problem: Simplify: $18 \div 6 + 4\left(5 - 2\right)$.

Solution:

	$18 \div 6 + 4(5 - 2)$
Parentheses? Yes, subtract first.	$18 \div 6 + 4(3)$
Exponents? No.	
Multiplication or division? Yes.	
Divide first because we multiply and divide left to right.	$3 + 4(3)$
Any other multiplication or division? Yes.	
Multiply.	$3 + 12$
Any other multiplication of division? No.	
Any addition or subtraction? Yes.	
Add.	15

Note:
Exercise:

Problem: Simplify: $30 \div 5 + 10\left(3 - 2\right)$.

Solution:

16

Note:
Exercise:

Problem: Simplify: $70 \div 10 + 4\left(6 - 2\right)$.

Solution:

23

When there are multiple grouping symbols, we simplify the innermost parentheses first and work outward.

Example:
Exercise:

Problem: Simplify: $5 + 2^3 + 3\left[6 - 3\left(4 - 2\right)\right]$.

Solution:

	$5 + 2^3 + 3[6 - 3(4 - 2)]$
Are there any parentheses (or other grouping symbols)? Yes.	$5 + 2^3 + 3[6 - 3(4 - 2)]$
Focus on the parentheses that are inside the brackets. Subtract.	$5 + 2^3 + 3[6 - 3(2)]$
Continue inside the brackets and multiply.	$5 + 2^3 + 3[6 - 6]$
Continue inside the brackets and subtract.	$5 + 2^3 + 3[0]$
The expression inside the brackets requires no further simplification.	
Are there any exponents? Yes. Simplify exponents.	$5 + 8 + 3[0]$
Is there any multiplication or division? Yes.	
Multiply.	$5 + 8 + 0$

Is there any addition of subtraction? Yes.	
Add.	$13+0$
Add.	13

Note:
Exercise:

Problem: Simplify: $9+5^3-[4(9+3)]$.

Solution:

86

Note:
Exercise:

Problem: Simplify: $7^2-2[4(5+1)]$.

Solution:

1

Evaluate an Expression

In the last few examples, we simplified expressions using the order of operations. Now we'll evaluate some expressions—again following the order of operations. To **evaluate an expression** means to find the value of the expression when the variable is replaced by a given number.

Note:
Evaluate an Expression
To **evaluate an expression** means to find the value of the expression when the variable is replaced by a given number.

To evaluate an expression, substitute that number for the variable in the expression and then simplify the expression.

Example:
Exercise:

Problem: Evaluate when $x = 4$: ⓐ x^2 ⓑ 3^x ⓒ $2x^2 + 3x + 8$.

Solution:

ⓐ

			x^2
Replace x with 4.			4^2
Use definition of exponent.			$4 \cdot 4$
Simplify.			16

ⓑ

		3^x
Replace x with 4.		3^4
Use definition of exponent.		$3 \cdot 3 \cdot 3 \cdot 3$
Simplify.		81

ⓒ

	$2x^2 + 3x + 8$
Replace x with 4.	$2(4)^2 + 3(4) + 8$
Follow the order of operations.	$2(16) + 3(4) + 8$
	$32 + 12 + 8$
	52

Note:
Exercise:

Problem: Evaluate when $x = 3$, ⓐ x^2 ⓑ 4^x ⓒ $3x^2 + 4x + 1$.

Solution:

ⓐ 9 ⓑ 64 ⓒ40

Note:
Exercise:

Problem: Evaluate when $x = 6$, ⓐ x^3 ⓑ 2^x ⓒ $6x^2 - 4x - 7$.

Solution:

ⓐ 216 ⓑ 64 ⓒ 185

Properties of Whole Number Operations

Addition and Multiplication
Both Addition and Multiplication have these properties:
Addition Multiplication
Identity Property: a + 0 = a a * 1 = a
Commutative Property: a + b = b + a a x b = b x a
Associative Property: (a + b) + c = a + (b + c) (a x b) x c = a x (b x c)

Subtraction and Division

The Commutative and Associative Properties do not work for division. All it takes is one example to prove that they are not true for all whole numbers.

Commutative 7 - 4 = 3 while 4 - 7 does not equal 3.
Associative (7 - 4) - 1 = 3 -1 = 2 while 7 - (4 - 1) = 7 - 3 = 4.
The Commutative and Associative Properties are false for Subtraction and Division.

The Commutative and Associative Properties are useful because they allow us to rearrange additions and multiplications into any convenient order to make work easier. We'll see later that negative numbers allows us to replace subtractions with additions and fractions allows us to replace divisions with multiplications. Making these replacements can make work much easier.

The Distributive Property of Multiplication over Addition

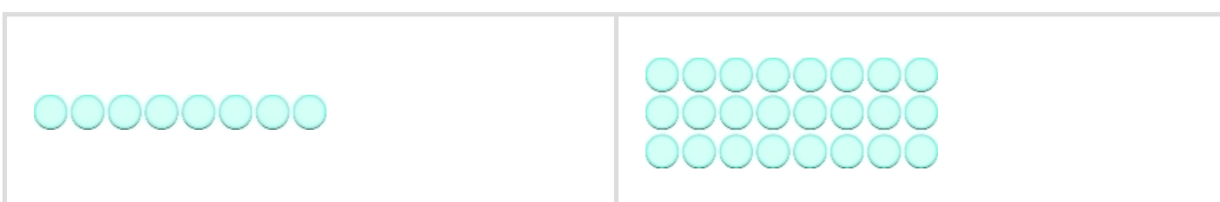
The Distributive Property of Multiplication over Addition is our first property that involves more than one operation. Normally it is just called the Distributive Property. The reason for the long name is that in more advanced math there are other distributive properties.

$a \cdot (b + c) = a \cdot b + a \cdot c$

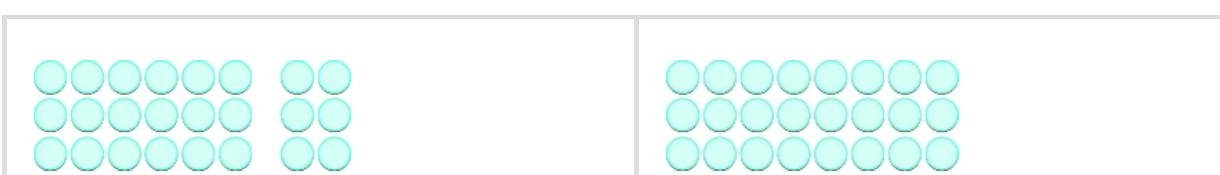
To help get a feeling for why this property is true, we'll use some numbers to see how it works.
For example, let a=3, b=6, and c=2. Then the equation becomes: 3 x (6 + 2) = (3 x 6) + (3 x 2)

The left hand side of the equation computes: 3 x (6 + 2) = 3 x 8 = 24.
The right hand side of the equation computes: (3 x 6) + (3 x 2) = 18 + 6 = 24. At least in this case, the property is true.

We can use counters to illustrate each side of the equation. The left hand side of the equation has one row of six counters with two counters added to it making one row of 8 counters. This is then multiplied to make 3 rows of 8 counters.



The right hand side of the equation starts with 3 rows of 6 counters followed by 3 rows of 2 counters. They are then added resulting in 3 rows of 8 counters.



Notice that in both cases that we end up with 24 counters arranged as 3 rows of 8.

This illustration is for a specific example, but this works for any number of rows no matter how long the rows are as long as they are all same length.

Multiplying Multi-digit Numbers with the Area Model

We are examining how the Distributive Property makes possible what we already know how to do so that we understand it deeply and can use the idea a little later in algebra. How we can multiply large numbers when we know only a few multiplication facts beyond a single digit times another single digit. Because our numbers are represented using place value this isn't necessary. We can master a procedure for multiplying large numbers that doesn't require learning any more facts than the basic ones we already know.

There is more than one correct way to do multi-digit multiplication. We will look at two related ways. The first way you might have learned in school. We'll call it the Compact Method or Standard Method. The second way uses the Area Model of Multiplication and so we'll call it the Area Method. It has the benefit of being easier to find your mistakes if you happen to make one. Let's look at the more common way first.

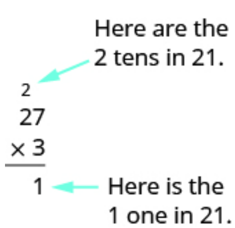
To multiply numbers with more than one digit, it is usually easier to write the numbers vertically in columns just as we did for addition and subtraction.
Equation:

$$\begin{array}{r} 27 \\ \underline{\times 3} \end{array}$$

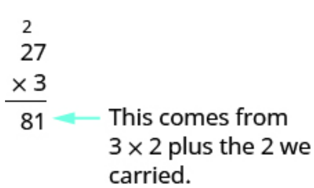
We start by multiplying 3 by 7.
Equation:

$$3 \times 7 = 21$$

We write the 1 in the ones place of the product. We carry the 2 tens by writing 2 above the tens place.

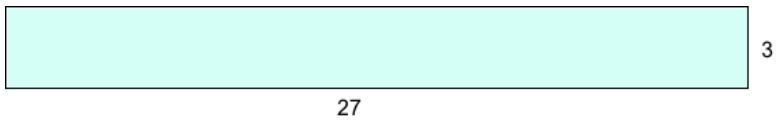


Then we multiply the 3 by the 2, and add the 2 above the tens place to the product. So $3 \times 2 = 6$, and $6 + 2 = 8$. Write the 8 in the tens place of the product.

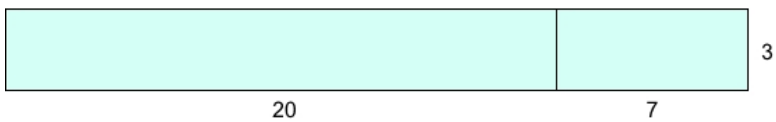


The product is 81.

Let's look at the same problem using the Area Model of Multiplication. Start by drawing a 27 by 3 rectangle. Occasionally, when we draw diagrams in math it helps to have the sizes fairly accurate. This is NOT one of those times, so don't worry about having the lengths perfect in relationship to each other.



The key to this method is to split any side that is has more than one digit into parts based on the Expanded Form for the number. In our example, we split 27 into 20 and 7. Because it is more familiar for us to have the 20 on the left and the 7 on the right, it is easiest if we do it that way. The other dimension, 3, is unchanged. Now we have two smaller rectangles rather than just one big rectangle.



We can find 27 x 3 if we can find the area of the 27 by 3 rectangle.
We can find the area of the large rectangle by finding the area of the two smaller rectangles and then adding them together.
This uses Expanded Form and then the Distributive Property.

3 x 27 = 3 x (20 + 7) = (3 x 20) + (3 x 7).

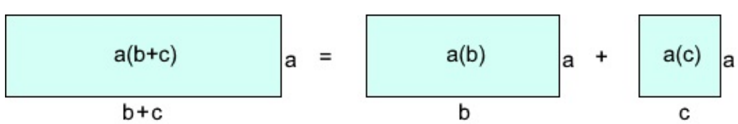
Calculating 3 x 7 is easy, it is just a basic multiplication fact.
Calculating 3 x 20 requires a little more work.
While you may already know that 3 x 20 = 60. Let's see how we know that using the properties of multiplication and easy multiplications.

Any number whose last digit is a "0" is the product of the number that has all the same digits in the same order but without the final "0" times 10.
In this case, 20 = 2 x 10. "2" has the same digits as "20" except for the final "0".

Substituting 2 x 10 for 20 gives: 3 x (2 x 10).
Using the Associative Property of Multiplication yields: (3 x 2) x 10.
Using a single digit multiplication fact: 6 x 10.
Finally using the Multiplication Property of 10 gives 60.

To finish the problem, we need to calculate 60 + 21 = 81.

Any multiplication can be thought of as an area of a rectangle where the sides are the factors being multiplied. Splitting the rectangle into two rectangles keeps the total area the same.



$a{\cdot}(b + c) = a \cdot b + a \cdot c$

Identify and Combine Like Terms

Algebraic expressions are made up of terms. A **term** is a constant, or the product of a constant and one or more variables.

Note:
Term
A **term** is a constant or the product of a constant and one or more variables.

Examples of terms are $7, y, 5x^2, 9a$, and b^5.

The constant that multiplies the variable is called the **coefficient**.

Note:
Coefficient
The **coefficient** of a term is the constant that multiplies the variable in a term.

Think of the coefficient as the number in front of the variable. The coefficient of the term $3x$ is 3. When we write x, the coefficient is 1, since $x = 1 \cdot x$.

Some terms share common traits. When two terms are constants or have the same variable and exponent, we say they are **like terms**.

Look at the following 6 terms. Which ones seem to have traits in common?
Equation:

$$5x \quad 7 \quad n^2 \quad 4 \quad 3x \quad 9n^2$$

We say,

7 and 4 are like terms.

$5x$ and $3x$ are like terms.

n^2 and $9n^2$ are like terms.

Note:
Like Terms
Terms that are either constants or have the same variables raised to the same powers are called **like terms.**

If there are like terms in an expression, you can simplify the expression by combining the like terms. We add the coefficients and keep the same variable.

The Distributive Property is why we can do this. Although equations have both a left hand side and a right hand side, it is only because in many other languages that we read from left to right that we often think of the right hand side replacing the left hand side. In some other languages, such as Arabic, Hebrew, and Persian, text

is read right to left. In mathematics it does not matter. Either side can replace the other.

Below is the Distributive Property written in the other direction. This isn't really needed because every math equation can be reversed so we don't normally bother to write it both ways.

$a \cdot b + a \cdot c = a{\cdot}(b + c)$

Factoring
When done in the opposite direction, the Distributive Property is called **Factoring**. This is because the variable a and the sum (b + c) are factors in the product $a{\cdot}(b + c)$.
Equation:

$$\begin{array}{ll} \text{Simplify.} & 2x + 5x \\ \text{Add the coefficients.} & 7x \end{array}$$

Here are the mathematical details that we normally would not bother to write.

$2x + 5x = x2 + x5 = x(2 + 5) = x7 = 7x.$

As long as the terms being added have exactly the same variable part then that part can be factored leaving just numbers in the parenthesis. These numbers can be added giving the final result.
This only works if the variable parts are exactly the same. That is why we say that we can add like terms. The only difference allowed is their coefficients.
Equation:

$$\begin{array}{ll} \text{Simplify.} & 4x + 7x + x \\ \text{Add the coefficients.} & 12x \end{array}$$

Example:
How To Combine Like Terms
Exercise:

Problem: Simplify: $2x^2 + 3x + 7 + x^2 + 4x + 5$.

Solution:

Step 1. Identify the like terms.	$2x^2 + 3x + 7 + x^2 + 4x + 5$ $\underline{2x^2} + \underline{3x} + \underline{7} + \underline{x^2} + \underline{4x} + \underline{5}$ $2x^2 + 3x + 7 + x^2 + 4x + 5$

Step 2. Rearrange the expression so the like terms are together.	$\underline{2x^2 + x^2} + \underline{3x + 4x} + \underline{7 + 5}$ $2x^2 + x^2 + 3x + 4x + 7 + 5$

Step 3. Combine like terms.	$3x^2 + 7x + 12$

Note:
Exercise:

Problem: Simplify: $3x^2 + 7x + 9 + 7x^2 + 9x + 8$.

Solution:

$10x^2 + 16x + 17$

Note:
Exercise:

Problem: Simplify: $4y^2 + 5y + 2 + 8y^2 + 4y + 5$.

Solution:

$12y^2 + 9y + 7$

Note:
Combine like terms.

Identify like terms.
Rearrange the expression so like terms are together.
Add or subtract the coefficients and keep the same variable for each group of like terms.

Translate an English Phrase to an Algebraic Expression

We listed many operation symbols that are used in algebra. Now, we will use them to translate English phrases into algebraic expressions. The symbols and variables we've talked about will help us do that. [link] summarizes them.

Operation	Phrase	Expression

Operation	Phrase	Expression
Addition	a plus b the sum of a and b a increased by b b more than a the total of a and b b added to a	$a+b$
Subtraction	a minus b the difference of a and b a decreased by b b less than a b subtracted from a	$a-b$
Multiplication	a times b the product of a and b twice a	$a \cdot b, ab, a(b), (a)(b)$ $2a$
Division	a divided by b the quotient of a and b the ratio of a and b b divided into a	$a \div b, a/b, \frac{a}{b}, b\overline{)a}$

Look closely at these phrases using the four operations:

the **sum** *of a and b*

the **difference** *of a and b*

the **product** *of a and b*

the **quotient** *of a and b*

Each phrase tells us to operate on two numbers. Look for the words *of* and *and* to find the numbers.

Example:
Exercise:

Problem: Translate each English phrase into an algebraic expression:

ⓐ the difference of $14x$ and 9
ⓑ the quotient of $8y^2$ and 3
ⓒ twelve more than y
ⓓ seven less than $49x^2$

Solution:

ⓐ The key word is *difference*, which tells us the operation is subtraction. Look for the words *of* and *and to* find the numbers to subtract.

the *difference of* 14*x* *and* 9

14*x* minus 9

$14x - 9$

ⓑ The key word is *quotient*, which tells us the operation is division.

the *quotient of* $8y^2$ *and* 3

divide $8y^2$ by 3

$8y^2 \div 3$

This can also be written $8y^2/3$ or $\frac{8y^2}{3}$.

ⓒ The key words are *more than*. They tell us the operation is addition. *More than* means "added to."
Equation:

$$\begin{array}{c}\text{twelve more than } y \\ \text{twelve added to } y \\ y + 12\end{array}$$

ⓓ The key words are *less than*. They tell us to subtract. *Less than* means "subtracted from."
Equation:

$$\begin{array}{c}\text{seven less than } 49x^2 \\ \text{seven subtracted from } 49x^2 \\ 49x^2 - 7\end{array}$$

Note:
Exercise:

Problem: Translate the English phrase into an algebraic expression:

ⓐ the difference of $14x^2$ and 13
ⓑ the quotient of $12x$ and 2
ⓒ 13 more than z
ⓓ 18 less than $8x$

Solution:

ⓐ $14x^2 - 13$ ⓑ $12x \div 2$
ⓒ $z + 13$ ⓓ $8x - 18$

Note:
Exercise:

Problem: Translate the English phrase into an algebraic expression:

ⓐ the sum of $17y^2$ and 19
ⓑ the product of 7 and *y*
ⓒ Eleven more than *x*
ⓓ Fourteen less than 11*a*

Solution:

ⓐ $17y^2 + 19$ ⓑ $7y$
ⓒ $x + 11$ ⓓ $11a - 14$

We look carefully at the words to help us distinguish between multiplying a sum and adding a product.

Example:
Exercise:

Problem: Translate the English phrase into an algebraic expression:

ⓐ eight times the sum of *x* and *y*
ⓑ the sum of eight times *x* and *y*

Solution:

There are two operation words—*times* tells us to multiply and *sum* tells us to add.

ⓐ Because we are multiplying 8 times the sum, we need parentheses around the sum of *x* and *y*, $(x + y)$. This forces us to determine the sum first. (Remember the order of operations.)
Equation:

$$\begin{array}{c}\text{eight times the sum of } x \text{ and } y \\ 8(x+y)\end{array}$$

ⓑ To take a sum, we look for the words *of* and *and* to see what is being added. Here we are taking the sum *of* eight times *x* and *y*.

the sum *of* eight times *x* *and* *y*

$8x + y$

Note:
Exercise:

Problem: Translate the English phrase into an algebraic expression:

ⓐ four times the sum of *p* and *q*
ⓑ the sum of four times *p* and *q*

Solution:

ⓐ $4\left(p+q\right)$ ⓑ $4p+q$

Note:
Exercise:

Problem: Translate the English phrase into an algebraic expression:

ⓐ the difference of two times *x* and 8
ⓑ two times the difference of *x* and 8

Solution:

ⓐ $2x-8$ ⓑ$2\left(x-8\right)$

Later in this course, we'll apply our skills in algebra to solving applications. The first step will be to translate an English phrase to an algebraic expression. We'll see how to do this in the next two examples.

Example:
Exercise:

Problem:

The length of a rectangle is 14 less than the width. Let *w* represent the width of the rectangle. Write an expression for the length of the rectangle.

Solution:

Write a phrase about the length of the rectangle.	14 less than the width
Substitute w for "the width."	w
Rewrite *less than* as *subtracted from*.	14 subtracted from w
Translate the phrase into algebra.	$w-14$

Note:
Exercise:

Problem:

The length of a rectangle is 7 less than the width. Let *w* represent the width of the rectangle. Write an expression for the length of the rectangle.

Solution:

$w-7$

Note:
Exercise:

Problem:

The width of a rectangle is 6 less than the length. Let l represent the length of the rectangle. Write an expression for the width of the rectangle.

Solution:

$l-6$

The expressions in the next example will be used in the typical coin mixture problems we will see soon.

Example:
Exercise:

Problem:

June has dimes and quarters in her purse. The number of dimes is seven less than four times the number of quarters. Let q represent the number of quarters. Write an expression for the number of dimes.

Solution:

Write a phrase about the number of dimes.	seven less than four times the number of quarters
Substitute q for the number of quarters.	7 less than 4 times q
Translate 4 times q.	7 less than $4q$
Translate the phrase into algebra.	$4q-7$

Note:
Exercise:

Problem:

Geoffrey has dimes and quarters in his pocket. The number of dimes is eight less than four times the number of quarters. Let q represent the number of quarters. Write an expression for the number of dimes.

Solution:

$4q-8$

Note:
Exercise:

Problem:

Lauren has dimes and nickels in her purse. The number of dimes is three more than seven times the number of nickels. Let *n* represent the number of nickels. Write an expression for the number of dimes.

Solution:

$7n+3$

Key Concepts

- **Divisibility Tests**
  A number is divisible by:
  2 if the last digit is 0, 2, 4, 6, or 8.
  3 if the sum of the digits is divisible by 3.
  5 if the last digit is 5 or 0.
  6 if it is divisible by both 2 and 3.
  10 if it ends with 0.
- **How to find the prime factorization of a composite number.**

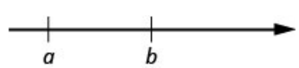
  Find two factors whose product is the given number, and use these numbers to create two branches.
  If a factor is prime, that branch is complete. Circle the prime, like a bud on the tree.
  If a factor is not prime, write it as the product of two factors and continue the process.
  Write the composite number as the product of all the circled primes.

- **How To Find the least common multiple using the prime factors method.**

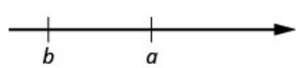
  Write each number as a product of primes.
  List the primes of each number. Match primes vertically when possible.
  Bring down the columns.
  Multiply the factors.

- **Equality Symbol**
  $a=b$ is read “a is equal to b.”
  The symbol “=” is called the equal sign.
- **Inequality**

  $a < b$ is read “a is less than b”
  a is to the left of b on the number line

  $a > b$ is read “a is greater than b”
  a is to the right of b on the number line

- **Inequality Symbols**

Inequality Symbols	Words

Inequality Symbols	Words
$a \neq b$	*a* is *not equal to b.*
$a < b$	*a* is *less than b.*
$a \leq b$	*a* is *less than or equal to b.*
$a > b$	*a* is *greater than b.*
$a \geq b$	*a* is *greater than or equal to b.*

- **Grouping Symbols**
  Parentheses $(\ )$
  Brackets $[\ ]$
  Braces $\{\ \}$
- **Exponential Notation**
  a^n means multiply *a* by itself, *n* times.
  The expression a^n is read *a* to the n^{th} power.
- **Simplify an Expression**
  To simplify an expression, do all operations in the expression.
- **How to use the order of operations.**

  Parentheses and Other Grouping Symbols
  - Simplify all expressions inside the parentheses or other grouping symbols, working on the innermost parentheses first.

  Exponents
  - Simplify all expressions with exponents.

  Multiplication and Division
  - Perform all multiplication and division in order from left to right. These operations have equal priority.

  Addition and Subtraction
  - Perform all addition and subtraction in order from left to right. These operations have equal priority.

- **How to combine like terms.**

  Identify like terms.
  Rearrange the expression so like terms are together.
  Add or subtract the coefficients and keep the same variable for each group of like terms.

Operation	Phrase	Expression

Operation	Phrase	Expression
Addition	a plus b the sum of a and b a increased by b b more than a the total of a and b b added to a	$a+b$
Subtraction	a minus b the difference of a and b a decreased by b b less than a b subtracted from a	$a-b$
Multiplication	a times b the product of a and b twice a	$a\cdot b, ab, a(b), (a)(b)$ $2a$
Division	a divided by b the quotient of a and b the ratio of a and b b divided into a	$a \div b, a/b, \frac{a}{b}, b\overline{)a}$

Practice Makes Perfect

Identify Multiples and Factors

In the following exercises, use the divisibility tests to determine whether each number is divisible by 2, by 3, by 5, by 6, and by 10.

Exercise:

Problem: 84

Solution:

Divisible by 2, 3, 6

Exercise:

Problem: 96

Exercise:

Problem: 896

Solution:

Divisible by 2

Exercise:

Problem: 942

Exercise:

Problem: 22,335

Solution:

Divisible by 3, 5

Exercise:

Problem: 39,075

Find Prime Factorizations and Least Common Multiples

In the following exercises, find the prime factorization.

Exercise:

Problem: 86

Solution:

$2 \cdot 43$

Exercise:

Problem: 78

Exercise:

Problem: 455

Solution:

$5 \cdot 7 \cdot 13$

Exercise:

Problem: 400

Exercise:

Problem: 432

Solution:

$2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$

Exercise:

Problem: 627

In the following exercises, find the least common multiple of each pair of numbers using the prime factors method.
Exercise:

Problem: 8, 12

Solution:

24

Exercise:

Problem: 12, 16

Exercise:

Problem: 28, 40

Solution:

420

Exercise:

Problem: 84, 90

Exercise:

Problem: 55, 88

Solution:

440

Exercise:

Problem: 60, 72

Simplify Expressions Using the Order of Operations

In the following exercises, simplify each expression.
Exercise:

Problem: $2^3 - 12 \div (9 - 5)$

Solution:

5

Exercise:

Problem: $3^2 - 18 \div (11 - 5)$

Exercise:

Problem: $2+8(6+1)$

Solution:

58

Exercise:

Problem: $4+6(3+6)$

Exercise:

Problem: $20 \div 4+6\,(5-1)$

Solution:

29

Exercise:

Problem: $33 \div 3+4\,(7-2)$

Exercise:

Problem: $3(1+9\cdot 6)-4^2$

Solution:

149

Exercise:

Problem: $5(2+8\cdot 4)-7^2$

Exercise:

Problem: $2\,[1+3\,(10-2)]$

Solution:

50

Exercise:

Problem: $5\,[2+4\,(3-2)]$

Exercise:

Problem: $8+2\,[7-2\,(5-3)]-3^2$

Solution:

5

Exercise:

Problem: $10+3\left[6-2\left(4-2\right)\right]-2^{4}$

Evaluate an Expression

In the following exercises, evaluate the following expressions.

Exercise:

When $x=2$,
ⓐ x^{6}
ⓑ 4^{x}
Problem: ⓒ $2x^{2}+3x-7$

Solution:

ⓐ 64 ⓑ 16 ⓒ 7

Exercise:

When $x=3$,
ⓐ x^{5}
ⓑ 5^{x}
Problem: ⓒ $3x^{2}-4x-8$

Exercise:

When $x=4,y=1$
Problem: $x^{2}+3xy-7y^{2}$

Solution:

21

Exercise:

When $x=3,y=2$
Problem: $6x^{2}+3xy-9y^{2}$

Exercise:

When $x=10,y=7$
Problem: $\left(x-y\right)^{2}$

Solution:

9

Exercise:

When $a=3,b=8$
Problem: $a^{2}+b^{2}$

Simplify Expressions by Combining Like Terms

In the following exercises, simplify the following expressions by combining like terms.
Exercise:

Problem: $7x + 2 + 3x + 4$

Solution:

$10x + 6$

Exercise:

Problem: $8y + 5 + 2y - 4$

Exercise:

Problem: $10a + 7 + 5a - 2 + 7a - 4$

Solution:

$22a + 1$

Exercise:

Problem: $7c + 4 + 6c - 3 + 9c - 1$

Exercise:

Problem: $3x^2 + 12x + 11 + 14x^2 + 8x + 5$

Solution:

$17x^2 + 20x + 16$

Exercise:

Problem: $5b^2 + 9b + 10 + 2b^2 + 3b - 4$

Translate an English Phrase to an Algebraic Expression

In the following exercises, translate the phrases into algebraic expressions.
Exercise:

Problem:
ⓐ the difference of $5x^2$ and $6xy$
ⓑ the quotient of $6y^2$ and $5x$
ⓒ Twenty-one more than y^2
ⓓ $6x$ less than $81x^2$

Solution:

ⓐ $5x^2 - 6xy$ ⓑ $\frac{6y^2}{5x}$
ⓒ $y^2 + 21$ ⓓ $81x^2 - 6x$

Exercise:

ⓐ the difference of $17x^2$ and $5xy$
ⓑ the quotient of $8y^3$ and $3x$
ⓒ Eighteen more than a^2;
Problem: ⓓ $11b$ less than $100b^2$

Exercise:

ⓐ the sum of $4ab^2$ and $3a^2b$
ⓑ the product of $4y^2$ and $5x$
ⓒ Fifteen more than m
Problem: ⓓ $9x$ less than $121x^2$

Solution:

ⓐ $4ab^2 + 3a^2b$ ⓑ $20xy^2$
ⓒ $m + 15$ ⓓ $121x^2 - 9x$

Exercise:

ⓐ the sum of $3x^2y$ and $7xy^2$
ⓑ the product of $6xy^2$ and $4z$
ⓒ Twelve more than $3x^2$
Problem: ⓓ $7x^2$ less than $63x^3$

Exercise:

ⓐ eight times the difference of y and nine
Problem: ⓑ the difference of eight times y and 9

Solution:

ⓐ $8(y-9)$ ⓑ $8y-9$

Exercise:

ⓐ seven times the difference of y and one
Problem: ⓑ the difference of seven times y and 1

Exercise:

ⓐ five times the sum of $3x$ and y
Problem: ⓑ the sum of five times $3x$ and y

Solution:

ⓐ $5(3x+y)$ ⓑ $15x+y$

Exercise:

ⓐ eleven times the sum of $4x^2$ and $5x$

Problem: ⓑ the sum of eleven times $4x^2$ and $5x$

Exercise:

Problem:

Eric has rock and country songs on his playlist. The number of rock songs is 14 more than twice the number of country songs. Let c represent the number of country songs. Write an expression for the number of rock songs.

Solution:

$14 > 2c$

Exercise:

Problem:

The number of women in a Statistics class is 8 more than twice the number of men. Let m represent the number of men. Write an expression for the number of women.

Exercise:

Problem:

Greg has nickels and pennies in his pocket. The number of pennies is seven less than three the number of nickels. Let n represent the number of nickels. Write an expression for the number of pennies.

Solution:

$3n - 7$

Exercise:

Problem:

Jeannette has $\$5$ and $\$10$ bills in her wallet. The number of fives is three more than six times the number of tens. Let t represent the number of tens. Write an expression for the number of fives.

Writing Exercises

Exercise:

Problem: Explain in your own words how to find the prime factorization of a composite number.

Solution:

Answers will vary.

Exercise:

Problem: Why is it important to use the order of operations to simplify an expression?

Exercise:

Problem: Explain how you identify the like terms in the expression $8a^2 + 4a + 9 - a^2 - 1$.

Solution:

Answers will vary.

Exercise:

Problem:

Explain the difference between the phrases "4 times the sum of *x* and *y*" and "the sum of 4 times *x* and *y*".

Self Check

ⓐ Use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
identify multiples and apply divisibility tests.			
find prime factorizations and least common multiples.			
use variables and algebraic symbols.			
simplify expressions using the order of operations.			
evaluate an expression.			
identify and combine like terms.			
translate English phrases to algebraic expressions.			

ⓑ If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

Glossary

coefficient
: The coefficient of a term is the constant that multiplies the variable in a term.

composite number
: A composite number is a counting number that is not prime. It has factors other than 1 and the number itself.

constant
A constant is a number whose value always stays the same.

divisible by a number
If a number m is a multiple of n, then m is divisible by n.

equation
An equation is two expressions connected by an equal sign.

evaluate an expression
To evaluate an expression means to find the value of the expression when the variables are replaced by a given number.

expression
An expression is a number, a variable, or a combination of numbers and variables using operation symbols.

factors
If $a \cdot b = m$, then a and b are factors of m.

least common multiple
The least common multiple (LCM) of two numbers is the smallest number that is a multiple of both numbers.

like terms
Terms that are either constants or have the same variables raised to the same powers are called like terms.

multiple of a number
A number is a multiple of n if it is the product of a counting number and n.

order of operations
The order of operations are established guidelines for simplifying an expression.

prime factorization
The prime factorization of a number is the product of prime numbers that equals the number.

prime number
A prime number is a counting number greater than 1 whose only factors are 1 and the number itself.

simplify an expression
To simplify an expression means to do all the math possible.

term
A term is a constant, or the product of a constant and one or more variables.

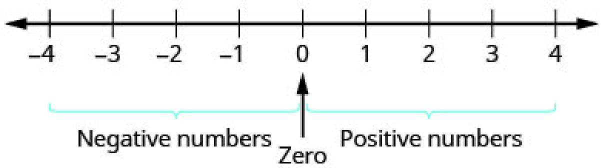
variable
A variable is a letter that represents a number whose value may change.

Integers: ASE
By the end of this section, you will be able to:

- Simplify expressions with absolute value
- Add and subtract integers
- Multiply and divide integers
- Simplify expressions with integers
- Evaluate variable expressions with integers
- Translate phrases to expressions with integers
- Use integers in applications

Simplify Expressions with Absolute Value

A **negative number** is a number less than 0. The negative numbers are to the left of zero on the number line. See [link].

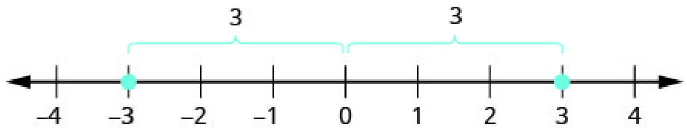


The number line shows the location of positive and negative numbers.

You may have noticed that, on the number line, the negative numbers are a mirror image of the positive numbers, with zero in the middle. Because the numbers 2 and -2 are the same distance from zero, each one is called the **opposite** of the other. The opposite of 2 is -2, and the opposite of -2 is 2.

Note:
Opposite
The **opposite** of a number is the number that is the same distance from zero on the number line but on the opposite side of zero.

[link] illustrates the definition.



The opposite of 3 is -3, and the opposite of -3 is 3.

Note:
Opposite Notation
Equation:

$$-a \text{ means the opposite of the number } a$$
$$\text{The notation } -a \text{ is read as "the opposite of } a\text{."}$$

Equation:

The Opposite of an Opposite is the Original Number

$$\text{-(-(a))} = a$$

Example:
-(-(3)) = 3.

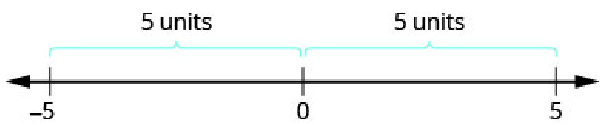
We saw that numbers such as 3 and -3 are opposites because they are on opposite sides of 0 and the same distance from 0 on the number line. They are both three units from 0. The distance between 0 and any number on the number line is called the **absolute value** of that number.

Note:
Absolute Value
The **absolute value** of a number is its distance from 0 on the number line.
The absolute value of a number n is written as $|n|$ and $|n| \geq 0$ for all numbers.
Absolute values are always greater than or equal to zero.

For example,
Equation:

$$\begin{array}{c}-5 \text{ is } 5 \text{ units away from } 0, \text{ so } |-5| = 5. \\ 5 \text{ is } 5 \text{ units away from } 0, \text{ so } |5| = 5.\end{array}$$

[link] illustrates this idea.



The numbers 5 and -5 are 5 units away from 0.

The absolute value of a number is never negative because distance cannot be negative. The only number with absolute value equal to zero is the number zero itself because the distance from 0 to 0 on the number line is zero units.

In the next example, we'll order expressions with absolute values.

Example:
Exercise:

Problem: Fill in $<$, $>$, or $=$ for each of the following pairs of numbers:

ⓐ $|-5|$ ___ $-|-5|$ ⓑ 8 ___ $-|-8|$ ⓒ -9 ___ $-|-9|$ ⓓ $-(-16)$ ___ $|-16|$.

Solution:

ⓐ

$$\begin{array}{lrcl} & |-5| & \_\_ & -|-5| \\ \text{Simplify.} & 5 & \_\_ & -5 \\ \text{Order.} & 5 & > & -5 \\ & |-5| & > & -|-5| \end{array}$$

ⓑ

	8	__	$-\|-8\|$
Simplify.	8	__	-8
Order.	8	$>$	-8
	8	$>$	$-\|-8\|$

ⓒ

	-9	__	$-\|-9\|$
Simplify.	-9	__	-9
Order.	-9	$=$	-9
	-9	$=$	$-\|-9\|$

ⓓ

	$-(-16)$	__	$\|-16\|$
Simplify.	16	__	16
Order.	16	$=$	16
	$-(-16)$	$=$	$\|-16\|$

Note:
Exercise:

Problem: Fill in $<, >$, or $=$ for each of the following pairs of numbers:

ⓐ -9 __ $-|-9|$ ⓑ 2 __ $-|-2|$ ⓒ -8 __ $|-8|$ ⓓ $-(-9)$ __ $|-9|$.

Solution:

ⓐ $>$ ⓑ $>$ ⓒ $<$
ⓓ $=$

Note:
Exercise:

Problem: Fill in $<, >$, or $=$ for each of the following pairs of numbers:

ⓐ 7 __ $-|-7|$ ⓑ $-(-10)$ __ $|-10|$ ⓒ $|-4|$ __ $-|-4|$ ⓓ -1 __ $|-1|$.

Solution:

ⓐ $>$ ⓑ $=$ ⓒ $>$
ⓓ $<$

We now add absolute value bars to our list of grouping symbols. When we use the order of operations, first we simplify inside the absolute value bars as much as possible, then we take the absolute value of the resulting number.

Note:
Grouping Symbols
Equation:

$$\begin{array}{llll} \text{Parentheses} & (\,) & \text{Braces} & \{\,\} \\ \text{Brackets} & [\,] & \text{Absolute value} & |\,| \end{array}$$

In the next example, we simplify the expressions inside absolute value bars first just like we do with parentheses.

Example:
Exercise:

Problem: Simplify: $24 - |19 - 3\,(6 - 2)|$.

Solution:

	$24 - \lvert 19 - 3\,(6 - 2)\rvert$
Work inside parentheses first: subtract 2 from 6.	$24 - \lvert 19 - 3\,(4)\rvert$
Multiply 3(4).	$24 - \lvert 19 - 12\rvert$
Subtract inside the absolute value bars.	$24 - \lvert 7\rvert$
Take the absolute value.	$24 - 7$
Subtract.	17

Note:
Exercise:

Problem: Simplify:$19 - |11 - 4\,(3 - 1)|$.

Solution:

16

Note:
Exercise:

Problem: Simplify: $9 - |8 - 4(7 - 5)|$.

Solution:

9

Adding Integers

Now that we have located positive and negative numbers on the number line, it is time to discuss arithmetic operations with integers.

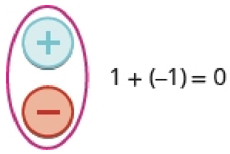
Most students are comfortable with the addition and subtraction facts for positive numbers. But doing addition or subtraction with both positive and negative numbers may be more difficult. This difficulty relates to the way the brain learns.

The brain learns best by working with objects in the real world and then generalizing to abstract concepts. Toddlers learn quickly that if they have two cookies and their older brother steals one, they have only one left. This is a concrete example of $2 - 1$. Children learn their basic addition and subtraction facts from experiences in their everyday lives. Eventually, they know the number facts without relying on cookies.

Addition and subtraction of negative numbers have fewer real world examples that are meaningful to us. Math teachers have several different approaches, such as number lines, banking, temperatures, and so on, to make these concepts real.

We will model addition and subtraction of negatives with two color counters. We let a blue counter represent a positive and a red counter will represent a negative. To make this easier to read when printed without color, we have placed a "+" in the positive counters and a "-" in the negative counters.

If we have one positive and one negative counter, the value of the pair is zero. They form a neutral pair. The value of this neutral pair is zero as summarized in [link].



A blue counter represents $+1$. A red counter represents -1. Together they add to zero.

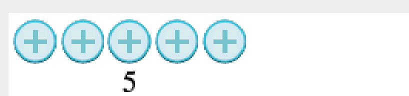
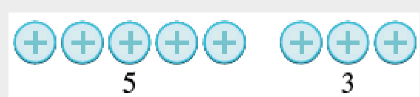
We will model four addition facts using the numbers $5, -5$ and 3 , -3.
Equation:

$$5+3 \quad -5+(-3) \quad -5+3 \quad 5+(-3)$$

Example:
Exercise:

Problem: Model: $5+3$.

Solution:
Solution

Interpret the expression.	$5+3$ means the sum of 5 and 3.
Model the first number. Start with 5 positives.	
Model the second number. Add 3 positives.	
Count the total number of counters.	

	
The sum of 5 and 3 is 8.	$5+3=8$

Note:
Exercise:

Problem: Model the expression.

$2+4$

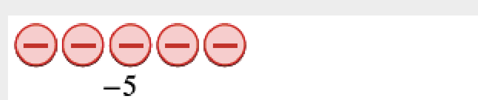
Solution:

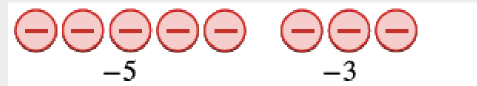
6

Example:
Exercise:

Problem: Model: $-5+(-3)$.

Solution:
Solution

Interpret the expression.	$-5+(-3)$ means the sum of -5 and -3.
Model the first number. Start with 5 negatives.	

Model the second number. Add 3 negatives.	
Count the total number of counters.	
The sum of −5 and −3 is −8.	$-5 + -3 = -8$

Note:
Exercise:

Problem: Model the expression.

$-2 + (-4)$

Solution:

−6

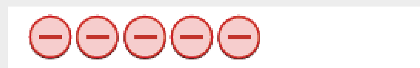
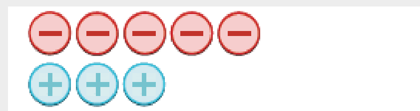
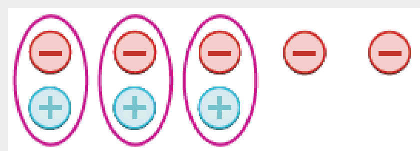
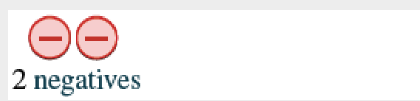
[link] and [link] are very similar. The first example adds 5 positives and 3 positives—both positives. The second example adds 5 negatives and 3 negatives—both negatives. In each case, we got a result of 8—either 8 positives or 8 negatives. When the signs are the same, the counters are all the same color.

Now let’s see what happens when the signs are different.

Example:
Exercise:

Problem: Model: $-5 + 3$.

Solution:
Solution

Interpret the expression.	$-5+3$ means the sum of -5 and 3.
Model the first number. Start with 5 negatives.	
Model the second number. Add 3 positives.	
Remove any neutral pairs.	
Count the result.	
The sum of −5 and 3 is −2.	$-5+3=-2$

Notice that there were more negatives than positives, so the result is negative.

Note:
Exercise:

Problem: Model the expression, and then simplify:

$2+(-4)$

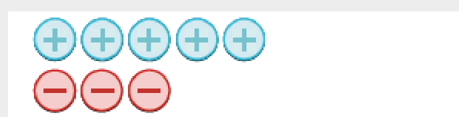
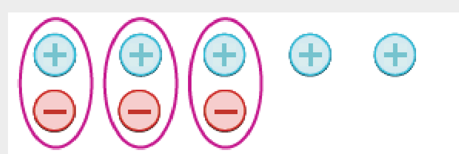
Solution:

−2

Example:
Exercise:

Problem: Model: $5+(-3)$.

Solution:
Solution

Interpret the expression.	$5+(-3)$ means the sum of 5 and -3.
Model the first number. Start with 5 positives.	
Model the second number. Add 3 negatives.	
Remove any neutral pairs.	
Count the result.	

The sum of 5 and −3 is 2.	$5 + (-3) = 2$

Note:
Exercise:

Problem: Model the expression, and then simplify:

$(-2) + 4$

Solution:

2

Subtract Integers

Remember the story in the last section about the toddler and the cookies? Children learn how to subtract numbers through their everyday experiences. Real-life experiences serve as models for subtracting positive numbers, and in some cases, such as temperature, for adding negative as well as positive numbers. But it is difficult to relate subtracting negative numbers to common life experiences. Most people do not have an intuitive understanding of subtraction when negative numbers are involved. Math teachers use several different models to explain subtracting negative numbers.

We will continue to use counters to model subtraction. Remember, the blue counters represent positive numbers and the red counters represent negative numbers.

Perhaps when you were younger, you read $5 - 3$ as *five take away three*. When we use counters, we can think of subtraction the same way.

We will model four subtraction facts using the numbers 5 and 3.
Equation:

$$5 - 3 \quad -5 - (-3) \quad -5 - 3 \quad 5 - (-3)$$

Example:
Exercise:

Problem: Model: $5 - 3$.

Solution:
Solution

Interpret the expression.	$5 - 3$ means 5 take away 3.
Model the first number. Start with 5 positives.	
Take away the second number. So take away 3 positives.	
Find the counters that are left.	
	$5 - 3 = 2$. The difference between 5 and 3 is 2.

Note:
Exercise:

Problem: Model the expression:

$6 - 4$

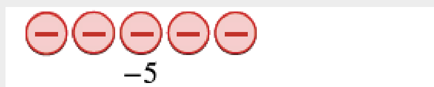
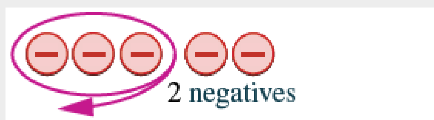
Solution:

2

Example:
Exercise:

Problem: Model: $-5-(-3)$.

Solution:
Solution

Interpret the expression.	$-5-(-3)$ means -5 take away -3.
Model the first number. Start with 5 negatives.	
Take away the second number. So take away 3 negatives.	
Find the number of counters that are left.	
	$-5-(-3)=-2$. The difference between -5 and -3 is -2.

Note:
Exercise:

Problem: Model the expression:

$-6 - (-4)$

Solution:

-2

Notice that [link] and [link] are very much alike.

- First, we subtracted 3 positives from 5 positives to get 2 positives.
- Then we subtracted 3 negatives from 5 negatives to get 2 negatives.

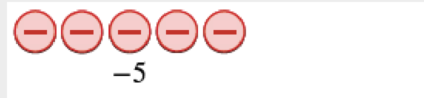
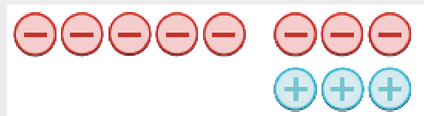
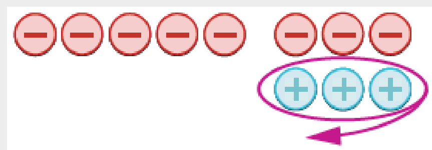
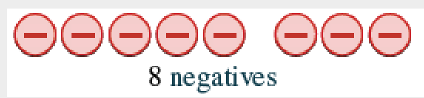
Each example used counters of only one color, and the “take away” model of subtraction was easy to apply.

Now let’s see what happens when we subtract one positive and one negative number. We will need to use both positive and negative counters and sometimes some neutral pairs, too. Adding a neutral pair does not change the value.

Example:
Exercise:

Problem: Model: $-5 - 3$.

Solution:
Solution

Interpret the expression.	$-5-3$ means -5 take away 3.
Model the first number. Start with 5 negatives.	
Take away the second number. So we need to take away 3 positives.	
But there are no positives to take away. Add neutral pairs until you have 3 positives.	
Now take away 3 positives.	
Count the number of counters that are left.	
	$-5-3=-8$. The difference of -5 and 3 is -8.

Note:
Exercise:

Problem: Model the expression:

$-6-4$

Solution:

-10

Example:
Exercise:

Problem: Model: $5-(-3)$.

Solution:
Solution

Interpret the expression.	$5-(-3)$ means 5 take away -3.
Model the first number. Start with 5 positives.	
Take away the second number, so take away 3 negatives.	
But there are no negatives to take away. Add neutral pairs until you have 3 negatives.	
Then take away 3 negatives.	
Count the number of counters that are left.	

	The difference of 5 and -3 is 8. $5-(-3)=8$

Note:
Exercise:

Problem: Model the expression:

$6-(-4)$

Solution:

10

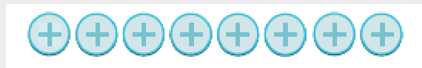
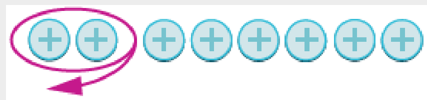
Example:
Exercise:

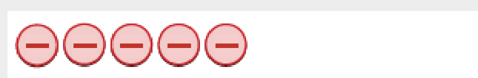
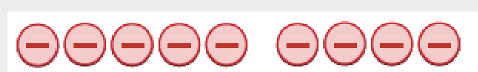
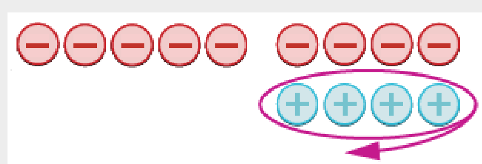
Problem: Model each subtraction.

ⓐ 8 − 2
ⓑ −5 − 4
ⓒ 6 − (−6)
ⓓ −8 − (−3)

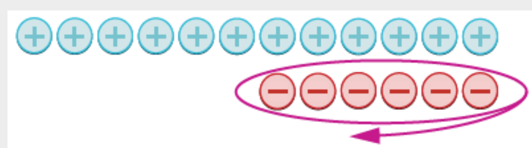
Solution:

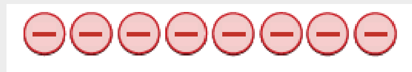
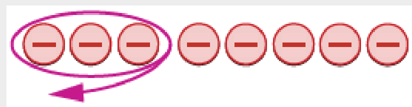
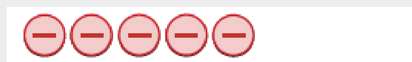
ⓐ	

	$8 - 2$ This means 8 take away 2.
Start with 8 positives.	
Take away 2 positives.	
How many are left?	6
	$8 - 2 = 6$

ⓑ	
	$-5 - 4$ This means -5 take away 4.
Start with 5 negatives.	
You need to take away 4 positives. Add 4 neutral pairs to get 4 positives.	 
Take away 4 positives.	

How many are left?	
	-9
	$-5 - 4 = -9$

ⓒ	
	$6 - (-6)$ This means 6 take away -6.
Start with 6 positives.	
Add 6 neutrals to get 6 negatives to take away.	
Remove 6 negatives.	
How many are left?	
	12
	$6 - (-6) = 12$

ⓓ	
	$-8-(-3)$ This means -8 take away -3.
Start with 8 negatives.	
Take away 3 negatives.	
How many are left?	
	-5
	$-8-(-3)=-5$

Note:
Exercise:

Problem: Model each subtraction.

ⓐ 7 - (-8)
ⓑ -2 - (-2)
ⓒ 4 - 1
ⓓ -6 - 8

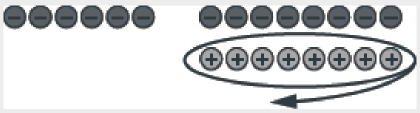
Solution:

ⓐ

ⓑ

ⓒ

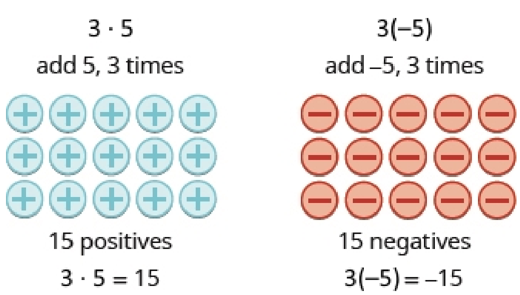
ⓓ



Multiply Integers

Since multiplication can be thought of as repeated addition, our counter model can show multiplication of integers. Let’s look at this concrete model to see what patterns we notice. We will use the same examples that we used for addition and subtraction.

We remember that $a \cdot b$ means add a addends of b. Here, we are using the model shown in [link] just to help us discover the pattern.



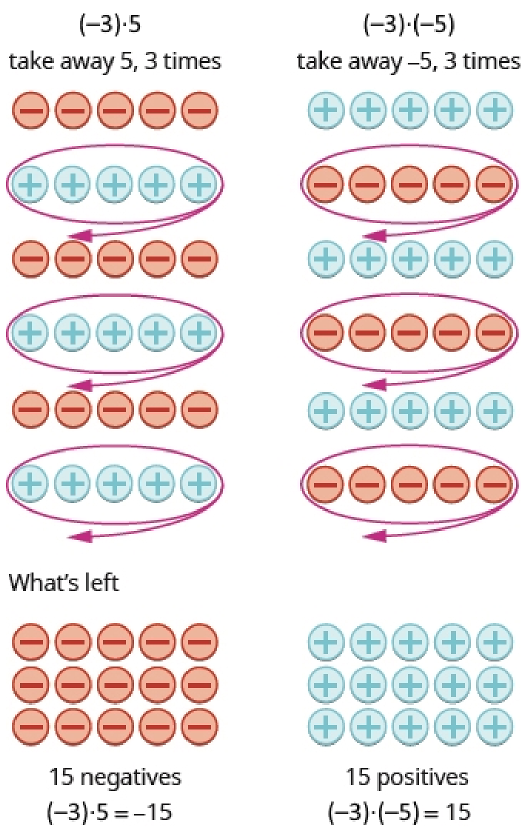
A positive number times a positive number is exactly the same as when we looked at multiplication of whole numbers. Therefore a positive times a positive results in a positive product. A positive number times a negative number is easy to understand as repeatedly adding the same negative addend. Since a negative plus a negative results in a negative sum, it follows that a positive times a negative must be negative.

What about a negative number times a positive number? Here our intuition is probably less certain because it does not immediately make sense how to do anything a negative number of times. One way this can make sense is to recognize that negative and positive numbers are opposites of each other, and that addition and subtraction are opposite operations or inverses of each other because adding and subtracting the same number results in no net change. Consequently, if multiplying by a

positive number is equivalent to repeated addition, multiplying by a negative number is equivalent to repeated subtraction.

Consider what it means to multiply -3 times 5. One meaning is subtract 5, 3 times. Starting with nothing, we cannot do the subtraction. Therefore, we start by adding neutral pairs as shown on the left side of the figure. This is one way of understanding why a negative number times a positive number results in a negative product.

What about a negative number times a negative number? The right hand side of the figure shows that it is positive. Notice that we are taking away the negative counters, leaving the positive counters. This is consistent with when we subtracted a negative value and saw that was the same as adding the positive value that was its opposite. Doing this repeatedly gives the same result as multiplying the opposites of the two negative values. Therefore a negative number times a negative number must result in a positive product.



In both cases, we started with $\mathbf{15}$ neutral pairs. In the case on the left, we took away $\mathbf{5}, \mathbf{3}$ times and the result was $-\mathbf{15}$. To multiply $(-3)\,(-5)$, we took away $-\mathbf{5}, \mathbf{3}$ times and the result was $\mathbf{15}$. So we found that

Equation:

$$\begin{array}{ll} 3 \cdot 5 = 15 & 3(-5) = -15 \\ -3(5) = -15 & (-3)(-5) = 15 \end{array}$$

Another way of appreciating why a negative number times a positive number results in a negative product relies on the commutative property. When the natural numbers are extended to become the integers, we want all of the properties that were true for natural numbers to still be true when we think of them as integers. This includes the commutative property of multiplication: a x b = b x a. If we require this property to be true for all integers, including the negative ones, then -3 x 5 = 5 x (-3), and a negative times a positive must be negative since we've already seen that a positive times a negative is negative.

Just as we needed a negative number times a positive number to result in a negative product in order for the commutative property to hold, we need a negative number times a negative number to have a positive product for the distributive property to hold. Consider the following equations and consider a reason why each one is true:

- 5 + (-5) = 0
  Reason: any number plus its opposite = 0
- (-3) x (5 + (-5)) = 0
  Reason: any number times 0 = 0
- (-3) x 5 + (-3) x (-5) = 0
  Reason: the distributive property of multiplication over addition
- -15 + (-3) x (-5) = 0
  Reason: multiplication fact and a negative times a positive is negative
- -15 and (-3) x (-5) are opposites
  Reason: they add to zero
- -(-15) = 15
  Reason: the opposite of an opposite is the number itself
- (-3) x (-5) = 15
  Reason: (-3) x (-5) is also the opposite of -15

This is another way of understanding why a negative number times a negative number results in a positive product.

Notice that for multiplication of two signed numbers, when the signs are the same, the product is positive, and when the signs are different, the product is negative.

Note:
Multiplication of Signed Numbers
The sign of the product of two numbers depends on their signs.

Same signs	**Product**
•Two positives •Two negatives	Positive Positive

Different signs	Product
•Positive • negative •Negative • positive	Negative Negative

Example:
Exercise:

Problem: Multiply each of the following:

ⓐ $-9 \cdot 3$
ⓑ $-2(-5)$
ⓒ $4(-8)$
ⓓ $7 \cdot 6$

Solution:
Solution

ⓐ	
	$-9 \cdot 3$
Multiply, noting that the signs are different and so the product is negative.	-27

ⓑ	
	$-2(-5)$
Multiply, noting that the signs are the same and so the product is positive.	10

ⓒ	
	$4(-8)$
Multiply, noting that the signs are different and so the product is negative.	-32

ⓓ	
	$7 \cdot 6$
The signs are the same, so the product is positive.	42

Note:
Exercise:

Problem: Multiply:

ⓐ $-6 \cdot 8$
ⓑ $-4(-7)$
ⓒ $9(-7)$
ⓓ $5 \cdot 12$

Solution:

ⓐ -48
ⓑ 28
ⓒ -63
ⓓ 60

When we multiply a number by 1, the result is the same number. This is the Identity Property of Multiplication. What happens when we multiply a number by -1? Let's multiply a positive number and then a negative number by -1 to see what we get.

Equation:

$-1 \cdot 4$	$-1(-3)$
-4	3
-4 is the opposite of **4**	**3** is the opposite of -3

Each time we multiply a number by -1, we get its opposite.

Note:
Multiplication by -1
Multiplying a number by -1 gives its opposite.
Equation:

$$-1a = -a$$

Example:
Exercise:

Problem: Multiply each of the following:

ⓐ $-1 \cdot 7$
ⓑ $-1(-11)$

Solution:
Solution

ⓐ	
The signs are different, so the product will be negative.	$-1 \cdot 7$
Notice that −7 is the opposite of 7.	-7

ⓑ	

The signs are the same, so the product will be positive.	$-1(-11)$
Notice that 11 is the opposite of −11.	11

Properties of Multiplication
All of the properties of multiplication for whole numbers hold for integers. That includes:
Identity Property of Multiplication: a x 1 = a
Zero Property of Multiplication: a x 0 = 0
Commutative Property of Multiplication: a x b = b x a
Associative Property of Multiplication: (a x b) x c = a x (b x c)
Distributive Property of Multiplication over Addition: a x (b + c) = a x b + a x c

Division of Integers

Division is the inverse operation of multiplication. So, $15 \div 3 = 5$ because $\mathbf{3 \cdot 5 = 15}$ In words, this expression says that $\mathbf{15}$ can be divided into $\mathbf{3}$ groups of $\mathbf{5}$ each because adding five three times gives $\mathbf{15}$. If we look at some examples of multiplying integers, we can discover the rules for dividing integers.
Equation:

$3 \cdot 5 = 15 \text{ so } 15 \div 3 = 5$ $\quad$ $3(-5) = -15 \text{ so } -15 \div 3 = -5$
$(-3)(\text{-}5) = 15 \text{ so } 15 \div (-3) = -5$ $\quad$ $-3(5) = -15 \text{ so } -15 \div (-3) = 5$

Division of signed numbers follows the same rules as multiplication. When the signs of the divisor and dividend are the same, the quotient is positive, and when the signs are different, the quotient is negative.

Note:
Division of Signed Numbers
The sign of the quotient of two numbers depends on the signs of the divisor and dividend.

Same signs	**Quotient**
•Two positives •Two negatives	Positive Positive

Different signs	Quotient
•Positive & negative •Negative & positive	Negative Negative

Notice that this is exactly the same summary we had for multiplication. Also, you can always check the answer to a division problem by multiplying.

Example:
Exercise:

Problem: Divide each of the following:

ⓐ $-27 \div 3$
ⓑ $-100 \div (-4)$

Solution:
Solution

ⓐ	
	$-27 \div 3$
Divide, noting that the signs are different and so the quotient is negative.	-9

ⓑ	
	$-100 \div (-4)$
Divide, noting that the signs are the same and so the quotient is positive.	25

Note:
Exercise:

Problem: Divide:

ⓐ $-42 \div 6$
ⓑ $-117 \div (-3)$

Solution:

ⓐ −7
ⓑ 39

Just as we saw with multiplication, when we divide a number by 1, the result is the same number. What happens when we divide a number by -1? Let's divide a positive number and then a negative number by -1 to see what we get.

Equation:

$$\begin{array}{ll} 8 \div (-1) & -9 \div (-1) \\ -8 & 9 \\ -8 \text{ is the opposite of } 8 & 9 \text{ is the opposite of } -9 \end{array}$$

When we divide a number by, -1 we get its opposite.

Note:
Division by -1
Dividing a number by -1 gives its opposite.
Equation:

$$a \div (-1) = -a$$

Example:
Exercise:

Problem: Divide each of the following:

ⓐ $16 \div (-1)$
ⓑ $-20 \div (-1)$

Solution:
Solution

ⓐ	
	$16 \div (-1)$
The dividend, 16, is being divided by –1.	–16
Dividing a number by –1 gives its opposite.	
Notice that the signs were different, so the result was negative.	

ⓑ	
	$-20 \div (-1)$
The dividend, –20, is being divided by –1.	20
Dividing a number by –1 gives its opposite.	

Notice that the signs were the same, so the quotient was positive.

Note:
Exercise:

Problem: Divide:

ⓐ $28 \div (-1)$
ⓑ $-52 \div (-1)$

Solution:

ⓐ −28
ⓑ 52

Properties of Division
Recall that with whole numbers division has some properties that multiplication has but other properties fail. For example, division is neither commutative or associative with whole numbers and since integers includes whole numbers, division with integers is neither commutative or associative.

Division by zero is still undefined
0 divided by any number other than 0 is 0
Any number divided by 1 is the same number
Any number divided by itself other than 0 is 1
The distributive property of division over addition: (a+b)/c = a/c + b/c

Simplify Expressions with Integers

What happens when there are more than two numbers in an expression? The order of operations still applies when negatives are included.

Let's try some examples. We'll simplify expressions that use all four operations with integers—addition, subtraction, multiplication, and division. Remember to follow the order of operations.

Example:
Exercise:

Problem: Simplify: ⓐ $(-2)^4$ ⓑ -2^4.

Solution:

Notice the difference in parts (a) and (b). In part (a), the exponent means to raise what is in the parentheses, the -2 to the 4th power. In part (b), the exponent means to raise just the 2 to the 4th power and then take the opposite.

ⓐ

	$(-2)^4$
Write in expanded form.	$(-2)(-2)(-2)(-2)$
Multiply.	$4(-2)(-2)$
Multiply.	$-8(-2)$
Multiply.	16

ⓑ

	-2^4
Write in expanded form. We are asked to find the opposite of 2^4.	$-(2\cdot2\cdot2\cdot2)$
Multiply.	$-(4\cdot2\cdot2)$
Multiply.	$-(8\cdot2)$
Multiply.	-16

Note:
Exercise:

Problem: Simplify: ⓐ $(-3)^4$ ⓑ -3^4.

Solution:

ⓐ 81 ⓑ -81

Note:
Exercise:

Problem: Simplify: ⓐ $(-7)^2$ ⓑ -7^2.

Solution:

ⓐ 49 ⓑ -49

The last example showed us the difference between $(-2)^4$ and -2^4. This distinction is important to prevent future errors. The next example reminds us to multiply and divide in order left to right.

Example:
Exercise:

Problem: Simplify: ⓐ $8(-9)\div(-2)^3$ ⓑ $-30\div2+(-3)(-7)$.

Solution:

ⓐ

	$8(-9) \div (-2)^3$
Exponents first.	$8(-9) \div (-8)$
Multiply.	$-72 \div (-8)$
Divide.	9

ⓑ

	$-30 \div 2 + (-3)(-7)$
Multiply and divide left to right, so divide first.	$-15 + (-3)(-7)$
Multiply.	$-15 + 21$
Add.	6

Note:
Exercise:

Problem: Simplify: ⓐ $12(-9) \div (-3)^3$ ⓑ $-27 \div 3 + (-5)(-6)$.

Solution:

ⓐ 4 ⓑ 21

Note:
Exercise:

Problem: Simplify: ⓐ $18(-4) \div (-2)^3$ ⓑ $-32 \div 4 + (-2)(-7)$.

Solution:

ⓐ 9 ⓑ 6

Evaluate Variable Expressions with Integers

Remember that to evaluate an expression means to substitute a number for the variable in the expression. Now we can use negative numbers as well as positive numbers.

Example:

Exercise:

Problem: Evaluate $4x^2 - 2xy + 3y^2$ when $x = 2, y = -1$.

Solution:

		$4x^2 - 2xy + 3y^2$
Substitute $x = 2$, $y = -1$. Use parentheses to show multiplication.		$4(2)^2 - 2(2)(-1) + 3(-1)^2$
Simplify exponents.		$4 \cdot 4 - (-4) + 3 \cdot 1$
Multiply.		$16 - (-4) + 3$
Subtract.		$20 + 3$
Add.		23

Note:
Exercise:

Problem: Evaluate: $3x^2 - 2xy + 6y^2$ when $x = 1, y = -2$.

Solution:

31

Note:
Exercise:

Problem: Evaluate: $4x^2 - xy + 5y^2$ when $x = -2, y = 3$.

Solution:

67

Translate Phrases to Expressions with Integers

Our earlier work translating English to algebra also applies to phrases that include both positive and negative numbers.

Example:
Exercise:

Problem: Translate and simplify: the sum of 8 and -12, increased by 3.

Solution:

	the **sum** of 8 and -12 increased by 3
Translate.	$[8 + (-12)] + 3$
Simplify. Be careful not to confuse the brackets with an absolute value sign.	$(-4) + 3$
Add.	-1

Note:
Exercise:

Problem: Translate and simplify the sum of 9 and -16, increased by 4.

Solution:

$(9 + (-16)) + 4; -3$

Note:
Exercise:

Problem: Translate and simplify the sum of -8 and -12, increased by 7.

Solution:

$(-8+(-12))+7; -13$

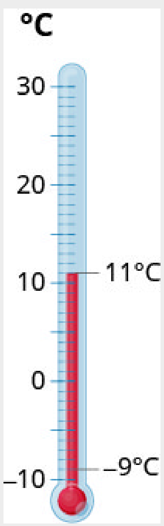
Use Integers in Applications

We’ll outline a plan to solve applications. It’s hard to find something if we don’t know what we’re looking for or what to call it! So when we solve an application, we first need to determine what the problem is asking us to find. Then we’ll write a phrase that gives the information to find it. We’ll translate the phrase into an expression and then simplify the expression to get the answer. Finally, we summarize the answer in a sentence to make sure it makes sense.

Example:
How to Solve Application Problems Using Integers
Exercise:

Problem:

The temperature in Kendallville, Indiana one morning was 11 degrees. By mid-afternoon, the temperature had dropped to -9 degrees. What was the difference in the morning and afternoon temperatures?



Solution:

Step 1. Read the problem. Make sure all the words and ideas are understood.	
Step 2. Identify what we are asked to find.	The difference of the morning and afternoon temperatures
Step 3. Write a phrase that gives the information to find it.	the difference *of* 11 *and* –9
Step 4. Translate the phrase to an expression.	$11-(-9)$
Step 5. Simplify the expression.	20
Step 6. Answer the question with a complete sentence.	The difference in temperatures was 20 degrees.

Note:
Exercise:

Problem:

The temperature in Anchorage, Alaska one morning was 15 degrees. By mid-afternoon the temperature had dropped to 30 degrees below zero. What was the difference in the morning and afternoon temperatures?

Solution:

The difference in temperatures was 45 degrees Fahrenheit.

Note:
Exercise:

Problem:

The temperature in Denver was -6 degrees at lunchtime. By sunset the temperature had dropped to -15 degrees. What was the difference in the lunchtime and sunset temperatures?

Solution:

The difference in temperatures was 9 degrees.

Note:
Use Integers in Applications.

Readthe problem. Make sure all the words and ideas are understood.
Identifywhat we are asked to find.
Write a phrasethat gives the information to find it.
Translatethe phrase to an expression.
Simplifythe expression.
Answerthe question with a complete sentence.

Note:
Access this online resource for additional instruction and practice with integers.

- Subtracting Integers with Counters

Key Concepts

- **Opposite Notation**
  Equation:

$$\begin{array}{l} -a \text{ means the opposite of the number } a \\ \text{The notation } -a \text{ is read as "the opposite of } a\text{."} \end{array}$$

- **Absolute Value**
  The absolute value of a number is its distance from 0 on the number line.
  The absolute value of a number n is written as $|n|$ and $|n| \geq 0$ for all numbers.
  Absolute values are always greater than or equal to zero.
- **Grouping Symbols**
  Equation:

$$\begin{array}{llll} \text{Parentheses} & (\,) & \text{Braces} & \{\,\} \\ \text{Brackets} & [\,] & \text{Absolute value} & |\,| \end{array}$$

- **Subtraction Property**
  $a - b = a + (-b)$
  Subtracting a number is the same as adding its opposite.
- **Multiplication and Division of Signed Numbers**
  For multiplication and division of two signed numbers:

Same signs	Result
• Two positives	Positive
• Two negatives	Positive

If the signs are the same, the result is positive.

Different signs	Result
• Positive and negative	Negative
• Negative and positive	Negative

If the signs are different, the result is negative.

- **Multiplication by** -1
  $$-1a = -a$$
  Multiplying a number by -1 gives its opposite.
- **How to Use Integers in Applications.**

  Read the problem. Make sure all the words and ideas are understood
  Identify what we are asked to find.
  Write a phrase that gives the information to find it.
  Translate the phrase to an expression.
  Simplify the expression.
  Answer the question with a complete sentence.

Practice Makes Perfect

Simplify Expressions with Absolute Value

In the following exercises, fill in $<$, $>$, or $=$ for each of the following pairs of numbers.
Exercise:

Problem:

ⓐ $|-7|$ ___ $-|-7|$
ⓑ 6 ___ $-|-6|$
ⓒ $|-11|$ ___ -11
ⓓ $-(-13)$ ___ $-|-13|$

Solution:

ⓐ $>$ ⓑ $>$ ⓒ $>$ ⓓ $>$

Exercise:

Problem:

ⓐ $-|-9|$ ___ $|-9|$
ⓑ -8 ___ $|-8|$
ⓒ $|-1|$ ___ -1
ⓓ $-(-14)$ ___ $-|-14|$

Exercise:

Problem:

ⓐ $-|2|$ ___ $-|-2|$
ⓑ -12 ___ $-|-12|$
ⓒ $|-3|$ ___ -3
ⓓ $|-19|$ ___ $-(-19)$

Solution:

ⓐ $=$ ⓑ $=$ ⓒ $>$ ⓓ $=$

Exercise:

Problem:

ⓐ $-|-4|$ ___ $-|4|$
ⓑ 5 ___ $-|-5|$
ⓒ $-|-10|$ ___ -10
ⓓ $-|-0|$ ___ $-(-0)$

In the following exercises, simplify.

Exercise:

Problem: $|15-7|-|14-6|$

Solution:

0

Exercise:

Problem: $|17-8|-|13-4|$

Exercise:

Problem: $18-|2\left(8-3\right)|$

Solution:

8

Exercise:

Problem: $15-|3\left(8-5\right)|$

Exercise:

Problem: $18-|12-4\left(4-1\right)+3|$

Solution:

15

Exercise:

Problem: $27-|19+4\left(3-1\right)-7|$

Exercise:

Problem: $10-3\left|9-3\left(3-1\right)\right|$

Solution:

1

Exercise:

Problem: $13-2\left|11-2\left(5-2\right)\right|$

Add and Subtract Integers

In the following exercises, simplify each expression.

Exercise:

Problem:
ⓐ $-7+\left(-4\right)$
ⓑ $-7+4$
ⓒ $7+\left(-4\right).$

Solution:

ⓐ -11 ⓑ -3 ⓒ 3

Exercise:

Problem: ⓐ $-5+(-9)$
ⓑ $-5+9$
ⓒ $5+(-9)$

Exercise:

Problem: $48+(-16)$

Solution:

32

Exercise:

Problem: $34+(-19)$

Exercise:

Problem: $-14+(-12)+4$

Solution:

-22

Exercise:

Problem: $-17+(-18)+6$

Exercise:

Problem: $19+2(-3+8)$

Solution:

29

Exercise:

Problem: $24+3(-5+9)$

Exercise:

ⓐ $13-7$
ⓑ $-13-(-7)$
ⓒ $-13-7$
Problem: ⓓ $13-(-7)$

Solution:

ⓐ 6 ⓑ -6 ⓒ -20 ⓓ 20

Exercise:

ⓐ $15-8$
ⓑ $-15-(-8)$
ⓒ $-15-8$
Problem: ⓓ $15-(-8)$

Exercise:

Problem: $-17-42$

Solution:

-59

Exercise:

Problem: $-58-(-67)$

Exercise:

Problem: $-14-(-27)+9$

Solution:

22

Exercise:

Problem: $64+(-17)-9$

Exercise:

Problem: ⓐ $44-28$ ⓑ $44+(-28)$

Solution:

ⓐ 16 ⓑ16

Exercise:

Problem: ⓐ $35-16$ ⓑ $35+(-16)$

Exercise:

Problem: ⓐ $27-(-18)$ ⓑ $27+18$

Solution:

ⓐ 45 ⓑ45

Exercise:

Problem: ⓐ $46-(-37)$ ⓑ $46+37$

Exercise:

Problem: $(2-7)-(3-8)$

Solution:

0

Exercise:

Problem: $(1-8)-(2-9)$

Exercise:

Problem: $-(6-8)-(2-4)$

Solution:

4

Exercise:

Problem: $-(4-5)-(7-8)$

Exercise:

Problem: $25-[10-(3-12)]$

Solution:

6

Exercise:

Problem: $32 - [5 - (15 - 20)]$

Multiply and Divide Integers

In the following exercises, multiply or divide.

Exercise:

Problem: ⓐ $-4 \cdot 8$
ⓑ $13\,(-5)$
ⓒ $-24 \div 6$
ⓓ $-52 \div (-4)$

Solution:

ⓐ -32 ⓑ -65 ⓒ -4
ⓓ 13

Exercise:

Problem: ⓐ $-3 \cdot 9$
ⓑ $9\,(-7)$
ⓒ $35 \div (-7)$
ⓓ $-84 \div (-6)$

Exercise:

Problem: ⓐ $-28 \div 7$
ⓑ $-180 \div 15$
ⓒ $3\,(-13)$
ⓓ $-1\,(-14)$

Solution:

ⓐ -4 ⓑ -12 ⓒ -39
ⓓ 14

Exercise:

Problem: ⓐ $-36 \div 4$
ⓑ $-192 \div 12$
ⓒ $9\,(-7)$
ⓓ $-1\,(-19)$

Simplify and Evaluate Expressions with Integers

In the following exercises, simplify each expression.

Exercise:

Problem: ⓐ $(-2)^6$ ⓑ -2^6

Solution:

ⓐ 64 ⓑ -64

Exercise:

Problem: ⓐ $(-3)^5$ ⓑ -3^5

Exercise:

Problem: $5(-6)+7(-2)-3$

Solution:

-47

Exercise:

Problem: $8(-4)+5(-4)-6$

Exercise:

Problem: $-3(-5)(6)$

Solution:

90

Exercise:

Problem: $-4(-6)(3)$

Exercise:

Problem: $(8-11)(9-12)$

Solution:

9

Exercise:

Problem: $(6-11)(8-13)$

Exercise:

Problem: $26 - 3\left(2 - 7\right)$

Solution:

41

Exercise:

Problem: $23 - 2\left(4 - 6\right)$

Exercise:

Problem: $65 \div \left(-5\right) + \left(-28\right) \div \left(-7\right)$

Solution:

-9

Exercise:

Problem: $52 \div \left(-4\right) + \left(-32\right) \div \left(-8\right)$

Exercise:

Problem: $9 - 2\left[3 - 8\left(-2\right)\right]$

Solution:

-29

Exercise:

Problem: $11 - 3\left[7 - 4\left(-2\right)\right]$

Exercise:

Problem: $8 - \left|2 - 4\left(4 - 1\right) + 3\right|$

Solution:

1

Exercise:

Problem: $7 - \left|5 - 3\left(4 - 1\right) - 6\right|$

Exercise:

Problem: $9-3\left|2\left(2-6\right)-\left(3-7\right)\right|$

Solution:

-3

Exercise:

Problem: $5-2\left|2\left(1-4\right)-\left(2-5\right)\right|$

Exercise:

Problem: $\left(-3\right)^{2}-24\div\left(8-2\right)$

Solution:

5

Exercise:

Problem: $\left(-4\right)^{2}-32\div\left(12-4\right)$

In the following exercises, evaluate each expression.

Exercise:

Problem: $y+\left(-14\right)$ when
ⓐ $y=-33$ ⓑ $y=30$

Solution:

ⓐ -47 ⓑ 16

Exercise:

Problem: $x+\left(-21\right)$ when
ⓐ $x=-27$ ⓑ $x=44$

Exercise:

Problem: $\left(x+y\right)^{2}$ when
$x=-3,y=14$

Solution:

121

Exercise:

Problem: $(y+z)^2$ when $y=-3, z=15$

Exercise:

Problem: $9a-2b-8$ when $a=-6$ and $b=-3$

Solution:

-56

Exercise:

Problem: $7m-4n-2$ when $m=-4$ and $n=-9$

Exercise:

Problem: $3x^2-4xy+2y^2$ when $x=-2, y=-3$

Solution:

6

Exercise:

Problem: $4x^2-xy+3y^2$ when $x=-3, y=-2$

Translate English Phrases to Algebraic Expressions

In the following exercises, translate to an algebraic expression and simplify if possible.

Exercise:

Problem: the sum of 3 and -15, increased by 7

Solution:

$(3+(-15))+7; -5$

Exercise:

Problem: the sum of -8 and -9, increased by 23

Exercise:

Problem: ⓐ the difference of 10 and -18
ⓑ subtract 11 from -25

Solution:

ⓐ $10-(-18); 28$
ⓑ$-25-11; -36$

Exercise:

Problem: ⓐ the difference of -5 and -30
ⓑ subtract -6 from -13

Exercise:

Problem: the quotient of -6 and the sum of a and b

Solution:

$\frac{-6}{a+b}$

Exercise:

Problem: the product of -13 and the difference of c and d

Use Integers in Applications

In the following exercises, solve.

Exercise:

Problem:

Temperature On January 15, the high temperature in Anaheim, California, was 84°. That same day, the high temperature in Embarrass, Minnesota, was -12°. What was the difference between the temperature in Anaheim and the temperature in Embarrass?

Solution:

96°

Exercise:

Problem:

Temperature On January 21, the high temperature in Palm Springs, California, was 89°, and the high temperature in Whitefield, New Hampshire, was -31°. What was the difference between the temperature in Palm Springs and the temperature in Whitefield?

Exercise:

Problem:

Football On the first down, the Chargers had the ball on their 25-yard line. On the next three downs, they lost 6 yards, gained 10 yards, and lost 8 yards. What was the yard line at the end of the fourth down?

Solution:

21

Exercise:

Problem:

Football On the first down, the Steelers had the ball on their 30-yard line. On the next three downs, they gained 9 yards, lost 14 yards, and lost 2 yards. What was the yard line at the end of the fourth down?

Exercise:

Problem:

Checking Account Mayra has $124 in her checking account. She writes a check for $152. What is the new balance in her checking account?

Solution:

$-\$28$

Exercise:

Problem:

Checking Account Reymonte has a balance of $-\$49$ in his checking account. He deposits $281 to the account. What is the new balance?

Writing Exercises

Exercise:

Problem: Explain why the sum of -8 and 2 is negative, but the sum of 8 and -2 is positive.

Solution:

Answers will vary.

Exercise:

Problem: Give an example from your life experience of adding two negative numbers.

Exercise:

Problem: In your own words, state the rules for multiplying and dividing integers.

Solution:

Answers will vary.

Exercise:

Problem: Why is $-4^3 = (-4)^3$?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
simplify expressions with absolute value.			
add and subtract integers.			
multiply and divide integers.			
simplify and evaluate expressions with integers.			
translate English phrases to algebraic expressions.			
use integers in applications.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

Glossary

absolute value
: The absolute value of a number is its distance from 0 on the number line.

integers
: The whole numbers and their opposites are called the integers.

negative numbers
: Numbers less than 0 are negative numbers.

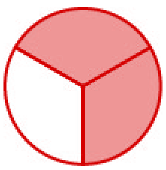
opposite
: The opposite of a number is the number that is the same distance from zero on the number line but on the opposite side of zero.

Fractions: ASE
By the end of this section, you will be able to:

- Simplify fractions
- Multiply and divide fractions
- Add and subtract fractions
- Use the order of operations to simplify fractions
- Evaluate variable expressions with fractions

Simplify Fractions

A **fraction** is a way to represent parts of a whole. The fraction $\frac{2}{3}$ represents two of three equal parts. See [link]. In the fraction $\frac{2}{3}$, the 2 is called the **numerator** and the 3 is called the **denominator**. The line is called the fraction bar.



In the circle, $\frac{2}{3}$ of the circle is shaded—2 of the 3 equal parts.

Note:
Fraction
A **fraction** is written $\frac{a}{b}$, where $b \neq 0$ and
a is the **numerator** and b is the **denominator**.
A fraction represents parts of a whole. The denominator b is the number of equal parts the whole has been divided into, and the numerator a indicates how many parts are included.

Fractions that have the same value are **equivalent fractions**. The Equivalent Fractions

Property allows us to find equivalent fractions and also simplify fractions.

Note:
Equivalent Fractions Property
If a, b, and c are numbers where $b \neq 0, c \neq 0$,
then $\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$ and $\frac{a \cdot c}{b \cdot c} = \frac{a}{b}$.

A fraction is considered simplified if there are no common factors, other than 1, in its numerator and denominator.

For example,

$\frac{2}{3}$ is simplified because there are no common factors of 2 and 3.

$\frac{10}{15}$ is not simplified because 5 is a common factor of 10 and 15.

We simplify, or reduce, a fraction by removing the common factors of the numerator and denominator. A fraction is not simplified until all common factors have been removed. If an expression has fractions, it is not completely simplified until the fractions are simplified.

Sometimes it may not be easy to find common factors of the numerator and denominator. When this happens, a good idea is to factor the numerator and the denominator into prime numbers. Then divide out the common factors using the Equivalent Fractions Property.

Example:
How To Simplify a Fraction
Exercise:

Problem: Simplify: $-\frac{315}{770}$.

Solution:

Step 1. Rewrite the numerator and denominator to show the common factors. If needed, use a factor tree.	Rewrite 315 and 770 as the product of the primes.	$-\frac{315}{770}$ $-\frac{3 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 5 \cdot 7 \cdot 11}$
Step 2. Simplify using the equivalent fractions property by dividing out common factors.	Mark the common factors 5 and 7. Divide out the common factors	$-\frac{3 \cdot 3 \cdot \cancel{5} \cdot \cancel{7}}{2 \cdot \cancel{5} \cdot \cancel{7} \cdot 11}$ $-\frac{3 \cdot 3}{2 \cdot 11}$
Step 3. Multiply the remaining factors, if necessary.		$-\frac{9}{22}$

Note:
Exercise:

Problem: Simplify: $-\frac{69}{120}$.

Solution:

$-\frac{23}{40}$

Note:
Exercise:

Problem: Simplify: $-\frac{120}{192}$.

Solution:

$-\frac{5}{8}$

We now summarize the steps you should follow to simplify fractions.

Note:
Simplify a fraction.

Rewrite the numerator and denominator to show the common factors. If needed, factor the numerator and denominator into prime numbers first.
Simplify using the Equivalent Fractions Property by dividing out common factors.
Multiply any remaining factors.

Multiply and Divide Fractions

Many people find multiplying and dividing fractions easier than adding and subtracting fractions.

To multiply fractions, we multiply the numerators and multiply the denominators.

Note:
Fraction Multiplication
If a, b, c, and d are numbers where $b \neq 0$, and $d \neq 0$, then
Equation:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

To multiply fractions, multiply the numerators and multiply the denominators.

When multiplying fractions, the properties of positive and negative numbers still apply, of course. It is a good idea to determine the sign of the product as the first step. In [link], we will multiply negative and a positive, so the product will be negative.

When multiplying a fraction by an integer, it may be helpful to write the integer as a fraction. Any integer, a, can be written as $\frac{a}{1}$. So, for example, $3 = \frac{3}{1}$.

Example:
Exercise:

Problem: Multiply: $-\frac{12}{5}(-20x)$.

Solution:

The first step is to find the sign of the product. Since the signs are the same, the product is positive.

	$-\frac{12}{5}(-20x)$
Determine the sign of the product. The signs are the same, so the product is positive.	$\frac{12}{5}(20x)$
Write $20x$ as a fraction.	$\frac{12}{5}\left(\frac{20x}{1}\right)$
Multiply.	$\frac{12 \cdot 20x}{5 \cdot 1}$
Rewrite 20 to show the common factor 5 and divide it out.	$\frac{12 \cdot 4 \cdot \cancel{5} \cdot x}{\cancel{5} \cdot 1}$

Simplify.	$48x$

Note:
Exercise:

Problem: Multiply: $\frac{11}{3}(-9a)$.

Solution:

$-33a$

Note:
Exercise:

Problem: Multiply: $\frac{13}{7}(-14b)$.

Solution:

$-26b$

Now that we know how to multiply fractions, we are almost ready to divide. Before we can do that, we need some vocabulary. The **reciprocal** of a fraction is found by inverting the fraction, placing the numerator in the denominator and the denominator in the numerator. The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$. Since 4 is written in fraction form as $\frac{4}{1}$, the reciprocal of 4 is $\frac{1}{4}$.

To divide fractions, we multiply the first fraction by the reciprocal of the second.

Note:
Fraction Division
If a, b, c, and d are numbers where $b \neq 0$, $c \neq 0$, and $d \neq 0$, then
Equation:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$

To divide fractions, we multiply the first fraction by the **reciprocal** of the second.

We need to say $b \neq 0$, $c \neq 0$, and $d \neq 0$, to be sure we don't divide by zero!

Example:
Exercise:

Problem: Find the quotient: $-\frac{7}{18} \div \left(-\frac{14}{27}\right)$.

Solution:

	$-\frac{7}{18} \div \left(-\frac{14}{27}\right)$
To divide, multiply the first fraction by the reciprocal of the second.	$-\frac{7}{18}\left(-\frac{27}{14}\right)$
Determine the sign of the product, and then multiply.	$\frac{7 \cdot 27}{18 \cdot 14}$
Rewrite showing common factors.	$\frac{\cancel{7} \cdot \cancel{9} \cdot 3}{\cancel{9} \cdot 2 \cdot \cancel{7} \cdot 2}$
Remove common factors.	$\frac{3}{2 \cdot 2}$
Simplify.	$\frac{3}{4}$

Note:
Exercise:

Problem: Divide: $-\frac{7}{27} \div \left(-\frac{35}{36}\right)$.

Solution:

$\frac{4}{15}$

Note:
Exercise:

Problem: Divide: $-\frac{5}{14} \div \left(-\frac{15}{28}\right)$.

Solution:

$\frac{2}{3}$

The numerators or denominators of some fractions contain fractions themselves. A fraction in which the numerator or the denominator is a fraction is called a **complex fraction**.

Note:
Complex Fraction
A **complex fraction** is a fraction in which the numerator or the denominator contains a fraction.

Some examples of complex fractions are:
Equation:

$$\frac{\frac{6}{7}}{3} \qquad \frac{\frac{3}{4}}{\frac{5}{8}} \qquad \frac{\frac{x}{2}}{\frac{5}{6}}$$

To simplify a complex fraction, remember that the fraction bar means division. For example, the complex fraction $\frac{\frac{3}{4}}{\frac{5}{8}}$ means $\frac{3}{4} \div \frac{5}{8}$.

Example:
Exercise:

Problem: Simplify: $\frac{\frac{x}{2}}{\frac{xy}{6}}$.

Solution:

	$\frac{\frac{x}{2}}{\frac{xy}{6}}$
Rewrite as division.	$\frac{x}{2} \div \frac{xy}{6}$
Multiply the first fraction by the reciprocal of the second.	$\frac{x}{2} \cdot \frac{6}{xy}$
Multiply.	$\frac{x \cdot 6}{2 \cdot xy}$
Look for common factors.	$\frac{\cancel{x} \cdot 3 \cdot \cancel{2}}{\cancel{2} \cdot \cancel{x} \cdot y}$
Divide common factors and simplify.	$\frac{3}{y}$

Note:
Exercise:

Problem: Simplify: $\frac{\frac{a}{8}}{\frac{ab}{6}}$.

Solution:

$\frac{3}{4b}$

Note:
Exercise:

Problem: Simplify: $\frac{\frac{p}{2}}{\frac{pq}{8}}$.

Solution:

$\frac{4}{q}$

Add and Subtract Fractions

When we multiplied fractions, we just multiplied the numerators and multiplied the denominators right straight across. To add or subtract fractions, they must have a common denominator.

Note:
Fraction Addition and Subtraction
If a, b, and c are numbers where $c \neq 0$, then
Equation:

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c} \text{ and } \frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$$

To add or subtract fractions, add or subtract the numerators and place the result over the common denominator.

The **least common denominator** (LCD) of two fractions is the smallest number that can be used as a common denominator of the fractions. The LCD of the two fractions is the least common multiple (LCM) of their denominators.

Note:
Least Common Denominator
The **least common denominator** (LCD) of two fractions is the least common multiple (LCM) of their denominators.

After we find the least common denominator of two fractions, we convert the fractions to equivalent fractions with the LCD. Putting these steps together allows us to add and subtract fractions because their denominators will be the same!

Example:
How to Add or Subtract Fractions
Exercise:

Problem: Add: $\frac{7}{12} + \frac{5}{18}$.

Solution:

Step 1. Do they have a common denominator?	No.	
• No—rewrite each fraction with the LCD (least common denominator).	Find the LCD of 12, 18.	$12 = 2 \cdot 2 \cdot 3$ $18 = 2 \cdot 3 \cdot 3$ $\text{LCD} = 2 \cdot 2 \cdot 3 \cdot 3$ $\text{LCD} = 36$ LCD is 36.
	We multiply the numerator and denominator of each fraction by the factor needed to get the denominator to be 36. Do not simplify the equivalent fractions! If you do, you'll get back to the original fractions and lose the common denominator!	$\frac{7}{12} + \frac{5}{18}$ $\frac{7 \cdot 3}{12 \cdot 3} + \frac{5 \cdot 2}{18 \cdot 2}$ $\frac{21}{36} + \frac{10}{36}$
Step 2. Add or subtract the fractions.	Add.	$\frac{31}{36}$
Step 3. Simplify, if possible.	Since 31 is prime, its only factors are 1 and 31. Since 31 does not go into 36, the answer is simplified.	

Note:
Exercise:

Problem: Add: $\frac{7}{12} + \frac{11}{15}$.

Solution:

$\frac{79}{60}$

Note:
Exercise:

Problem: Add: $\frac{13}{15} + \frac{17}{20}$.

Solution:

$\frac{103}{60}$

Note:
Add or subtract fractions.

Do they have a common denominator?

- Yes—go to step 2.
- No—rewrite each fraction with the LCD (least common denominator).
  - Find the LCD.
  - Change each fraction into an equivalent fraction with the LCD as its denominator.

Add or subtract the fractions.
Simplify, if possible.

We now have all four operations for fractions. [link] summarizes fraction operations.

Fraction Multiplication	**Fraction Division**
$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$	$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$
Multiply the numerators and multiply the denominators	Multiply the first fraction by the reciprocal of the second.
Fraction Addition	**Fraction Subtraction**
$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$	$\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$
Add the numerators and place the sum over the common denominator.	Subtract the numerators and place the difference over the common denominator.
To multiply or divide fractions, an LCD is NOT needed. To add or subtract fractions, an LCD is needed.	

When starting an exercise, always identify the operation and then recall the methods needed for that operation.

Example:
Exercise:

Problem: Simplify: ⓐ $\frac{5x}{6}-\frac{3}{10}$ ⓑ $\frac{5x}{6}\cdot\frac{3}{10}$.

Solution:

First ask, “What is the operation?” Identifying the operation will determine whether or not we need a common denominator. Remember, we need a common denominator to add or subtract, but not to multiply or divide.

ⓐ

What is the operation? The operation is subtraction.

Do the fractions have a common denominator? No. $\frac{5x}{6}-\frac{3}{10}$

Find the LCD of 6 and 10 — The LCD is 30.

$$\begin{array}{r} 6=2\cdot 3 \\ \underline{10=2\cdot 5} \\ \text{LCD}=2\cdot 3\cdot 5 \\ \text{LCD}=30 \end{array}$$

Rewrite each fraction as an equivalent fraction with the LCD. $\frac{5x\cdot5}{6\cdot5}-\frac{3\cdot3}{10\cdot3}$

$\frac{25x}{30}-\frac{9}{30}$

Subtract the numerators and place the difference over the common denominators. $\frac{25x-9}{30}$

Simplify, if possible. There are no common factors. The fraction is simplified.

ⓑ

What is the operation? Multiplication. $\frac{25x}{6}\cdot\frac{3}{10}$

To multiply fractions, multiply the numerators and multiply the denominators. $\frac{25x\cdot3}{6\cdot10}$

Rewrite, showing common factors. Remove common factors. $\frac{\cancel{5}x\cdot\cancel{3}}{2\cdot\cancel{3}\cdot2\cdot\cancel{5}}$

Simplify. $\frac{x}{4}$

Notice, we needed an LCD to add $\frac{25x}{6}-\frac{3}{10}$, but not to multiply $\frac{25x}{6}\cdot\frac{3}{10}$.

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{3a}{4}-\frac{8}{9}$ ⓑ $\frac{3a}{4}\cdot\frac{8}{9}$.

Solution:

ⓐ $\frac{27a-32}{36}$ ⓑ $\frac{2a}{3}$

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{4k}{5} - \frac{1}{6}$ ⓑ $\frac{4k}{5} \cdot \frac{1}{6}$.

Solution:

ⓐ $\frac{24k-5}{30}$ ⓑ $\frac{2k}{15}$

Use the Order of Operations to Simplify Fractions

The fraction bar in a fraction acts as grouping symbol. The order of operations then tells us to simplify the numerator and then the denominator. Then we divide.

Note:
Simplify an expression with a fraction bar.

Simplify the expression in the numerator. Simplify the expression in the denominator.
Simplify the fraction.

Where does the negative sign go in a fraction? Usually the negative sign is in front of the fraction, but you will sometimes see a fraction with a negative numerator, or sometimes with a negative denominator. Remember that fractions represent division. When the numerator and denominator have different signs, the quotient is negative.
Equation:

$$\frac{-1}{3} = -\frac{1}{3} \qquad \frac{\text{negative}}{\text{positive}} = \text{negative}$$

Equation:

$$\frac{1}{-3} = -\frac{1}{3} \qquad \frac{\text{positive}}{\text{negative}} = \text{negative}$$

Note:
Placement of Negative Sign in a Fraction
For any positive numbers *a* and *b*,
Equation:

$$\frac{-a}{b}=\frac{a}{-b}=-\frac{a}{b}$$

Example:
Exercise:

Problem: Simplify: $\frac{4(-3)+6(-2)}{-3(2)-2}$.

Solution:

The fraction bar acts like a grouping symbol. So completely simplify the numerator and the denominator separately.

	$\frac{4(-3)+6(-2)}{-3(2)-2}$
Multiply.	$\frac{-12+(-12)}{-6-2}$
Simplify.	$\frac{-24}{-8}$
Divide.	3

Note:
Exercise:

Problem: Simplify: $\frac{8(-2)+4(-3)}{-5(2)+3}$.

Solution:

4

Note:
Exercise:

Problem: Simplify: $\frac{7(-1)+9(-3)}{-5(3)-2}$.

Solution:

2

Now we’ll look at complex fractions where the numerator or denominator contains an expression that can be simplified. So we first must completely simplify the numerator and denominator separately using the order of operations. Then we divide the numerator by the denominator as the fraction bar means division.

Example:
How to Simplify Complex Fractions
Exercise:

Problem: Simplify: $\frac{\left(\frac{1}{2}\right)^2}{4+3^2}$.

Solution:

Step 1. Simplify the numerator.	$\frac{\left(\frac{1}{2}\right)^2}{4+3^2}$ $\frac{\frac{1}{4}}{4+3^2}$
Step 2. Simplify the denominator.	$\frac{\frac{1}{4}}{4+9}$ $\frac{\frac{1}{4}}{13}$
Step 3. Divide the numerator by the denominator. Simplify if possible	$\frac{1}{4} \div \frac{13}{1}$ $\frac{1}{4} \cdot \frac{1}{13}$ $\frac{1}{52}$

Note:
Exercise:

Problem: Simplify: $\frac{\left(\frac{1}{3}\right)^2}{2^3+2}$.

Solution:

$\frac{1}{90}$

Note:
Exercise:

Problem: Simplify: $\frac{1+4^2}{\left(\frac{1}{4}\right)^2}$.

Solution:

272

Note:
Simplify complex fractions.

Simplify the numerator.
Simplify the denominator.
Divide the numerator by the denominator. Simplify if possible.

Example:
Exercise:

Problem: Simplify: $\frac{\frac{1}{2}+\frac{2}{3}}{\frac{3}{4}-\frac{1}{6}}$.

Solution:

It may help to put parentheses around the numerator and the denominator.

	$\frac{\left(\frac{1}{2}+\frac{2}{3}\right)}{\left(\frac{3}{4}-\frac{1}{6}\right)}$
Simplify the numerator $(\text{LCD} = 6)$ and simplify the denominator $(\text{LCD} = 12)$.	$\frac{\left(\frac{3}{6}+\frac{4}{6}\right)}{\left(\frac{9}{12}-\frac{2}{12}\right)}$
Simplify.	$\frac{\left(\frac{7}{6}\right)}{\left(\frac{7}{12}\right)}$
Divide the numerator by the denominator.	$\frac{7}{6} \div \frac{7}{12}$
Simplify.	$\frac{7}{6} \cdot \frac{12}{7}$
Divide out common factors.	$\frac{\cancel{7} \cdot \cancel{6} \cdot 2}{\cancel{6} \cdot \cancel{7} \cdot 1}$
Simplify.	2

Note:
Exercise:

Problem: Simplify: $\frac{\frac{1}{3}+\frac{1}{2}}{\frac{3}{4}-\frac{1}{3}}$.

Solution:

2

Note:
Exercise:

Problem: Simplify: $\frac{\frac{2}{3}-\frac{1}{2}}{\frac{1}{4}+\frac{1}{3}}$.

Solution:

$\frac{2}{7}$

Evaluate Variable Expressions with Fractions

We have evaluated expressions before, but now we can evaluate expressions with fractions. Remember, to evaluate an expression, we substitute the value of the variable into the expression and then simplify.

Example:

Exercise:

Problem: Evaluate $2x^2y$ when $x = \frac{1}{4}$ and $y = -\frac{2}{3}$.

Solution:

Substitute the values into the expression.

	$2x^2y$
Substitute $\frac{1}{4}$ for x and $-\frac{2}{3}$ for y.	$2\left(\frac{1}{4}\right)^2\left(-\frac{2}{3}\right)$
Simplify exponents first.	$2\left(\frac{1}{16}\right)\left(-\frac{2}{3}\right)$
Multiply; divide out the common factors. Notice we write 16 as $2 \cdot 2 \cdot 4$ to make it easy to remove common factors.	$-\frac{\cancel{2} \cdot 1 \cdot \cancel{2}}{\cancel{2} \cdot \cancel{2} \cdot 4 \cdot 3}$
Simplify.	$-\frac{1}{12}$

Note:
Exercise:

Problem: Evaluate $3ab^2$ when $a = -\frac{2}{3}$ and $b = -\frac{1}{2}$.

Solution:

$-\frac{1}{2}$

Note:
Exercise:

Problem: Evaluate $4c^3d$ when $c=-\frac{1}{2}$ and $d=-\frac{4}{3}$.

Solution:

$\frac{2}{3}$

Note:
Access this online resource for additional instruction and practice with fractions.

- Adding Fractions with Unlike Denominators

Key Concepts

- **Equivalent Fractions Property**
  If a, b, and c are numbers where $b \neq 0, c \neq 0$, then
  $\frac{a}{b}=\frac{a \cdot c}{b \cdot c}$ and $\frac{a \cdot c}{b \cdot c}=\frac{a}{b}$.
- **How to simplify a fraction.**

  Rewrite the numerator and denominator to show the common factors. If needed, factor the numerator and denominator into prime numbers first.
  Simplify using the Equivalent Fractions Property by dividing out common factors.
  Multiply any remaining factors.

- **Fraction Multiplication**
  If a, b, c, and d are numbers where $b \neq 0$, and $d \neq 0$, then
  $\frac{a}{b} \cdot \frac{c}{d}=\frac{ac}{bd}$.
  To multiply fractions, multiply the numerators and multiply the denominators.
- **Fraction Division**
  If a, b, c, and d are numbers where $b \neq 0, c \neq 0$, and $d \neq 0$, then
  $\frac{a}{b} \div \frac{c}{d}=\frac{a}{b} \cdot \frac{d}{c}$.
  To divide fractions, we multiply the first fraction by the reciprocal of the second.
- **Fraction Addition and Subtraction**
  If a, b, and c are numbers where $c \neq 0$, then
  $\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$ and $\frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}$.
  To add or subtract fractions, add or subtract the numerators and place the result over the common denominator.
- **How to add or subtract fractions.**

  Do they have a common denominator?
    - Yes—go to step 2.

- No—rewrite each fraction with the LCD (least common denominator).
  - Find the LCD.
  - Change each fraction into an equivalent fraction with the LCD as its denominator.

Add or subtract the fractions.
Simplify, if possible.

- **How to simplify an expression with a fraction bar.**

  Simplify the expression in the numerator. Simplify the expression in the denominator.
  Simplify the fraction.

- **Placement of Negative Sign in a Fraction**
  For any positive numbers *a* and *b*,
  $\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$.
- **How to simplify complex fractions.**

  Simplify the numerator.
  Simplify the denominator.
  Divide the numerator by the denominator. Simplify if possible.

Practice Makes Perfect

Simplify Fractions

In the following exercises, simplify.
Exercise:

Problem: $-\frac{108}{63}$

Solution:

$-\frac{12}{7}$

Exercise:

Problem: $-\frac{104}{48}$

Exercise:

Problem: $\frac{120}{252}$

Solution:

$\frac{10}{21}$

Exercise:

Problem: $\frac{182}{294}$

Exercise:

Problem: $\frac{14x^2}{21y}$

Solution:

$\frac{2x^2}{3y}$

Exercise:

Problem: $\frac{24a}{32b^2}$

Exercise:

Problem: $-\frac{210a^2}{110b^2}$

Solution:

$-\frac{21a^2}{11b^2}$

Exercise:

Problem: $-\frac{30x^2}{105y^2}$

Multiply and Divide Fractions

In the following exercises, perform the indicated operation.

Exercise:

Problem: $-\frac{3}{4}\left(-\frac{4}{9}\right)$

Solution:

$\frac{1}{3}$

Exercise:

Problem: $-\frac{3}{8} \cdot \frac{4}{15}$

Exercise:

Problem: $\left(-\frac{14}{15}\right)\left(\frac{9}{20}\right)$

Solution:

$-\frac{21}{50}$

Exercise:

Problem: $\left(-\frac{9}{10}\right)\left(\frac{25}{33}\right)$

Exercise:

Problem: $\left(-\frac{63}{84}\right)\left(-\frac{44}{90}\right)$

Solution:

$\frac{11}{30}$

Exercise:

Problem: $\left(-\frac{33}{60}\right)\left(-\frac{40}{88}\right)$

Exercise:

Problem: $\frac{3}{7}\cdot 21n$

Solution:

$9n$

Exercise:

Problem: $\frac{5}{6}\cdot 30m$

Exercise:

Problem: $\frac{3}{4}\div\frac{x}{11}$

Solution:

$\frac{33}{4x}$

Exercise:

Problem: $\frac{2}{5}\div\frac{y}{9}$

Exercise:

Problem: $\frac{5}{18}\div\left(-\frac{15}{24}\right)$

Solution:

$-\frac{4}{9}$

Exercise:

Problem: $\frac{7}{18} \div \left(-\frac{14}{27}\right)$

Exercise:

Problem: $\frac{8u}{15} \div \frac{12v}{25}$

Solution:

$\frac{10u}{9v}$

Exercise:

Problem: $\frac{12r}{25} \div \frac{18s}{35}$

Exercise:

Problem: $\frac{3}{4} \div (-12)$

Solution:

$-\frac{1}{16}$

Exercise:

Problem: $-15 \div \left(-\frac{5}{3}\right)$

In the following exercises, simplify.

Exercise:

Problem: $\frac{-\frac{8}{21}}{\frac{12}{35}}$

Solution:

$-\frac{10}{9}$

Exercise:

Problem: $\frac{-\frac{9}{16}}{\frac{33}{40}}$

Exercise:

Problem: $\frac{-\frac{4}{5}}{2}$

Solution:

$-\frac{2}{5}$

Exercise:

Problem: $\frac{\frac{5}{3}}{10}$

Exercise:

Problem: $\frac{\frac{m}{3}}{\frac{n}{2}}$

Solution:

$\frac{2m}{3n}$

Exercise:

Problem: $\frac{-\frac{3}{8}}{-\frac{y}{12}}$

Add and Subtract Fractions

In the following exercises, add or subtract.

Exercise:

Problem: $\frac{7}{12}+\frac{5}{8}$

Solution:

$\frac{29}{24}$

Exercise:

Problem: $\frac{5}{12}+\frac{3}{8}$

Exercise:

Problem: $\frac{7}{12}-\frac{9}{16}$

Solution:

$\frac{1}{48}$

Exercise:

Problem: $\frac{7}{16}-\frac{5}{12}$

Exercise:

Problem: $-\frac{13}{30}+\frac{25}{42}$

Solution:

$\frac{17}{105}$

Exercise:

Problem: $-\frac{23}{30}+\frac{5}{48}$

Exercise:

Problem: $-\frac{39}{56}-\frac{22}{35}$

Solution:

$-\frac{53}{40}$

Exercise:

Problem: $-\frac{33}{49}-\frac{18}{35}$

Exercise:

Problem: $-\frac{2}{3}-\left(-\frac{3}{4}\right)$

Solution:

$\frac{1}{12}$

Exercise:

Problem: $-\frac{3}{4}-\left(-\frac{4}{5}\right)$

Exercise:

Problem: $\frac{x}{3}+\frac{1}{4}$

Solution:

$\frac{4x+3}{12}$

Exercise:

Problem: $\frac{x}{5}-\frac{1}{4}$

Exercise:

Problem: ⓐ $\frac{2}{3} + \frac{1}{6}$
ⓑ $\frac{2}{3} \div \frac{1}{6}$

Solution:

ⓐ $\frac{5}{6}$ ⓑ 4

Exercise:

Problem: ⓐ $-\frac{2}{5} - \frac{1}{8}$
ⓑ $-\frac{2}{5} \cdot \frac{1}{8}$

Exercise:

Problem: ⓐ $\frac{5n}{6} \div \frac{8}{15}$
ⓑ $\frac{5n}{6} - \frac{8}{15}$

Solution:

ⓐ $\frac{25n}{16}$ ⓑ $\frac{25n-16}{30}$

Exercise:

Problem: ⓐ $\frac{3a}{8} \div \frac{7}{12}$
ⓑ $\frac{3a}{8} - \frac{7}{12}$

Exercise:

Problem: ⓐ $-\frac{4x}{9} - \frac{5}{6}$
ⓑ $-\frac{4k}{9} \cdot \frac{5}{6}$

Solution:

ⓐ $\frac{-8x-15}{18}$ ⓑ $-\frac{10k}{27}$

Exercise:

Problem: ⓐ $-\frac{3y}{8} - \frac{4}{3}$
ⓑ $-\frac{3y}{8} \cdot \frac{4}{3}$

Exercise:

Problem: ⓐ $-\frac{5a}{3} + \left(-\frac{10}{6}\right)$
ⓑ $-\frac{5a}{3} \div \left(-\frac{10}{6}\right)$

Solution:

ⓐ $\frac{-5(a+1)}{3}$ ⓑ a

Exercise:

Problem: ⓐ $\frac{2b}{5} + \frac{8}{15}$
ⓑ $\frac{2b}{5} \div \frac{8}{15}$

Use the Order of Operations to Simplify Fractions

In the following exercises, simplify.

Exercise:

Problem: $\frac{5 \cdot 6 - 3 \cdot 4}{4 \cdot 5 - 2 \cdot 3}$

Solution:

$\frac{9}{7}$

Exercise:

Problem: $\frac{8 \cdot 9 - 7 \cdot 6}{5 \cdot 6 - 9 \cdot 2}$

Exercise:

Problem: $\frac{5^2 - 3^2}{3 - 5}$

Solution:

-8

Exercise:

Problem: $\frac{6^2 - 4^2}{4 - 6}$

Exercise:

Problem: $\frac{7 \cdot 4 - 2(8 - 5)}{9 \cdot 3 - 3 \cdot 5}$

Solution:

$\frac{11}{6}$

Exercise:

Problem: $\frac{9 \cdot 7 - 3(12 - 8)}{8 \cdot 7 - 6 \cdot 6}$

Exercise:

Problem: $\frac{9(8 - 2) - 3(15 - 7)}{6(7 - 1) - 3(17 - 9)}$

Solution:

$\frac{5}{2}$

Exercise:

Problem: $\frac{8(9 - 2) - 4(14 - 9)}{7(8 - 3) - 3(16 - 9)}$

Exercise:

Problem: $\frac{2^3 + 4^2}{\left(\frac{2}{3}\right)^2}$

Solution:

54

Exercise:

Problem: $\frac{3^3 - 3^2}{\left(\frac{3}{4}\right)^2}$

Exercise:

Problem: $\frac{\left(\frac{3}{5}\right)^2}{\left(\frac{3}{7}\right)^2}$

Solution:

$\frac{49}{25}$

Exercise:

Problem: $\frac{\left(\frac{3}{4}\right)^2}{\left(\frac{5}{8}\right)^2}$

Exercise:

Problem: $\frac{2}{\frac{1}{3} + \frac{1}{5}}$

Solution:

$\frac{15}{4}$

Exercise:

Problem: $\frac{5}{\frac{1}{4}+\frac{1}{3}}$

Exercise:

Problem: $\frac{\frac{7}{8}-\frac{2}{3}}{\frac{1}{2}+\frac{3}{8}}$

Solution:

$\frac{5}{21}$

Exercise:

Problem: $\frac{\frac{3}{4}-\frac{3}{5}}{\frac{1}{4}+\frac{2}{5}}$

Mixed Practice

In the following exercises, simplify.

Exercise:

Problem: $-\frac{3}{8} \div \left(-\frac{3}{10}\right)$

Solution:

$\frac{5}{4}$

Exercise:

Problem: $-\frac{3}{12} \div \left(-\frac{5}{9}\right)$

Exercise:

Problem: $-\frac{3}{8} + \frac{5}{12}$

Solution:

$\frac{1}{24}$

Exercise:

Problem: $-\frac{1}{8} + \frac{7}{12}$

Exercise:

Problem: $-\frac{7}{15}-\frac{y}{4}$

Solution:

$\frac{-28-15y}{60}$

Exercise:

Problem: $-\frac{3}{8}-\frac{x}{11}$

Exercise:

Problem: $\frac{11}{12a}\cdot\frac{9a}{16}$

Solution:

$\frac{33}{64}$

Exercise:

Problem: $\frac{10y}{13}\cdot\frac{8}{15y}$

Exercise:

Problem: $\frac{1}{2}+\frac{2}{3}\cdot\frac{5}{12}$

Solution:

$\frac{7}{9}$

Exercise:

Problem: $\frac{1}{3}+\frac{2}{5}\cdot\frac{3}{4}$

Exercise:

Problem: $1-\frac{3}{5}\div\frac{1}{10}$

Solution:

-5

Exercise:

Problem: $1-\frac{5}{6}\div\frac{1}{12}$

Exercise:

Problem: $\frac{3}{8} - \frac{1}{6} + \frac{3}{4}$

Solution:

$\frac{23}{24}$

Exercise:

Problem: $\frac{2}{5} + \frac{5}{8} - \frac{3}{4}$

Exercise:

Problem: $12\left(\frac{9}{20} - \frac{4}{15}\right)$

Solution:

$\frac{11}{5}$

Exercise:

Problem: $8\left(\frac{15}{16} - \frac{5}{6}\right)$

Exercise:

Problem: $\frac{\frac{5}{8}+\frac{1}{6}}{\frac{19}{24}}$

Solution:

1

Exercise:

Problem: $\frac{\frac{1}{6}+\frac{3}{10}}{\frac{14}{30}}$

Exercise:

Problem: $\left(\frac{5}{9} + \frac{1}{6}\right) \div \left(\frac{2}{3} - \frac{1}{2}\right)$

Solution:

$\frac{13}{3}$

Exercise:

Problem: $\left(\frac{3}{4} + \frac{1}{6}\right) \div \left(\frac{5}{8} - \frac{1}{3}\right)$

Evaluate Variable Expressions with Fractions

In the following exercises, evaluate.

Exercise:

Problem: $\frac{7}{10} - w$ when
ⓐ $w = \frac{1}{2}$ ⓑ $w = -\frac{1}{2}$

Solution:

ⓐ $\frac{1}{5}$ ⓑ $\frac{6}{5}$

Exercise:

Problem: $\frac{5}{12} - w$ when
ⓐ $w = \frac{1}{4}$ ⓑ $w = -\frac{1}{4}$

Exercise:

Problem: $2x^2y^3$ when
$x = -\frac{2}{3}$ and $y = -\frac{1}{2}$

Solution:

$-\frac{1}{9}$

Exercise:

Problem: $8u^2v^3$ when
$u = -\frac{3}{4}$ and $v = -\frac{1}{2}$

Exercise:

Problem: $\frac{a+b}{a-b}$ when
$a = -3, b = 8$

Solution:

$-\frac{5}{11}$

Exercise:

Problem: $\frac{r-s}{r+s}$ when
$r = 10, s = -5$

Writing Exercises

Exercise:

Problem: Why do you need a common denominator to add or subtract fractions? Explain.

Solution:

Answers will vary.

Exercise:

Problem: How do you find the LCD of 2 fractions?

Exercise:

Problem: Explain how you find the reciprocal of a fraction.

Solution:

Answers will vary.

Exercise:

Problem: Explain how you find the reciprocal of a negative number.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
simplify fractions.			
multiply and divide fractions.			
add and subtract fractions.			
use the order of operations to simplify fractions.			
evaluate variable expressions with fractions.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Glossary

complex fraction
: A fraction in which the numerator or the denominator is a fraction is called a complex fraction.

denominator

In a fraction, written $\frac{a}{b}$, where $b \neq 0$, the denominator b is the number of equal parts the whole has been divided into.

equivalent fractions
: Equivalent fractions are fractions that have the same value.

fraction
: A fraction is written $\frac{a}{b}$, where $b \neq 0$, and a is the numerator and b is the denominator. A fraction represents parts of a whole.

least common denominator
: The least common denominator (LCD) of two fractions is the least common multiple (LCM) of their denominators.

numerator
: In a fraction, written $\frac{a}{b}$, where $b \neq 0$, the numerator a indicates how many parts are included.

reciprocal
: The reciprocal of a fraction is found by inverting the fraction, placing the numerator in the denominator and the denominator in the numerator.

Decimals: ASE
By the end of this section, you will be able to:

- Round decimals
- Add and subtract decimals
- Multiply and divide decimals
- Convert decimals, fractions, and percents
- Simplify expressions with square roots
- Identify integers, rational numbers, irrational numbers, and real numbers
- Locate fractions and decimals on the number line

Round Decimals

Decimals are another way of writing fractions whose denominators are powers of ten.
Equation:

$$\begin{array}{rcll} 0.1 & = & \frac{1}{10} & \text{is “one tenth”} \\ 0.01 & = & \frac{1}{100} & \text{is “one hundredth”} \\ 0.001 & = & \frac{1}{1000} & \text{is “one thousandth”} \\ 0.0001 & = & \frac{1}{10,000} & \text{is “one ten-thousandth”} \end{array}$$

Just as in whole numbers, each digit of a decimal corresponds to the place value based on the powers of ten. [link] shows the names of the place values to the left and right of the decimal point.

Place Value											
Hundred thousands	Ten thousands	Thousands	Hundreds	Tens	Ones		Tenths	Hundredths	Thousandths	Ten-thousandths	Hundred-thousandths
						•					

When we work with decimals, it is often necessary to round the number to the nearest required place value. We summarize the steps for rounding a decimal here.

Note:

Round decimals.

Locate the given place value and mark it with an arrow.
Underline the digit to the right of the place value.
Is the underlined digit greater than or equal to 5?

- Yes: add 1 to the digit in the given place value.
- No: do <u>not</u> change the digit in the given place value

Rewrite the number, deleting all digits to the right of the rounding digit.

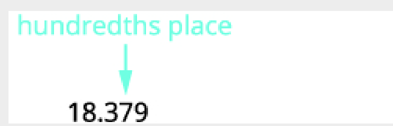
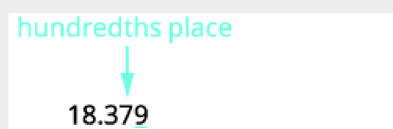
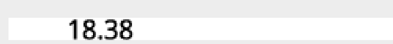
Example:
Exercise:

Problem: Round 18.379 to the nearest ⓐ hundredth ⓑ tenth ⓒ whole number.

Solution:

Round 18.379.

ⓐ to the nearest hundredth

Locate the hundredths place with an arrow.	
Underline the digit to the right of the given place value.	
Because 9 is greater than or equal to 5, add 1 to the 7.	
Rewrite the number, deleting all digits to the	

right of the rounding digit.	
Notice that the deleted digits were NOT replaced with zeros.	So, 18.379 rounded to the nearest hundredth is 18.38.

ⓑ to the nearest tenth

Locate the tenths place with an arrow.	tenths place 18.379
Underline the digit to the right of the given place value.	tenths place $18.3\underline{7}9$
Because 7 is greater than or equal to 5, add 1 to the 3.	$18.3\underbrace{79}_{\text{delete}}$ add 1
Rewrite the number, deleting all digits to the right of the rounding digit.	18.4
Notice that the deleted digits were NOT replaced with zeros.	So, 18.379 rounded to the nearest tenth is 18.4.

ⓒ to the nearest whole number

Locate the ones place with an arrow.	

	ones place 18.379
Underline the digit to the right of the given place value.	ones place 18.<u>3</u>79
Since 3 is not greater than or equal to 5, do not add 1 to the 8.	18.379 delete do not add 1
Rewrite the number, deleting all digits to the right of the rounding digit.	18
	So, 18.379 rounded to the nearest whole number is 18.

Note:
Exercise:

Problem: Round 6.582 to the nearest ⓐ hundredth ⓑ tenth ⓒ whole number.

Solution:

ⓐ 6.58 ⓑ 6.6 ⓒ 7

Note:
Exercise:

Problem: Round 15.2175 to the nearest ⓐ thousandth ⓑ hundredth ⓒ tenth.

Solution:

ⓐ 15.218 ⓑ 15.22
ⓒ 15.2

Add and Subtract Decimals

To add or subtract decimals, we line up the decimal points. By lining up the decimal points this way, we can add or subtract the corresponding place values. We then add or subtract the numbers as if they were whole numbers and then place the decimal point in the sum.

Note:
Add or subtract decimals.

Determine the sign of the sum or difference.
Write the numbers so the decimal points line up vertically.
Use zeros as placeholders, as needed.
Add or subtract the numbers as if they were whole numbers. Then place the decimal point in the answer under the decimal points in the given numbers.
Write the sum or difference with the appropriate sign.

Example:
Exercise:

Problem: Add or subtract: ⓐ $-23.5 - 41.38$ ⓑ $14.65 - 20$.

Solution:

ⓐ

$$-23.5 - 41.38$$

The difference will be negative. To subtract, we add the numerals. Write the numbers so the decimal points line up vertically.

$$\begin{array}{r} 23.5 \\ \underline{+41.38} \end{array}$$

Put 0 as a placeholder after the 5 in 23.5.
Remember, $\frac{5}{10} = \frac{50}{100}$ so $0.5 = 0.50$.

$$\begin{array}{r} 23.50 \\ \underline{+41.38} \end{array}$$

Add the numbers as if they were whole numbers.
Then place the decimal point in the sum.

$$\begin{array}{r} 23.50 \\ \underline{+41.38} \\ 64.88 \end{array}$$

Write the result with the correct sign.

$$-23.5 - 41.38 = -64.88$$

ⓑ

$$14.65 - 20$$

The difference will be negative. To subtract, we subtract 14.65 from 20.

Write the numbers so the decimal points line up vertically.

$$\begin{array}{r} 20 \\ \underline{-14.65} \end{array}$$

Remember, 20 is a whole number, so place the decimal point after the 0.

Put in zeros to the right as placeholders.

$$\begin{array}{r} 20.00 \\ \underline{-14.65} \end{array}$$

Subtract and place the decimal point in the answer.

$$\begin{array}{cccccc} & 9 & & 9 & \\ 1 & \cancel{10} & & \cancel{10} & 10 \\ 2 & 0 & . & 0 & 0 \\ -1 & 4 & . & 6 & 5 \\ \hline & 5 & . & 3 & 5 \end{array}$$

Write the result with the correct sign.

$$14.65 - 20 = -5.35$$

Note:

Exercise:

Problem: Add or subtract: ⓐ $-4.8 - 11.69$ ⓑ$9.58 - 10.$

Solution:

ⓐ -16.49 ⓑ -0.42

Note:
Exercise:

Problem: Add or subtract: ⓐ $-5.123 - 18.47$ ⓑ$37.42 - 50.$

Solution:

ⓐ -23.593 ⓑ-12.58

Multiply and Divide Decimals

When we multiply signed decimals, first we determine the sign of the product and then multiply as if the numbers were both positive. We multiply the numbers temporarily ignoring the decimal point and then count the number of decimal points in the factors and that sum tells us the number of decimal places in the product. Finally, we write the product with the appropriate sign.

Note:
Multiply decimals.

Determine the sign of the product.
Write in vertical format, lining up the numbers on the right. Multiply the numbers as if they were whole numbers, temporarily ignoring the decimal points.
Place the decimal point. The number of decimal places in the product is the sum of the number of decimal places in the factors.
Write the product with the appropriate sign.

Example:
Exercise:

Problem: Multiply: $(-3.9)\,(4.075).$

Solution:

	$(-3.9)(4.075)$
The signs are different. The product will be negative.	The product will be negative.
Write in vertical format, lining up the numbers on the right.	$\begin{array}{r} 4.075 \\ \underline{\times\ 3.9} \end{array}$
Multiply.	$\begin{array}{r} 4.075 \\ \underline{\times\ 3.9} \\ 36675 \\ \underline{12225} \\ 158925 \end{array}$
Add the number of decimal places in the factors (1 + 3). Place the decimal point 4 places from the right. $\underbrace{(-3.9)}_{\text{1 place}}\quad\underbrace{(4.075)}_{\text{3 places}}$	$\begin{array}{r} 4.075 \\ \underline{\times\ 3.9} \\ 36675 \\ \underline{12225} \\ 15.\underbrace{8925}_{\text{4 places}} \end{array}$
The signs are the different, so the product is negative.	$(-3.9)(4.075) = -15.8925$

Note:
Exercise:

Problem: Multiply: $-4.5\,(6.107)$.

Solution:

-27.4815

Note:
Exercise:

Problem: Multiply: $-10.79\,(8.12)$.

Solution:

-87.6148

Often, especially in the sciences, you will multiply decimals by powers of 10 (10, 100, 1000, etc). If you multiply a few products on paper, you may notice a pattern relating the number of zeros in the power of 10 to number of decimal places we move the decimal point to the right to get the product.

Note:
Multiply a decimal by a power of ten.

Move the decimal point to the right the same number of places as the number of zeros in the power of 10.
Add zeros at the end of the number as needed.

Example:
Exercise:

Problem: Multiply: 5.63 by ⓐ 10 ⓑ 100 ⓒ 1000.

Solution:

By looking at the number of zeros in the multiple of ten, we see the number of places we need to move the decimal to the right.

ⓐ

		5.63 (10)
There is 1 zero in 10, so move the decimal point 1 place to the right.		5.63
		56.3

ⓑ

	5.63(100)
There are 2 zeroes in 100, so move the decimal point 2 places to the right.	5.63
	563

ⓒ

	5.63(1,000)
There are 3 zeroes in 1,000, so move the decimal point 3 place to the right.	5.63
A zero must be added to the end.	5,630

Note:
Exercise:

Problem: Multiply 2.58 by ⓐ 10 ⓑ 100 ⓒ 1000.

Solution:

ⓐ 25.8 ⓑ 258 ⓒ 2,580

Note:
Exercise:

Problem: Multiply 14.2 by ⓐ 10 ⓑ 100 ⓒ 1000.

Solution:

ⓐ 142 ⓑ 1,420 ⓒ 14,200

Just as with multiplication, division of signed decimals is very much like dividing whole numbers. We just have to figure out where the decimal point must be placed and the sign of the quotient. When dividing signed decimals, first determine the sign of the quotient and then divide as if the numbers were both positive. Finally, write the quotient with the appropriate sign.

We review the notation and vocabulary for division:

$$\underset{\text{dividend}}{a} \div \underset{\text{divisor}}{b} = \underset{\text{quotient}}{c} \qquad \underset{\text{divisor}}{b}\overset{\overset{c}{\text{quotient}}}{\overline{\left)\underset{\text{dividend}}{a}\right.}}$$

We’ll write the steps to take when dividing decimals for easy reference.

Note:
Divide decimals.

Determine the sign of the quotient.
Make the divisor a whole number by “moving” the decimal point all the way to the right.
“Move” the decimal point in the dividend the same number of places—adding zeros as needed.
Divide. Place the decimal point in the quotient above the decimal point in the dividend.

Write the quotient with the appropriate sign.

Example:
Exercise:

Problem: Divide: $-25.65 \div (-0.06)$.

Solution:

Remember, you can “move” the decimals in the divisor and dividend because of the Equivalent Fractions Property.

	$-25.65 \div (-0.06)$
The signs are the same.	The quotient is positive.
Make the divisor a whole number by “moving” the decimal point all the way to the right.	
“Move” the decimal point in the dividend the same number of places.	$0.06\overline{)25.65}$
Divide. Place the decimal point in the quotient above the decimal point in the dividend.	427.5 006.)2565.0 −24 16 −12 45 −42 30 30
Write the quotient with the appropriate sign.	$-25.65 \div (-0.06) = 427.5$

Note:
Exercise:

Problem: Divide: $-23.492 \div (-0.04)$.

Solution:

587.3

Note:
Exercise:

Problem: Divide: $-4.11 \div (-0.12)$.

Solution:

34.25

Convert Decimals, Fractions, and Percents

In our work, it is often necessary to change the form of a number. We may have to change fractions to decimals or decimals to percent.

We convert decimals into fractions by identifying the place value of the last (farthest right) digit. In the decimal $0.03.$ the 3 is in the hundredths place, so 100 is the denominator of the fraction equivalent to 0.03.
Equation:

$$0.03 = \frac{3}{100}$$

The steps to take to convert a decimal to a fraction are summarized in the procedure box.

Note:
Convert a decimal to a proper fraction and a fraction to a decimal.

To convert a decimal to a proper fraction, determine the place value of the final digit.
Write the fraction.

- numerator—the “numbers” to the right of the decimal point
- denominator—the place value corresponding to the final digit

To convert a fraction to a decimal, divide the numerator of the fraction by the denominator of the fraction.

Example:
Exercise:

Problem: Write: ⓐ 0.374 as a fraction ⓑ $-\frac{5}{8}$ as a decimal.

Solution:

ⓐ

	0.374
Determine the place value of the final digit.	0.3 tenths, 7 hundredths, 4 thousandths
Write the fraction for 0.374: The numerator is 374. The denominator is 1,000.	$\frac{374}{1000}$
Simplify the fraction.	$\frac{2 \cdot 187}{2 \cdot 500}$
Divide out the common factors.	$\frac{187}{500}$
	so, $0.374 = \frac{187}{500}$

ⓑ Since a fraction bar means division, we begin by writing the fraction $\frac{5}{8}$ as $8\overline{)5.}$ Now divide.

$$\begin{array}{r} 0.625 \\ 8\overline{)5.000} \\ \underline{48} \\ 20 \\ \underline{16} \\ 40 \\ \underline{40} \end{array}$$

so, $-\frac{5}{8} = -0.625$

Note:
Exercise:

Problem: Write: ⓐ 0.234 as a fraction ⓑ $-\frac{7}{8}$ as a decimal.

Solution:

ⓐ $\frac{117}{500}$ ⓑ -0.875

Note:
Exercise:

Problem: Write: ⓐ 0.024 as a fraction ⓑ $-\frac{3}{8}$ as a decimal.

Solution:

ⓐ $\frac{3}{125}$ ⓑ -0.375

A **percent** is a ratio whose denominator is 100. Percent means per hundred. We use the percent symbol, %, to show percent. Since a percent is a ratio, it can easily be expressed as a fraction. Percent means per 100, so the denominator of the fraction is 100. We then change the fraction to a decimal by dividing the numerator by the denominator. After doing this many times, you may see the pattern.

To convert a percent number to a decimal number, we move the decimal point two places to the left.

6%	78%	2.7%	135%
0.06	0.78	0.027	1.35

To convert a decimal to a percent, remember that percent means per hundred. If we change the decimal to a fraction whose denominator is 100, it is easy to change that fraction to a percent. After many conversions, you may recognize the pattern.

To convert a decimal to a percent, we move the decimal point two places to the right and then add the percent sign.

0.05	0.83	1.05	0.075	0.3
5%	83%	105%	7.5%	30%

Note:
Convert a percent to a decimal and a decimal to a percent.

To convert a percent to a decimal, move the decimal point two places to the left after removing the percent sign.
To convert a decimal to a percent, move the decimal point two places to the right and then add the percent sign.

Example:
Exercise:

Problem: Convert each:

ⓐ percent to a decimal: 62%, 135%, and 13.7%.

ⓑ decimal to a percent: 0.51, 1.25, and 0.093.

Solution:

ⓐ

	62% 135% 35.7%
Move the decimal point two places to the left.	0.62 1.35 0.357

ⓑ

	0.51 1.25 0.093
Move the decimal point two places to the right.	51% 125% 9.3%

Note:
Exercise:

Problem: Convert each:

ⓐ percent to a decimal: 9%, 87%, and 3.9%.

ⓑ decimal to a percent: 0.17, 1.75, and 0.0825.

Solution:

ⓐ 0.09, 0.87, 0.039 ⓑ 17%, 175%, 8.25%

Note:
Exercise:

Problem: Convert each:

ⓐ percent to a decimal: 3%, 91%, and 8.3%.

ⓑ decimal to a percent: 0.41, 2.25, and 0.0925.

Solution:

ⓐ 0.03, 0.91, 0.083 ⓑ 41%, 225%, 9.25%

Simplify Expressions with Square Roots

Remember that when a number n is multiplied by itself, we write n^2 and read it "n squared." The result is called the **square of a number** n. For example, 8^2 is read "8 squared" and 64 is called the *square* of 8. Similarly, 121 is the square of 11 because 11^2 is 121. It will be helpful to learn to recognize the perfect square numbers.

Note:
Square of a number
If $n^2 = m$, then m is the **square** of n.

What about the squares of negative numbers? We know that when the signs of two numbers are the same, their product is positive. So the square of any negative number is also positive.
Equation:

$$(-3)^2 = 9 \qquad (-8)^2 = 64 \qquad (-11)^2 = 121 \qquad (-15)^2 = 225$$

Because $10^2 = 100$, we say 100 is the square of 10. We also say that 10 is a *square root* of 100. A number whose square is m is called a **square root of a number** m.

Note:
Square Root of a Number
If $n^2 = m$, then n is a **square root** of m.

Notice $(-10)^2 = 100$ also, so -10 is also a square root of 100. Therefore, both 10 and -10 are square roots of 100. So, every positive number has two square roots—one positive and one negative. The radical sign, $\sqrt{m}$, denotes the positive square root. The positive square root is called the **principal square root**. When we use the radical sign that always means we want the principal square root.

Note:
Square Root Notation
$\sqrt{m}$ is read "the square root of m."

If $m = n^2$, then $\sqrt{m} = n$, for $n \geq 0$.
The square root of m, $\sqrt{m}$, is the positive number whose square is m.

We know that every positive number has two square roots and the radical sign indicates the positive one. We write $\sqrt{100} = 10$. If we want to find the negative square root of a number, we place a negative in front of the radical sign. For example, $-\sqrt{100} = -10$. We read $-\sqrt{100}$ as "the opposite of the principal square root of 10."

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt{25}$ ⓑ $\sqrt{121}$ ⓒ $-\sqrt{144}$.

Solution:

ⓐ

	$\sqrt{25}$
Since $5^2 = 25$	5

ⓑ

	$\sqrt{121}$
Since $11^2 = 121$	11

ⓒ

	$-\sqrt{144}$
The negative is in front of the radical sign.	-12

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{36}$ ⓑ $\sqrt{169}$ ⓒ $-\sqrt{225}$.

Solution:

ⓐ 6 ⓑ 13 ⓒ -15

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{16}$ ⓑ $\sqrt{196}$ ⓒ $-\sqrt{100}$.

Solution:

ⓐ 4 ⓑ 14 ⓒ -10

Identify Integers, Rational Numbers, Irrational Numbers, and Real Numbers

We have already described numbers as *counting numbers*, *whole numbers*, and *integers*. What is the difference between these types of numbers? Difference could be confused with subtraction. How about asking how we distinguish between these types of numbers?

Equation:

$$\begin{array}{ll} \text{Counting numbers} & 1,2,3,4,\ldots.. \\ \text{Whole numbers} & 0,1,2,3,4,\ldots. \\ \text{Integers} & \ldots.-3,-2,-1,0,1,2,3,\ldots. \end{array}$$

What type of numbers would we get if we started with all the integers and then included all the fractions? The numbers we would have form the set of rational numbers. A **rational number** is a number that can be written as a ratio of two integers.

In general, any decimal that ends after a number of digits (such as 7.3 or -1.2684) is a rational number. We can use the place value of the last digit as the denominator when writing the decimal as a fraction. The decimal for $\frac{1}{3}$ is the number $0.\bar{3}$. The bar over the 3 indicates that the number 3 repeats infinitely. Continuously has an important meaning in calculus. The number(s) under the bar is called the repeating block and it repeats continuously.

Since all integers can be written as a fraction whose denominator is 1, the integers (and so also the counting and whole numbers. are rational numbers.

Every rational number can be written both as a ratio of integers $\frac{p}{q}$*, where p and q are integers and* $q \neq 0$*, and as a decimal that stops or repeats.*

Note:
Rational Number
A **rational number** is a number of the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.
Its decimal form stops or repeats.

Are there any decimals that do not stop or repeat? Yes! The number π (the Greek letter *pi*, pronounced "pie"), which is very important in describing circles, has a decimal form that does not stop or repeat. We use three dots (…) to indicate the decimal does not stop or repeat.

Equation:

$$\pi = 3.141592654...$$

The square root of a number that is not a perfect square is a decimal that does not stop or repeat.

A numbers whose decimal form does not stop or repeat cannot be written as a fraction of integers. We call this an **irrational number**.

Note:
Irrational Number
An **irrational number** is a number that cannot be written as the ratio of two integers.
Its decimal form does not stop and does not repeat.
An irrational number has a corresponding point on the number line as do rational numbers.

Let's summarize a method we can use to determine whether a number is rational or irrational.

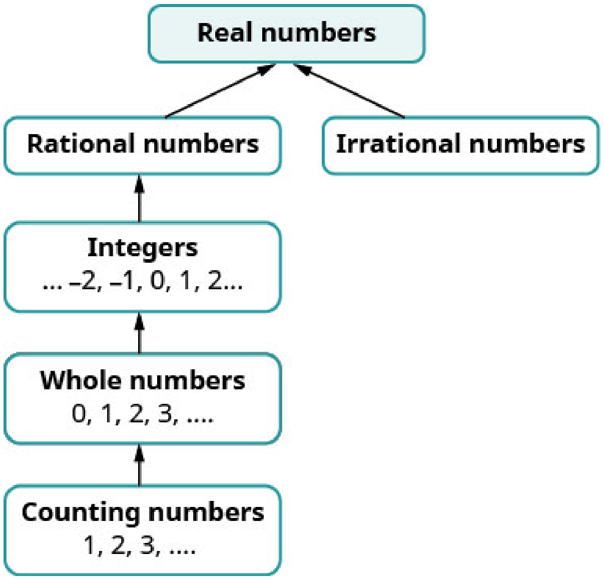
Note:
Rational or Irrational
If the decimal form of a number

- *repeats or stops*, the number is a **rational number**.
- *does not repeat and does not stop*, the number is an **irrational number.**

We have seen that all counting numbers are whole numbers, all whole numbers are integers, and all integers are rational numbers. The irrational numbers are numbers whose decimal form does not stop and does not repeat. When we put together the rational numbers and the irrational numbers, we get the set of **real numbers**.

Note:
Real Number
A **real number** is a number that is either rational or irrational. For every real number there is a corresponding point on the number line.

Later in this course we will introduce numbers beyond the real numbers. [link] illustrates how the number sets we’ve used so far fit together.



This chart shows the number sets that make up the set of real numbers.

Does the term “real numbers” seem strange to you? Are there any numbers that are not “real,” and, if so, what could they be? Can we simplify $\sqrt{-25}$? Is there a number whose square is -25?
Equation:

$$(\ )^2 = -25?$$

None of the numbers that we have dealt with so far has a square that is -25. Why? Any positive number squared is positive. Any negative number squared is positive. So we say there is no real number equal to $\sqrt{-25}$. The square root of a negative number is not a real number.

Example:
Exercise:

Problem:

Given the numbers $-7, \frac{14}{5}, 8, \sqrt{5}, 5.9, -\sqrt{64}$, list the ⓐ whole numbers ⓑ integers ⓒ rational numbers ⓓ irrational numbers ⓔ real numbers.

Solution:

ⓐ Remember, the whole numbers are $0, 1, 2, 3, \ldots$, so 8 is the only whole number given.

ⓑ The integers are the whole numbers and their opposites (which includes 0). So the whole number 8 is an integer, and -7 is the opposite of a whole number so it is an integer, too. Also, notice that 64 is the square of 8 so $-\sqrt{64} = -8$. So the integers are $-7, 8$, and $-\sqrt{64}$.

ⓒ Since all integers are rational, then $-7, 8$, and $-\sqrt{64}$ are rational. Rational numbers also include fractions and decimals that repeat or stop, so $\frac{14}{5}$ and 5.9 are rational. So the list of rational numbers is $-7, \frac{14}{5}, 8, 5.9,$ and $-\sqrt{64}$.

ⓓ Remember that 5 is not a perfect square, so $\sqrt{5}$ is irrational.

ⓔ All the numbers listed are real numbers.

Note:
Exercise:

Problem:

Given the numbers $-3, -\sqrt{2}, 0.\overline{3}, \frac{9}{5}, 4, \sqrt{49}$, list the ⓐ whole numbers ⓑ integers ⓒ rational numbers
ⓓ irrational numbers ⓔ real numbers.

Solution:

ⓐ $4, \sqrt{49}$ ⓑ $-3, 4, \sqrt{49}$
ⓒ $-3, 0.\overline{3}, \frac{9}{5}, 4, \sqrt{49}$ ⓓ $-\sqrt{2}$
ⓔ $-3, -\sqrt{2}, 0.\overline{3}, \frac{9}{5}, 4, \sqrt{49}$

Note:
Exercise:

Problem:

Given numbers $-\sqrt{25}, -\frac{3}{8}, -1, 6, \sqrt{121}, 2.041975\ldots$, list the ⓐ whole numbers ⓑ integers ⓒ rational numbers ⓓ irrational numbers ⓔ real numbers.

Solution:

ⓐ $6, \sqrt{121}$
ⓑ $-\sqrt{25}, -1, 6, \sqrt{121}$

ⓒ $-\sqrt{25}, -\frac{3}{8}, -1, 6, \sqrt{121}$
ⓓ $2.041975...$
ⓔ $-\sqrt{25}, -\frac{3}{8}, -1, 6, \sqrt{121}, 2.041975...$

Locate Fractions and Decimals on the Number Line

We now want to include fractions and decimals on the number line. Let's start with fractions and locate $\frac{1}{5}, -\frac{4}{5}, 3, \frac{7}{4}, -\frac{9}{2}, -5$ and $\frac{8}{3}$ on the number line.

We'll start with the whole numbers 3 and -5 because they are the easiest to plot. See [link].

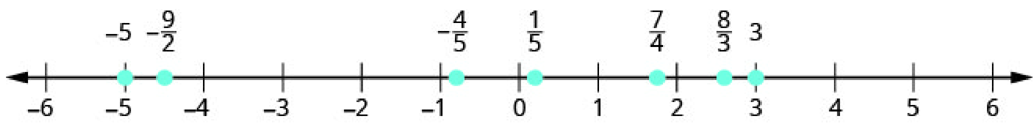
The proper fractions listed are $\frac{1}{5}$ and $-\frac{4}{5}$. We know the proper fraction $\frac{1}{5}$ has value less than one and so would be located between 0 and 1. The denominator is 5, so we divide the unit from 0 to 1 into 5 equal parts $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$. We plot $\frac{1}{5}$.

Similarly, $-\frac{4}{5}$ is between 0 and -1. After dividing the unit into 5 equal parts we plot $-\frac{4}{5}$.

Finally, look at the improper fractions $\frac{7}{4}, \frac{9}{2}, \frac{8}{3}$. Locating these points may be easier if you change each of them to a mixed number.
Equation:

$$\frac{7}{4} = 1\frac{3}{4} \qquad -\frac{9}{2} = -4\frac{1}{2} \qquad \frac{8}{3} = 2\frac{2}{3}$$

[link] shows the number line with all the points plotted.



Example:
Exercise:

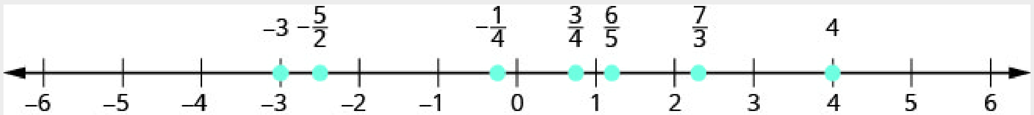
Problem: Locate and label the following on a number line: $4, \frac{3}{4}, -\frac{1}{4}, -3, \frac{6}{5}, -\frac{5}{2}$, and $\frac{7}{3}$.

Solution:

Locate and plot the integers, $4, -3$.

Locate the proper fraction $\frac{3}{4}$ first. The fraction $\frac{3}{4}$ is between 0 and 1. Divide the distance between 0 and 1 into four equal parts, then we plot $\frac{3}{4}$. Similarly plot $-\frac{1}{4}$.

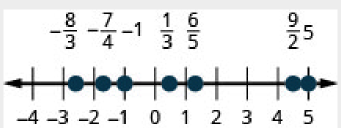
Now locate the improper fractions $\frac{6}{5}$, $-\frac{5}{2}$, and $\frac{7}{3}$. It is easier to plot them if we convert them to mixed numbers and then plot them as described above:
$\frac{6}{5} = 1\frac{1}{5}, -\frac{5}{2} = -2\frac{1}{2}, \frac{7}{3} = 2\frac{1}{3}$.



Note:
Exercise:

Problem: Locate and label the following on a number line: $-1, \frac{1}{3}, \frac{6}{5}, -\frac{7}{4}, \frac{9}{2}, 5, -\frac{8}{3}$.

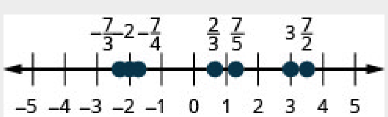
Solution:



Note:
Exercise:

Problem: Locate and label the following on a number line: $-2, \frac{2}{3}, \frac{7}{5}, -\frac{7}{4}, \frac{7}{2}, 3, -\frac{7}{3}$.

Solution:



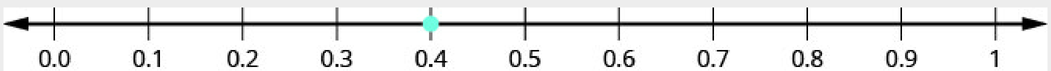
Since decimals are forms of fractions, locating decimals on the number line is similar to locating fractions on the number line.

Example:
Exercise:

Problem: Locate on the number line: ⓐ 0.4 ⓑ -0.74.

Solution:

ⓐ The decimal number 0.4 is equivalent to $\frac{4}{10}$, a proper fraction, so 0.4 is located between 0 and 1. On a number line, divide the interval between 0 and 1 into 10 equal parts. Now label the parts 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0. We write 0 as 0.0 and 1 as 1.0, so that the numbers are consistently in tenths. Finally, mark 0.4 on the number line.



ⓑ The decimal -0.74 is equivalent to $-\frac{74}{100}$, so it is located between 0 and -1. On a number line, mark off and label the hundredths in the interval between 0 and -1.

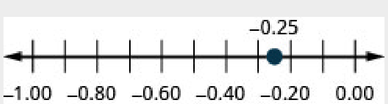
Note:
Exercise:

Problem: Locate on the number line: ⓐ 0.6 ⓑ -0.25.

Solution:

ⓐ

ⓑ

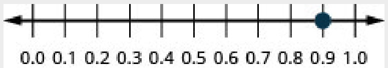


Note:
Exercise:

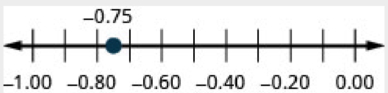
Problem: Locate on the number line: ⓐ 0.9 ⓑ -0.75.

Solution:

ⓐ



ⓑ



Note:
Access this online resource for additional instruction and practice with decimals.

- Arithmetic Basics: Dividing Decimals

Key Concepts

- **How to round decimals.**

  Locate the given place value and mark it with an arrow.
  Underline the digit to the right of the place value.
  Is the underlined digit greater than or equal to $5?$
    - Yes: add 1 to the digit in the given place value.
    - No: do not change the digit in the given place value

  Rewrite the number, deleting all digits to the right of the rounding digit.

- **How to add or subtract decimals.**

  Determine the sign of the sum or difference.
  Write the numbers so the decimal points line up vertically.
  Use zeros as placeholders, as needed.
  Add or subtract the numbers as if they were whole numbers. Then place the decimal point in

the answer under the decimal points in the given numbers.
Write the sum or difference with the appropriate sign

- **How to multiply decimals.**

  Determine the sign of the product.
  Write in vertical format, lining up the numbers on the right. Multiply the numbers as if they were whole numbers, temporarily ignoring the decimal points.
  Place the decimal point. The number of decimal places in the product is the sum of the number of decimal places in the factors.
  Write the product with the appropriate sign.

- **How to multiply a decimal by a power of ten.**

  Move the decimal point to the right the same number of places as the number of zeros in the power of 10.
  Add zeros at the end of the number as needed.

- **How to divide decimals.**

  Determine the sign of the quotient.
  Make the divisor a whole number by "moving" the decimal point all the way to the right. "Move" the decimal point in the dividend the same number of places—adding zeros as needed.
  Divide. Place the decimal point in the quotient above the decimal point in the dividend.
  Write the quotient with the appropriate sign.

- **How to convert a decimal to a proper fraction and a fraction to a decimal.**

  To convert a decimal to a proper fraction, determine the place value of the final digit.
  Write the fraction.
    - numerator—the "numbers" to the right of the decimal point
    - denominator—the place value corresponding to the final digit

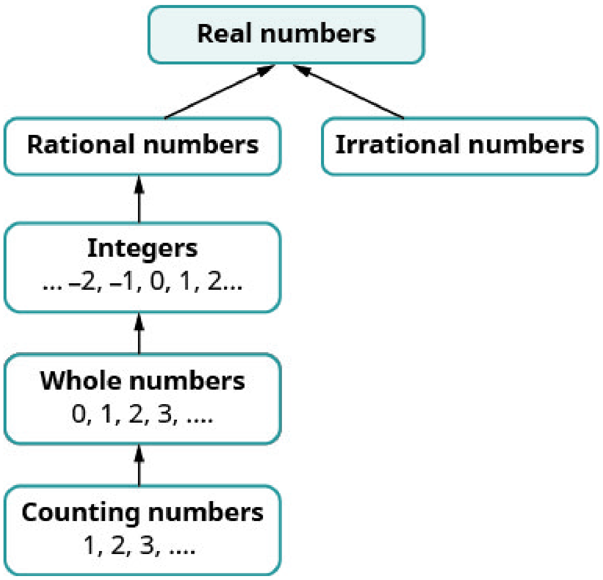
  To convert a fraction to a decimal, divide the numerator of the fraction by the denominator of the fraction.

- **How to convert a percent to a decimal and a decimal to a percent.**

  To convert a percent to a decimal, move the decimal point two places to the left after removing the percent sign.
  To convert a decimal to a percent, move the decimal point two places to the right and then add the percent sign.

- **Square Root Notation**
  $\sqrt{m}$ is read "the square root of *m*."
  If $m = n^2$, then $\sqrt{m} = n$, for $n \geq 0$.
  The square root of *m*, $\sqrt{m}$, is the positive number whose square is *m*.

- **Rational or Irrational**
  If the decimal form of a number
  - *repeats or stops*, the number is a rational number.
  - *does not repeat and does not stop*, the number is an irrational number.
- **Real Numbers**



Practice Makes Perfect

Round Decimals

In the following exercises, round each number to the nearest ⓐ hundredth ⓑ tenth ⓒ whole number.

Exercise:

Problem: 5.781

Solution:

ⓐ 5.78 ⓑ 5.8 ⓒ 6

Exercise:

Problem: 1.638

Exercise:

Problem: 0.299

Solution:

ⓐ 0.30 ⓑ 0.3 ⓒ 0

Exercise:

Problem: 0.697

Exercise:

Problem: 63.479

Solution:

ⓐ 63.48 ⓑ 63.5 ⓒ 63

Exercise:

Problem: 84.281

Add and Subtract Decimals

In the following exercises, add or subtract.

Exercise:

Problem: $-16.53 - 24.38$

Solution:

-40.91

Exercise:

Problem: $-19.47 - 32.58$

Exercise:

Problem: $-38.69 + 31.47$

Solution:

-7.22

Exercise:

Problem: $-29.83 + 19.76$

Exercise:

Problem: $72.5 - 100$

Solution:

-27.5

Exercise:

Problem: $86.2-100$

Exercise:

Problem: $91.75-(-10.462)$

Solution:

02.212

Exercise:

Problem: $94.69-(-12.678)$

Exercise:

Problem: $55.01-3.7$

Solution:

51.31

Exercise:

Problem: $59.08-4.6$

Exercise:

Problem: $2.51-7.4$

Solution:

-4.89

Exercise:

Problem: $3.84-6.1$

Multiply and Divide Decimals

In the following exercises, multiply.

Exercise:

Problem: $94.69 - (-12.678)$

Solution:

-11.653

Exercise:

Problem: $(-8.5)(1.69)$

Exercise:

Problem: $(-5.18)(-65.23)$

Solution:

337.8914

Exercise:

Problem: $(-9.16)(-68.34)$

Exercise:

Problem: $(0.06)(21.75)$

Solution:

1.305

Exercise:

Problem: $(0.08)(52.45)$

Exercise:

Problem: $(9.24)(10)$

Solution:

92.4

Exercise:

Problem: $(6.531)(10)$

Exercise:

Problem: $(0.025)\,(100)$

Solution:

2.5

Exercise:

Problem: $(0.037)\,(100)$

Exercise:

Problem: $(55.2)\,(1000)$

Solution:

55200

Exercise:

Problem: $(99.4)\,(1000)$

In the following exercises, divide. Round money monetary answers to the nearest cent.

Exercise:

Problem: $\$117.25 \div 48$

Solution:

$\$2.44$

Exercise:

Problem: $\$109.24 \div 36$

Exercise:

Problem: $1.44 \div (-0.3)$

Solution:

-4.8

Exercise:

Problem: $-1.15 \div (-0.05)$

Exercise:

Problem: $5.2 \div 2.5$

Solution:

2.08

Exercise:

Problem: $14 \div 0.35$

Convert Decimals, Fractions and Percents

In the following exercises, write each decimal as a fraction.

Exercise:

Problem: 0.04

Solution:

$\frac{1}{25}$

Exercise:

Problem: 1.464

Exercise:

Problem: 0.095

Solution:

$\frac{19}{200}$

Exercise:

Problem: -0.375

In the following exercises, convert each fraction to a decimal.

Exercise:

Problem: $\frac{17}{20}$

Solution:

0.85

Exercise:

Problem: $\frac{17}{4}$

Exercise:

Problem: $-\frac{310}{25}$

Solution:

-12.4

Exercise:

Problem: $-\frac{18}{11}$

In the following exercises, convert each percent to a decimal.

Exercise:

Problem: 71%

Solution:

0.71

Exercise:

Problem: 150%

Exercise:

Problem: 39.3%

Solution:

0.393

Exercise:

Problem: 7.8%

In the following exercises, convert each decimal to a percent.

Exercise:

Problem: 1.56

Solution:

156%

Exercise:

Problem: 3

Exercise:

Problem: 0.0625

Solution:

6.25%

Exercise:

Problem: 2.254

Simplify Expressions with Square Roots

In the following exercises, simplify.

Exercise:

Problem: $\sqrt{64}$

Solution:

8

Exercise:

Problem: $\sqrt{169}$

Exercise:

Problem: $\sqrt{144}$

Solution:

12

Exercise:

Problem: $-\sqrt{4}$

Exercise:

Problem: $-\sqrt{100}$

Solution:

-10

Exercise:

Problem: $-\sqrt{121}$

Identify Integers, Rational Numbers, Irrational Numbers, and Real Numbers

In the following exercises, list the ⓐ whole numbers, ⓑ integers, ⓒ rational numbers, ⓓ irrational numbers, ⓔ real numbers for each set of numbers.

Exercise:

Problem: $-8, 0, 1.95286..., \frac{12}{5}, \sqrt{36}, 9$

Solution:

ⓐ $0, \sqrt{36}, 9$ ⓑ $-8, 0, \sqrt{36}, 9$ ⓒ $-8, 0, \sqrt{36}, 9$ ⓓ $1.95286...,$ ⓔ $-8, 0, 1.95286..., \frac{12}{5}, \sqrt{36}, 9$

Exercise:

Problem: $-9, -3\frac{4}{9}, -\sqrt{9}, 0.4\overline{09}, \frac{11}{6}, 7$

Exercise:

Problem: $-\sqrt{100}, -7, -\frac{8}{3}, -1, 0.77, 3\frac{1}{4}$

Solution:

ⓐ none ⓑ $-\sqrt{100}, -7, -1$
ⓒ $-\sqrt{100}, -7, -\frac{8}{3}, -1, 0.77, 3\frac{1}{4}$
ⓓ none
ⓔ $-\sqrt{100}, -7, -\frac{8}{3}, -1, 0.77, 3\frac{1}{4}$

Exercise:

Problem: $-6, -\frac{5}{2}, 0, 0.\overline{714285}, 2\frac{1}{5}, \sqrt{14}$

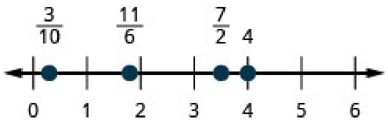
Locate Fractions and Decimals on the Number Line

In the following exercises, locate the numbers on a number line.

Exercise:

Problem: $\frac{3}{10}, \frac{7}{2}, \frac{11}{6}, 4$

Solution:



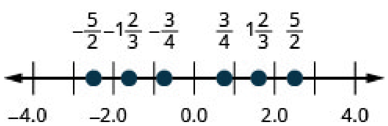
Exercise:

Problem: $\frac{7}{10}, \frac{5}{2}, \frac{13}{8}, 3$

Exercise:

Problem: $\frac{3}{4}, -\frac{3}{4}, 1\frac{2}{3}, -1\frac{2}{3}, \frac{5}{2}, -\frac{5}{2}$

Solution:



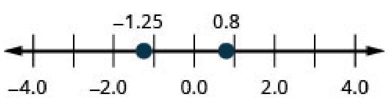
Exercise:

Problem: $\frac{2}{5}, -\frac{2}{5}, 1\frac{3}{4}, -1\frac{3}{4}, \frac{8}{3}, -\frac{8}{3}$

Exercise:

Problem: ⓐ 0.8 ⓑ -1.25

Solution:



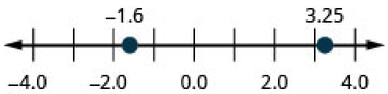
Exercise:

Problem: ⓐ -0.9 ⓑ -2.75

Exercise:

Problem: ⓐ -1.6 ⓑ 3.25

Solution:



Exercise:

Problem: ⓐ 3.1 ⓑ -3.65

Writing Exercises

Exercise:

Problem: How does knowing about U.S. money help you learn about decimals?

Solution:

Answers will vary.

Exercise:

Problem:

When the Szetos sold their home, the selling price was 500% of what they had paid for the house 30 years ago. Explain what 500% means in this context.

Exercise:

Problem:

In your own words, explain the difference between a rational number and an irrational number.

Solution:

Answers will vary.

Exercise:

Problem:

Explain how the sets of numbers (counting, whole, integer, rational, irrationals, reals) are related to each other.

Self Check

ⓐ Use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
round decimals.			
add and subtract decimals.			
multiply and divide decimals.			
convert decimals, fractions and percents.			
simplify expressions with square roots.			
identify integers, rational numbers, irrational numbers and real numbers.			
locate fractions and decimals on the number line.			

ⓑ On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Glossary

irrational number
: An irrational number is a number that cannot be written as the ratio of two integers. Its decimal form does not stop and does not repeat.

percent
: A percent is a ratio whose denominator is 100.

principal square root
: The positive square root is called the principal square root.

rational number
: A rational number is a number of the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. Its decimal form stops or repeats.

real number
: A real number is a number that is either rational or irrational.

square of a number
: If $n^2 = m$, then m is the square of n.

square root of a number
: If $n^2 = m$, then n is a square root of m.

Properties of Real Numbers: ASE
By the end of this section, you will be able to:

- Use the commutative and associative properties
- Use the properties of identity, inverse, and zero
- Simplify expressions using the Distributive Property

Use the Commutative and Associative Properties

The order we add two numbers doesn't affect the result. If we add $8+9$ or $9+8$, the results are the same—they both equal 17. So, $8+9=9+8$. The order in which we add does not matter!

Similarly, when multiplying two numbers, the order does not affect the result. If we multiply $9\cdot 8$ or $8\cdot 9$ the results are the same—they both equal 72. So, $9\cdot 8=8\cdot 9$. The order in which we multiply does not matter!

These examples illustrate the Commutative Property.

Note:
Commutative Property
Equation:

$$\begin{array}{lll}\textbf{of Addition} & \text{If } a \text{ and } b \text{ are real numbers, then} & a+b=b+a. \\ \textbf{of Multiplication} & \text{If } a \text{ and } b \text{ are real numbers, then} & a\cdot b=b\cdot a.\end{array}$$

When adding or multiplying, changing the *order* gives the same result.

The Commutative Property has to do with order. We subtract $9-8$ and $8-9$, and see that $9-8\neq 8-9$. Since changing the order of the subtraction does not give the same result, we know that *subtraction is not commutative*.

Division is not commutative either. Since $12\div 3\neq 3\div 12$, changing the order of the division did not give the same result. The commutative properties apply only to addition and multiplication!

Addition and multiplication *are* commutative.

Subtraction and division are *not* commutative.

When adding three numbers, changing the grouping of the numbers gives the same result. For example, $(7+8)+2=7+(8+2)$, since each side of the equation equals 17.

This is true for multiplication, too. For example, $\left(5\cdot \frac{1}{3}\right)\cdot 3=5\cdot \left(\frac{1}{3}\cdot 3\right)$, since each side of the equation equals 5.

These examples illustrate the Associative Property.

Note:
Associative Property
Equation:

of Addition	If $a, b,$ and c are real numbers, then	$(a+b)+c=a+(b+c)$.
of Multiplication	If a, b, and c are real numbers, then	$(a \cdot b) \cdot c=a \cdot(b \cdot c)$.

When adding or multiplying, changing the *grouping* gives the same result.

The Associative Property has to do with grouping. If we change how the numbers are grouped, the result will be the same. Notice it is the same three numbers in the same order—the only difference is the grouping.

We saw that subtraction and division were not commutative. They are not associative either.
Equation:

$$\begin{array}{cc} (10-3)-2 \neq 10-(3-2) & (24 \div 4) \div 2 \neq 24 \div(4 \div 2) \\ 7-2 \neq 10-1 & 6 \div 2 \neq 24 \div 2 \\ 5 \neq 9 & 3 \neq 12 \end{array}$$

Removing Subtraction and Division
Subtraction and division can be more difficult to deal with than addition and multiplication because subtraction and division are not commutative or associative. This is not a problem because every subtraction can be replaced by an addition of the opposite value and every division can be replaced by a multiplication by the reciprocal.

Example:
Simplify

a. $93-25+7+125$

$93+(-25)+7+125$

$93+7+(-25)+125$

$100+((-25)+125)$

$100+100$

200

b. $25 \cdot 3 \cdot 7 \div 25$

$25 \cdot 3 \cdot 7 \cdot \frac{1}{25}$

$25 \cdot \frac{1}{25} \cdot 3 \cdot 7$

$1 \cdot 3 \cdot 7$

$3 \cdot 7$

21

When simplifying an expression, it is always a good idea to plan what the steps will be. In order to combine like terms in the next example, we will use the Commutative Property of addition to write the like terms together.

Example:
Exercise:

Problem: Simplify: $18p + 6q + 15p + 5q$.

Solution:

	$18p + 6q + 15p + 5q$
Use the Commutative Property of addition to reorder so that like terms are together.	$18p + 15p + 6q + 5q$
Add like terms.	$33p + 11q$

Note:
Exercise:

Problem: Simplify: $23r + 14s + 9r + 15s$.

Solution:

$32r + 29s$

Note:
Exercise:

Problem: Simplify:$37m + 21n + 4m - 15n$.

Solution:

$41m + 6n$

When we have to simplify algebraic expressions, we can often make the work easier by applying the Commutative Property or Associative Property first.

Example:
Exercise:

Problem: Simplify: $\left(\frac{5}{13} + \frac{3}{4}\right) + \frac{1}{4}$.

Solution:

	$\left(\frac{5}{13}+\frac{3}{4}\right)+\frac{1}{4}$
Notice that the last 2 terms have a common denominator, so change the grouping.	$\frac{5}{13}+\left(\frac{3}{4}+\frac{1}{4}\right)$
Add in parentheses first.	$\frac{5}{13}+\left(\frac{4}{4}\right)$
Simplify the fraction.	$\frac{5}{13}+1$
Add.	$1\frac{5}{13}$
Convert to an improper fraction.	$\frac{18}{13}$

Note:
Exercise:

Problem: Simplify: $\left(\frac{7}{15}+\frac{5}{8}\right)+\frac{3}{8}$.

Solution:

$1\frac{7}{15}$

Note:
Exercise:

Problem: Simplify: $\left(\frac{2}{9}+\frac{7}{12}\right)+\frac{5}{12}$.

Solution:

$1\frac{2}{9}$

Use the Properties of Identity, Inverse, and Zero

What happens when we add 0 to any number? Adding 0 doesn't change the value. For this reason, we call 0 the **additive identity**. The **Identity Property of Addition** that states that for any real number $a, a+0=a$ and $0+a=a$.

What happens when we multiply any number by one? Multiplying by 1 doesn't change the value. So we call 1 the **multiplicative identity**. The **Identity Property of Multiplication** that states that for any real number $a, a\cdot 1=a$ and $1\cdot a=a$.

We summarize the Identity Properties here.

Note:
Identity Property
Equation:

of Addition	For any real number a :	$a+0=a$	$0+a=a$
	0 is the **additive identity**		
of Multiplication	For any real number a :	$a \cdot 1=a$	$1 \cdot a=a$
	1 is the **multiplicative identity**		

What number added to 5 gives the additive identity, 0? We know

$5+(-5)=0$

The missing number was the opposite of the number!

We call $-a$ the **additive inverse** of a. *The opposite of a number is its additive inverse.* A number and its opposite add to zero, which is the additive identity. This leads to the **Inverse Property of Addition** that states for any real number $a, a+(-a)=0$.

What number multiplied by $\frac{2}{3}$ gives the multiplicative identity, 1? In other words, $\frac{2}{3}$ times what results in 1? We know

$\frac{2}{3} \cdot \frac{3}{2}=1$

The missing number was the reciprocal of the number!

We call $\frac{1}{a}$ the **multiplicative inverse** of a. *The reciprocal of a number is its multiplicative inverse.* This leads to the **Inverse Property of Multiplication** that states that for any real number $a, a \neq 0, a \cdot \frac{1}{a}=1$.

We'll formally state the inverse properties here.

Note:
Inverse Property
Equation:

of Addition	For any real number a,	$a+(-a)=0$
	$-a$ is the **additive inverse** of a	
	A number and its *opposite* add to zero.	
of Multiplication	For any real number a,	$a \neq 0, \quad a \cdot \frac{1}{a}=1.$
	$\frac{1}{a}$ is the **multiplicative inverse** of a.	
	A number and its *reciprocal* multiply to one.	

The Identity Property of addition says that when we add 0 to any number, the result is that same number. What happens when we multiply a number by 0? Multiplying by 0 makes the product equal zero.

What about division involving zero? What is $0 \div 3$? Think about a real example: If there are no cookies in the cookie jar and 3 people are to share them, how many cookies does each person get? There are no cookies to share, so each person gets 0 cookies. So, $0 \div 3=0$.

We can check division with the related multiplication fact. So we know $0 \div 3 = 0$ because $0 \cdot 3 = 0$.

Now think about dividing *by* zero. What is the result of dividing 4 by 0? Think about the related multiplication fact:

$4 \div 0 = \boxed{?}$ means $\boxed{?} \cdot 0 = 4$

Is there a number that multiplied by 0 gives 4? Since any real number multiplied by 0 gives 0, there is no real number that can be multiplied by 0 to obtain 4. We conclude that there is no answer to $4 \div 0$ and so we say that division by 0 is **undefined**.

We summarize the properties of zero here.

Note:
Properties of Zero
Multiplication by Zero: For any real number *a*,

$$a \cdot 0 = 0 \quad 0 \cdot a = 0 \qquad \text{The product of any number and 0 is 0.}$$

Division by Zero: For any real number *a*, $a \neq 0$
Equation:

$$\begin{array}{cl} \frac{0}{a} = 0 & \text{Zero divided by any real number, except itself, is zero.} \\ \frac{a}{0} \text{ is undefined} & \text{Division by zero is undefined.} \end{array}$$

We will now practice using the properties of identities, inverses, and zero to simplify expressions.

Example:
Exercise:

Problem: Simplify: $-84n + (-73n) + 84n$.

Solution:

	$-84n + (-73n) + 84n$
Notice that the first and third terms are opposites; use the Commutative Property of addition to re-order the terms.	$-84n + 84n + (-73n)$
Add left to right.	$0 + (-73n)$
Add.	$-73n$

Note:
Exercise:

Problem: Simplify: $-27a + (-48a) + 27a$.

Solution:

$-48a$

Note:
Exercise:

Problem: Simplify: $39x + (-92x) + (-39x)$.

Solution:

$-92x$

Now we will see how recognizing reciprocals is helpful. Before multiplying left to right, look for reciprocals—their product is 1.

Example:
Exercise:

Problem: Simplify: $\frac{7}{15} \cdot \frac{8}{23} \cdot \frac{15}{7}$.

Solution:

	$\frac{7}{15} \cdot \frac{8}{23} \cdot \frac{15}{7}$
Notice the first and third terms are reciprocals, so use the Commutative Property of multiplication to re-order the factors.	$\frac{7}{15} \cdot \frac{15}{7} \cdot \frac{8}{23}$
Multiply left to right.	$1 \cdot \frac{8}{23}$
Multiply.	$\frac{8}{23}$

Note:
Exercise:

Problem: Simplify: $\frac{9}{16} \cdot \frac{5}{49} \cdot \frac{16}{9}$.

Solution:

$\frac{5}{49}$

Note:
Exercise:

Problem: Simplify: $\frac{6}{17} \cdot \frac{11}{25} \cdot \frac{17}{6}$.

Solution:

$\frac{11}{25}$

The next example makes us aware of the distinction between dividing 0 by some number or some number being divided by 0.

Example:
Exercise:

Problem: Simplify: ⓐ $\frac{0}{n+5}$, where $n \neq -5$ ⓑ $\frac{10-3p}{0}$, where $10 - 3p \neq 0$.

Solution:

ⓐ

	$\frac{0}{n+5}$
Zero divided by any real number except itself is 0.	0

ⓑ

	$\frac{10-3p}{0}$
Division by 0 is undefined.	undefined

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{0}{m+7}$, where $m \neq -7$ ⓑ $\frac{18-6c}{0}$, where $18 - 6c \neq 0$.

Solution:

ⓐ 0 ⓑ undefined

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{0}{d-4}$, where $d \neq 4$ ⓑ $\frac{15-4q}{0}$, where $15 - 4q \neq 0$.

Solution:

ⓐ 0 ⓑ undefined

Simplify Expressions Using the Distributive Property

Suppose that three friends are going to the movies. They each need \$9.25—that's 9 dollars and 1 quarter—to pay for their tickets. How much money do they need all together?

You can think about the dollars separately from the quarters. They need 3 times \$9 so \$27 and 3 times 1 quarter, so 75 cents. In total, they need \$27.75. If you think about doing the math in this way, you are using the Distributive Property.

Note:
Distributive Property

If $a, b,$ and c are real numbers, then

$$\begin{aligned} a\left(b+c\right) &= ab+ac \\ (b+c)a &= ba+ca \\ a\left(b-c\right) &= ab-ac \\ (b-c)a &= ba-ca \end{aligned}$$

In algebra, we use the Distributive Property to remove parentheses as we simplify expressions.

Example:
Exercise:

Problem: Simplify: $3\left(x+4\right)$.

Solution:

	$3\left(x+4\right)$
Distribute.	$3\cdot x+3\cdot 4$
Multiply.	$3x+12$

Note:
Exercise:

Problem: Simplify: $4\left(x+2\right)$.

Solution:

$4x+8$

Note:
Exercise:

Problem: Simplify: $6\left(x+7\right)$.

Solution:

$6x+42$

Some students find it helpful to draw in arrows to remind them how to use the Distributive Property. Then the first step in [link] would look like this:

$3(x+4)$

Other students like to think of the similarity to multi-digit multiplication:

$$\begin{array}{r} (x+4) \\ 3 \\ \hline 12 \\ +\ 3\ x \\ \hline 3x+12 \end{array}$$

Example:
Exercise:

Problem: Simplify: $8\left(\frac{3}{8}x+\frac{1}{4}\right)$.

Solution:

	$8\left(\frac{3}{8}x+\frac{1}{4}\right)$
Distribute.	$8\cdot\frac{3}{8}x+8\cdot\frac{1}{4}$
Multiply.	$3x+2$

Note:
Exercise:

Problem: Simplify: $6\left(\frac{5}{6}y+\frac{1}{2}\right)$.

Solution:

$5y+3$

Note:
Exercise:

Problem: Simplify: $12\left(\frac{1}{3}n+\frac{3}{4}\right)$.

Solution:

$4n+9$

Using the Distributive Property as shown in the next example will be very useful when we solve money applications in later chapters.

Example:
Exercise:

Problem: Simplify: $100\left(0.3+0.25q\right)$.

Solution:

	$100(0.3+0.25q)$
Distribute.	$100(0.3)+100(0.25q)$
Multiply.	$30+25q$

Note:
Exercise:

Problem: Simplify: $100\left(0.7+0.15p\right)$.

Solution:

$70+15p$

Note:
Exercise:

Problem: Simplify: $100\,(0.04 + 0.35d)$.

Solution:

$4 + 35d$

When we distribute a negative number, we need to be extra careful to get the signs correct!

Example:
Exercise:

Problem: Simplify: $-11\,(4 - 3a)$.

Solution:

	$-11\,(4 - 3a)$
Distribute.	$-11 \cdot 4 - (-11) \cdot 3a$
Multiply.	$-44 - (-33a)$
Simplify.	$-44 + 33a$

Notice that you could also write the result as $33a - 44$. Do you know why?

Note:
Exercise:

Problem: Simplify: $-5\,(2 - 3a)$.

Solution:

$-10 + 15a$

Note:
Exercise:

Problem: Simplify: $-7\,(8 - 15y)$.

Solution:

$-56 + 105y$

In the next example, we will show how to use the Distributive Property to find the opposite of an expression.

Example:
Exercise:

Problem: Simplify: $-(y+5)$.

Solution:

	$-(y+5)$
Multiplying by -1 results in the opposite.	$-1(y+5)$
Distribute.	$-1 \cdot y + (-1) \cdot 5$
Simplify.	$-y + (-5)$
Simplify.	$-y - 5$

Note:
Exercise:

Problem: Simplify: $-(z-11)$.

Solution:

$-z + 11$

Note:
Exercise:

Problem: Simplify: $-(x-4)$.

Solution:

$-x + 4$

There will be times when we'll need to use the Distributive Property as part of the order of operations. Start by looking at the parentheses. If the expression inside the parentheses cannot be simplified, the next step would be multiply using the Distributive Property, which removes the parentheses. The next two examples will illustrate this.

Example:
Exercise:

Problem: Simplify: $8 - 2(x+3)$

Solution:

We follow the order of operations. Multiplication comes before subtraction, so we will distribute the 2 first and then subtract.

	$8-2(x+3)$
Distribute.	$8-2\cdot x-2\cdot 3$
Multiply.	$8-2x-6$
Combine like terms.	$-2x+2$

Note:
Exercise:

Problem: Simplify: $9-3(x+2)$.

Solution:

$3-3x$

Note:
Exercise:

Problem: Simplify: $7x-5(x+4)$.

Solution:

$2x-20$

Example:
Exercise:

Problem: Simplify: $4(x-8)-(x+3)$.

Solution:

	$4(x-8)-(x+3)$
Distribute.	$4x-32-x-3$
Combine like terms.	$3x-35$

Note:
Exercise:

Problem: Simplify: $6(x-9)-(x+12)$.

Solution:

$5x-66$

Note:
Exercise:

Problem: Simplify: $8\left(x-1\right)-\left(x+5\right)$.

Solution:

$7x-13$

All the properties of real numbers we have used in this chapter are summarized here.

Commutative Property
When adding or multiplying, changing the *order* gives the same result

$$\begin{array}{lrcl}\textbf{of addition}\text{ If }a,b\text{ are real numbers, then} & a+b & = & b+a\\ \textbf{of multiplication}\text{ If }a,b\text{ are real numbers, then} & a\cdot b & = & b\cdot a\end{array}$$

Associative Property
When adding or multiplying, changing the *grouping* gives the same result.

$$\begin{array}{lrcl}\textbf{of addition}\text{ If }a,b,\text{ and }c\text{ are real numbers, then} & (a+b)+c & = & a+(b+c)\\ \textbf{of multiplication}\text{ If }a,b,\text{ and }c\text{ are real numbers, then} & (a\cdot b)\cdot c & = & a\cdot(b\cdot c)\end{array}$$

Distributive Property

$$\begin{array}{lrcl}\text{If }a,b,\text{ and }c\text{ are real numbers, then} & a\left(b+c\right) & = & ab+ac\\ & (b+c)a & = & ba+ca\\ & a\left(b-c\right) & = & ab-ac\\ & (b-c)a & = & ba-ca\end{array}$$

Identity Property

$$\begin{array}{ll}\textbf{of addition}\text{ For any real number }a: & a+0=a\\ \qquad\textbf{0}\text{ is the }\textbf{additive identity} & 0+a=a\\ \textbf{of multiplication}\text{ For any real number }a: & a\cdot 1=a\\ \qquad\textbf{1}\text{ is the }\textbf{multiplicative identity} & 1\cdot a=a\end{array}$$

Inverse Property	
of addition For any real number a,	$a+(-a)=0$
$-a$ is the **additive inverse** of a	
A number and its *opposite* add to zero.	
of multiplication For any real number $a, a \neq 0$	$a \cdot \frac{1}{a}=1$
$\frac{1}{a}$ is the **multiplicative inverse** of a	
A number and its *reciprocal* multiply to one.	
Properties of Zero	
For any real number a,	$a \cdot 0=0$
	$0 \cdot a=0$
For any real number $a, a \neq 0$,	$\frac{0}{a}=0$
For any real number a,	$\frac{a}{0}$ is undefined

Key Concepts

Commutative Property	
When adding or multiplying, changing the *order* gives the same result	
of addition If a, b are real numbers, then	$a+b = b+a$
of multiplication If a, b are real numbers, then	$a \cdot b = b \cdot a$
Associative Property	
When adding or multiplying, changing the *grouping* gives the same result.	
of addition If $a, b,$ and c are real numbers, then	$(a+b)+c = a+(b+c)$
of multiplication If $a, b,$ and c are real numbers, then	$(a \cdot b) \cdot c = a \cdot (b \cdot c)$
Distributive Property	
If $a, b,$ and c are real numbers, then	$a\,(b+c) = ab+ac$
	$(b+c)a = ba+ca$
	$a\,(b-c) = ab-ac$
	$(b-c)a = ba-ca$
Identity Property	
of addition For any real number a:	$a+0=a$
0 is the **additive identity**	$0+a=a$
of multiplication For any real number a:	$a \cdot 1=a$
1 is the **multiplicative identity**	$1 \cdot a=a$
Inverse Property	

of addition For any real number a,	$a + (-a) = 0$
$-a$ is the **additive inverse** of a	
A number and its *opposite* add to zero.	
of multiplication For any real number $a, a \neq 0$	$a \cdot \frac{1}{a} = 1$
$\frac{1}{a}$ is the **multiplicative inverse** of a	
A number and its *reciprocal* multiply to one.	
Properties of Zero	
For any real number a,	$a \cdot 0 = 0$
	$0 \cdot a = 0$
For any real number $a, a \neq 0$,	$\frac{0}{a} = 0$
For any real number a,	$\frac{a}{0}$ is undefined

Section Exercises

Practice Makes Perfect

Use the Commutative and Associative Properties

In the following exercises, simplify.

Exercise:

Problem: $43m + (-12n) + (-16m) + (-9n)$

Solution:

$27m + (-21n)$

Exercise:

Problem: $-22p + 17q + (-35p) + (-27q)$

Exercise:

Problem: $\frac{3}{8}g + \frac{1}{12}h + \frac{7}{8}g + \frac{5}{12}h$

Solution:

$\frac{5}{4}g + \frac{1}{2}h$

Exercise:

Problem: $\frac{5}{6}a + \frac{3}{10}b + \frac{1}{6}a + \frac{9}{10}b$

Exercise:

Problem: $6.8p + 9.14q + (-4.37p) + (-0.88q)$

Solution:

$2.43p + 8.26q$

Exercise:

Problem: $9.6m + 7.22n + (-2.19m) + (-0.65n)$

Exercise:

Problem: $-24 \cdot 7 \cdot \frac{3}{8}$

Solution:

-63

Exercise:

Problem: $-36 \cdot 11 \cdot \frac{4}{9}$

Exercise:

Problem: $\left(\frac{5}{6} + \frac{8}{15}\right) + \frac{7}{15}$

Solution:

$1\,\frac{5}{6}$

Exercise:

Problem: $\left(\frac{11}{12} + \frac{4}{9}\right) + \frac{5}{9}$

Exercise:

Problem: $17\,(0.25)\,(4)$

Solution:

17

Exercise:

Problem: $36\,(0.2)\,(5)$

Exercise:

Problem: $[2.48\,(12)]\,(0.5)$

Solution:

14.88

Exercise:

Problem: $[9.731\,(4)]\,(0.75)$

Exercise:

Problem: $12\left(\frac{5}{6}p\right)$

Solution:

$10p$

Exercise:

Problem: $20\left(\frac{3}{5}q\right)$

Use the Properties of Identity, Inverse and Zero

In the following exercises, simplify.

Exercise:

Problem: $19a+44-19a$

Solution:

44

Exercise:

Problem: $27c+16-27c$

Exercise:

Problem: $\frac{1}{2}+\frac{7}{8}+\left(-\frac{1}{2}\right)$

Solution:

$\frac{7}{8}$

Exercise:

Problem: $\frac{2}{5}+\frac{5}{12}+\left(-\frac{2}{5}\right)$

Exercise:

Problem: $10\left(0.1d\right)$

Solution:

d

Exercise:

Problem: $100\left(0.01p\right)$

Exercise:

Problem: $\frac{3}{20}\cdot\frac{49}{11}\cdot\frac{20}{3}$

Solution:

$\frac{49}{11}$

Exercise:

Problem: $\frac{13}{18}\cdot\frac{25}{7}\cdot\frac{18}{13}$

Exercise:

Problem: $\frac{0}{u-4.99}$, where $u \neq 4.99$

Solution:

0

Exercise:

Problem: $0 \div \left(y - \frac{1}{6}\right)$, where $x \neq \frac{1}{6}$

Exercise:

Problem: $\frac{32-5a}{0}$, where $32 - 5a \neq 0$

Solution:

undefined

Exercise:

Problem: $\frac{28-9b}{0}$, where $28 - 9b \neq 0$

Exercise:

Problem: $\left(\frac{3}{4} + \frac{9}{10}m\right) \div 0$, where $\frac{3}{4} + \frac{9}{10}m \neq 0$

Solution:

undefined

Exercise:

Problem: $\left(\frac{5}{16}n - \frac{3}{7}\right) \div 0$, where $\frac{5}{16}n - \frac{3}{7} \neq 0$

Simplify Expressions Using the Distributive Property

In the following exercises, simplify using the Distributive Property.

Exercise:

Problem: $8\left(4y + 9\right)$

Solution:

$32y + 72$

Exercise:

Problem: $9\left(3w + 7\right)$

Exercise:

Problem: $6\left(c - 13\right)$

Solution:

$6c-78$

Exercise:

Problem: $7\left(y-13\right)$

Exercise:

Problem: $\frac{1}{4}\left(3q+12\right)$

Solution:

$\frac{3}{4}q+3$

Exercise:

Problem: $\frac{1}{5}\left(4m+20\right)$

Exercise:

Problem: $9\left(\frac{5}{9}y-\frac{1}{3}\right)$

Solution:

$5y-3$

Exercise:

Problem: $10\left(\frac{3}{10}x-\frac{2}{5}\right)$

Exercise:

Problem: $12\left(\frac{1}{4}+\frac{2}{3}r\right)$

Solution:

$3+8r$

Exercise:

Problem: $12\left(\frac{1}{6}+\frac{3}{4}s\right)$

Exercise:

Problem: $15\cdot\frac{3}{5}\left(4d+10\right)$

Solution:

$36d+90$

Exercise:

Problem: $18\cdot\frac{5}{6}\left(15h+24\right)$

Exercise:

Problem: $r\left(s-18\right)$

Solution:

$rs-18r$

Exercise:

Problem: $u\left(v-10\right)$

Exercise:

Problem: $(y+4)p$

Solution:

$yp+4p$

Exercise:

Problem: $(a+7)x$

Exercise:

Problem: $-7\left(4p+1\right)$

Solution:

$-28p-7$

Exercise:

Problem: $-9\left(9a+4\right)$

Exercise:

Problem: $-3\left(x-6\right)$

Solution:

$-3x+18$

Exercise:

Problem: $-4\left(q-7\right)$

Exercise:

Problem: $-\left(3x-7\right)$

Solution:

$-3x+7$

Exercise:

Problem: $-\left(5p-4\right)$

Exercise:

Problem: $16 - 3\left(y+8\right)$

Solution:

$-3y-8$

Exercise:

Problem: $18 - 4\left(x+2\right)$

Exercise:

Problem: $4 - 11\left(3c-2\right)$

Solution:

$-33c+26$

Exercise:

Problem: $9 - 6\left(7n-5\right)$

Exercise:

Problem: $22 - \left(a+3\right)$

Solution:

$-a+19$

Exercise:

Problem: $8 - \left(r-7\right)$

Exercise:

Problem: $\left(5m-3\right)-\left(m+7\right)$

Solution:

$4m-10$

Exercise:

Problem: $\left(4y-1\right)-\left(y-2\right)$

Exercise:

Problem: $9\left(8x-3\right)-\left(-2\right)$

Solution:

$72x-25$

Exercise:

Problem: $4\left(6x-1\right)-\left(-8\right)$

Exercise:

Problem: $5\left(2n+9\right)+12\left(n-3\right)$

Solution:

$22n+9$

Exercise:

Problem: $9\left(5u+8\right)+2\left(u-6\right)$

Exercise:

Problem: $14\left(c-1\right)-8\left(c-6\right)$

Solution:

$6c+34$

Exercise:

Problem: $11\left(n-7\right)-5\left(n-1\right)$

Exercise:

Problem: $6\left(7y+8\right)-\left(30y-15\right)$

Solution:

$12y+63$

Exercise:

Problem: $7\left(3n+9\right)-\left(4n-13\right)$

Writing Exercises

Exercise:

Problem: In your own words, state the Associative Property of addition.

Solution:

Answers will vary.

Exercise:

Problem: What is the difference between the additive inverse and the multiplicative inverse of a number?

Exercise:

Problem: Simplify $8\left(x-\frac{1}{4}\right)$ using the Distributive Property and explain each step.

Solution:

Answers will vary.

Exercise:

Problem:

Explain how you can multiply $4\,(\$5.97)$ without paper or calculator by thinking of $\$5.97$ as $6-0.03$ and then using the Distributive Property.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
use the commutative and associative properties.			
use the properties of identity, inverse and zero.			
simplify expressions using the distributive property.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

Chapter Review Exercises

Use the Language of Algebra

Identify Multiples and Factors

Exercise:

Problem: Use the divisibility tests to determine whether 180 is divisible by 2, by 3, by 5, by 6, and by 10.

Solution:

Divisible by $2, 3, 5, 6$

Exercise:

Problem: Find the prime factorization of 252.

Exercise:

Problem: Find the least common multiple of 24 and 40.

Solution:

120

In the following exercises, simplify each expression.

Exercise:

Problem: $24 \div 3 + 4\,(5-2)$

Exercise:

Problem: $7+3\left[6-4\left(5-4\right)\right]-3^{2}$

Solution:

4

Evaluate an Expression

In the following exercises, evaluate the following expressions.
Exercise:

Problem: When $x=4$, ⓐ x^{3} ⓑ 5^{x} ⓒ $2x^{2}-5x+3$

Exercise:

Problem: $2x^{2}-4xy-3y^{2}$ when $x=3$, $y=1$

Solution:

3

Simplify Expressions by Combining Like Terms

In the following exercises, simplify the following expressions by combining like terms.
Exercise:

Problem: $12y+7+2y-5$

Exercise:

Problem: $14x^{2}-9x+11-8x^{2}+8x-6$

Solution:

$6x^{2}-x+5$

Translate an English Phrase to an Algebraic Expression

In the following exercises, translate the phrases into algebraic expressions.
Exercise:

Problem:
ⓐ the sum of $4ab^{2}$ and $7a^{3}b^{2}$
ⓑ the product of $6y^{2}$ and $3y$
ⓒ twelve more than $5x$
ⓓ $5y$ less than $8y^{2}$

Exercise:

Problem:
ⓐ eleven times the difference of y and two
ⓑ the difference of eleven times y and two

Solution:

ⓐ $11\left(y-2\right)$ ⓑ $11y-2$

Exercise:

Problem:

Dushko has nickels and pennies in his pocket. The number of pennies is four less than five the number of nickels. Let n represent the number of nickels. Write an expression for the number of pennies.

Integers

Simplify Expressions with Absolute Value

In the following exercise, fill in $<$, $>$, or $=$ for each of the following pairs of numbers.

Exercise:

Problem:
ⓐ $-\left|7\right|$ ___ $-\left|-7\right|$
ⓑ -8 ___ $-\left|-8\right|$
ⓒ $\left|-13\right|$ ___ -13
ⓓ $\left|-12\right|$ ___ $-\left(-12\right)$

Solution:

ⓐ $=$ ⓑ $=$ ⓒ $>$ ⓓ $=$

In the following exercises, simplify.

Exercise:

Problem: $9-\left|3\left(4-8\right)\right|$

Exercise:

Problem: $12-3\left|1-4\left(4-2\right)\right|$

Solution:

-9

Add and Subtract Integers

In the following exercises, simplify each expression.

Exercise:

Problem: $-12+\left(-8\right)+7$

Exercise:

Problem:
ⓐ $15-7$
ⓑ $-15-\left(-7\right)$
ⓒ $-15-7$
ⓓ $15-\left(-7\right)$

Solution:

ⓐ 8 ⓑ -8 ⓒ -22 ⓓ 22

Exercise:

Problem: $-11-(-12)+5$

Exercise:

Problem: ⓐ $23-(-17)$ ⓑ $23+17$

Solution:

ⓐ 40 ⓑ 40

Exercise:

Problem: $-(7-11)-(3-5)$

Multiply and Divide Integers

In the following exercise, multiply or divide.

Exercise:

Problem: ⓐ $-27\div 9$ ⓑ$120\div(-8)$ ⓒ $4(-14)$ ⓓ $-1(-17)$

Solution:

ⓐ-3 ⓑ-15 ⓒ-56 ⓓ 17

Simplify and Evaluate Expressions with Integers

In the following exercises, simplify each expression.

Exercise:

Problem: ⓐ $(-7)^3$ ⓑ -7^3

Exercise:

Problem: $(7-11)(6-13)$

Solution:

16

Exercise:

Problem: $63\div(-9)+(-36)\div(-4)$

Exercise:

Problem: $6-3\,|4(1-2)-(7-5)|$

Solution:

-12

Exercise:

Problem: $(-2)^4 - 24 \div (13 - 5)$

For the following exercises, evaluate each expression.

Exercise:

Problem: $(y + z)^2$ when $y = -4, z = 7$

Solution:

9

Exercise:

Problem: $3x^2 - 2xy + 4y^2$ when $x = -2, y = -3$

Translate English Phrases to Algebraic Expressions

In the following exercises, translate to an algebraic expression and simplify if possible.

Exercise:

Problem: the sum of -4 and -9, increased by 23

Solution:

$(-4 + (-9)) + 23; 10$

Exercise:

Problem: ⓐ the difference of 17 and -8 ⓑ subtract 17 from -25

Use Integers in Applications

In the following exercise, solve.

Exercise:

Problem:

Temperature On July 10, the high temperature in Phoenix, Arizona, was 109°, and the high temperature in Juneau, Alaska, was 63°. What was the difference between the temperature in Phoenix and the temperature in Juneau?

Solution:

46°

Fractions

Simplify Fractions

In the following exercises, simplify.
Exercise:

Problem: $\frac{204}{228}$

Exercise:

Problem: $-\frac{270x^3}{198y^2}$

Solution:

$-\frac{15x^3}{11y^2}$

Multiply and Divide Fractions

In the following exercises, perform the indicated operation.
Exercise:

Problem: $\left(-\frac{14}{15}\right)\left(\frac{10}{21}\right)$

Exercise:

Problem: $\frac{6x}{25} \div \frac{9y}{20}$

Solution:

$\frac{8x}{15y}$

Exercise:

Problem: $\frac{-\frac{4}{9}}{\frac{8}{21}}$

Add and Subtract Fractions

In the following exercises, perform the indicated operation.
Exercise:

Problem: $\frac{5}{18} + \frac{7}{12}$

Solution:

$\frac{31}{36}$

Exercise:

Problem: $\frac{11}{36} - \frac{15}{48}$

Exercise:

Problem: ⓐ $\frac{5}{8} + \frac{3}{4}$ ⓑ $\frac{5}{8} \div \frac{3}{4}$

Solution:

ⓐ $\frac{11}{8}$ ⓑ $\frac{5}{6}$

Exercise:

Problem: ⓐ $-\frac{3y}{10}-\frac{5}{6}$ ⓑ $-\frac{3y}{10}\cdot\frac{5}{6}$

Use the Order of Operations to Simplify Fractions

In the following exercises, simplify.

Exercise:

Problem: $\frac{4\cdot 3-2\cdot 5}{-6\cdot 3+2\cdot 3}$

Solution:

$-\frac{1}{6}$

Exercise:

Problem: $\frac{4(7-3)-2(4-9)}{-3(4+2)+7(3-6)}$

Exercise:

Problem: $\frac{4^3-4^2}{\left(\frac{4}{5}\right)^2}$

Solution:

75

Evaluate Variable Expressions with Fractions

In the following exercises, evaluate.

Exercise:

Problem: $4x^2y^2$ when $x=\frac{2}{3}$ and $y=-\frac{3}{4}$

Exercise:

Problem: $\frac{a+b}{a-b}$ when $a=-4, b=6$

Solution:

$-\frac{1}{5}$

Decimals

Round Decimals

Exercise:

Problem: Round 6.738 to the nearest ⓐ hundredth ⓑ tenth ⓒ whole number.

Add and Subtract Decimals

In the following exercises, perform the indicated operation.
Exercise:

Problem: $-23.67+29.84$

Solution:

6.17

Exercise:

Problem: $54.3-100$

Exercise:

Problem: $79.38-\left(-17.598\right)$

Solution:

96.978

Multiply and Divide Decimals

In the following exercises, perform the indicated operation.
Exercise:

Problem: $\left(-2.8\right)\left(3.97\right)$

Exercise:

Problem: $\left(-8.43\right)\left(-57.91\right)$

Solution:

488.1813

Exercise:

Problem: $\left(53.48\right)\left(10\right)$

Exercise:

Problem: $\left(0.563\right)\left(100\right)$

Solution:

56.3

Exercise:

Problem: $\$118.35÷2.6$

Exercise:

Problem: $1.84 \div (-0.8)$

Solution:

-23

Convert Decimals, Fractions and Percents

In the following exercises, write each decimal as a fraction.
Exercise:

Problem: $\frac{13}{20}$

Exercise:

Problem: $-\frac{240}{25}$

Solution:

-9.6

In the following exercises, convert each fraction to a decimal.
Exercise:

Problem: $-\frac{5}{8}$

Exercise:

Problem: $\frac{14}{11}$

Solution:

1.27

In the following exercises, convert each decimal to a percent.
Exercise:

Problem: 2.43

Exercise:

Problem: 0.0475

Solution:

4.75%

Simplify Expressions with Square Roots

In the following exercises, simplify.
Exercise:

Problem: $\sqrt{289}$

Exercise:

Problem: $\sqrt{-121}$

Solution:

no real number

Identify Integers, Rational Numbers, Irrational Numbers, and Real Numbers

In the following exercise, list the ⓐ whole numbers ⓑ integers ⓒ rational numbers ⓓ irrational numbers ⓔ real numbers for each set of numbers

Exercise:

Problem: $-8, 0, 1.95286..., \frac{12}{5}, \sqrt{36}, 9$

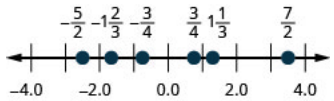
Locate Fractions and Decimals on the Number Line

In the following exercises, locate the numbers on a number line.

Exercise:

Problem: $\frac{3}{4}, -\frac{3}{4}, 1\frac{1}{3}, -1\frac{2}{3}, \frac{7}{2}, -\frac{5}{2}$

Solution:



Exercise:

Problem: ⓐ 3.2 ⓑ -1.35

[Properties of Real Numbers]

Use the Commutative and Associative Properties

In the following exercises, simplify.

Exercise:

Problem: $\frac{5}{8}x + \frac{5}{12}y + \frac{1}{8}x + \frac{7}{12}y$

Solution:

$\frac{3}{4}x + y$

Exercise:

Problem: $-32 \cdot 9 \cdot \frac{5}{8}$

Exercise:

Problem: $\left(\frac{11}{15} + \frac{3}{8}\right) + \frac{5}{8}$

Solution:

$1\frac{11}{15}$

Use the Properties of Identity, Inverse and Zero

In the following exercises, simplify.

Exercise:

Problem: $\frac{4}{7} + \frac{8}{15} + \left(-\frac{4}{7}\right)$

Exercise:

Problem: $\frac{13}{15} \cdot \frac{9}{17} \cdot \frac{15}{13}$

Solution:

$\frac{9}{17}$

Exercise:

Problem: $\frac{0}{x-3}, x \neq 3$

Exercise:

Problem: $\frac{5x-7}{0}, 5x - 7 \neq 0$

Solution:

undefined

Simplify Expressions Using the Distributive Property

In the following exercises, simplify using the Distributive Property.

Exercise:

Problem: $8\left(a - 4\right)$

Exercise:

Problem: $12\left(\frac{2}{3}b + \frac{5}{6}\right)$

Solution:

$8b + 10$

Exercise:

Problem: $18 \cdot \frac{5}{6}(2x-5)$

Exercise:

Problem: $(x-5)p$

Solution:

$xp-5p$

Exercise:

Problem: $-4\,(y-3)$

Exercise:

Problem: $12-6\,(x+3)$

Solution:

$-6x-6$

Exercise:

Problem: $6\,(3x-4)-(-5)$

Exercise:

Problem: $5\,(2y+3)-(4y-1)$

Solution:

$y+16$

Practice Test

Exercise:

Problem: Find the prime factorization of 756.

Exercise:

Problem: Combine like terms: $5n+8+2n-1$

Solution:

$7n+7$

Exercise:

Problem: Evaluate when $x=-2$ and $y=3$: $\frac{|3x-4y|}{6}$

Exercise:

Problem: Translate to an algebraic expression and simplify:

ⓐ eleven less than negative eight

ⓑ the difference of -8 and -3, increased by 5

Solution:

$-8-11;-19$

$(-8-(-3))+5;0$

Exercise:

Problem:

Dushko has nickels and pennies in his pocket. The number of pennies is seven less than four times the number of nickels. Let n represent the number of nickels. Write an expression for the number of pennies.

Exercise:

Problem: Round 28.1458 to the nearest

ⓐ hundredth ⓑ thousandth

Solution:

ⓐ 28.15 ⓑ 28.146

Exercise:

Problem: Convert

ⓐ $\frac{5}{11}$ to a decimal ⓑ 1.15 to a percent

Exercise:

Problem: Locate $\frac{3}{5}$, 2.8, and $-\frac{5}{2}$ on a number line.

Solution:

In the following exercises, simplify each expression.

Exercise:

Problem: $8+3\left[6-3\left(5-2\right)\right]-4^2$

Exercise:

Problem: $-\left(4-9\right)-\left(9-5\right)$

Solution:

1

Exercise:

Problem: $56 \div (-8) + (-27) \div (-3)$

Exercise:

Problem: $16 - 2\left|3\left(1-4\right) - \left(8-5\right)\right|$

Solution:

-8

Exercise:

Problem: $-5 + 2(-3)^2 - 9$

Exercise:

Problem: $\frac{180}{204}$

Solution:

$\frac{15}{17}$

Exercise:

Problem: $-\frac{7}{18} + \frac{5}{12}$

Exercise:

Problem: $\frac{4}{5} \div \left(-\frac{12}{25}\right)$

Solution:

$-\frac{5}{3}$

Exercise:

Problem: $\frac{9-3\cdot 9}{15-9}$

Exercise:

Problem: $\frac{4(-3+2(3-6))}{3(11-3(2+3))}$

Solution:

3

Exercise:

Problem: $\frac{5}{13} \cdot 47 \cdot \frac{13}{5}$

Exercise:

Problem: $\frac{-\frac{5}{9}}{\frac{10}{21}}$

Solution:

$-\frac{7}{6}$

Exercise:

Problem: $-4.8+(-6.7)$

Exercise:

Problem: $34.6-100$

Solution:

-65.4

Exercise:

Problem: $-12.04\cdot(4.2)$

Exercise:

Problem: $-8\div 0.05$

Solution:

160

Exercise:

Problem: $\sqrt{-121}$

Exercise:

Problem: $\left(\frac{8}{13}+\frac{5}{7}\right)+\frac{2}{7}$

Solution:

$1\frac{8}{13}$

Exercise:

Problem: $5x+(-8y)-6x+3y$

Exercise:

Problem: ⓐ $\frac{0}{9}$ ⓑ $\frac{11}{0}$

Solution:

ⓐ 0 ⓑ undefined

Exercise:

Problem: $-3\left(8x-5\right)$

Exercise:

Problem: $6\left(3y-1\right)-\left(5y-3\right)$

Solution:

$13y-3$

Glossary

additive identity
: The number 0 is the additive identity because adding 0 to any number does not change its value.

additive inverse
: The opposite of a number is its additive inverse.

multiplicative identity
: The number 1 is the multiplicative identity because multiplying 1 by any number does not change its value.

multiplicative inverse
: The reciprocal of a number is its multiplicative inverse.

Introduction
class="introduction"

This drone is flying high in the sky while its pilot remains safely on the ground. (credit: “Unsplash” / Pixabay)

Imagine being a pilot, but not just any pilot—a drone pilot. Drones, or unmanned aerial vehicles, are devices that can be flown remotely. They contain sensors that can relay information to a command center where the pilot is located. Larger drones can also carry cargo. In the near future, several companies hope to use drones to deliver materials and piloting a drone will become an important career. Law enforcement and the military are using drones rather than send personnel into dangerous situations.

Building and piloting a drone requires the ability to program a set of actions, including taking off, turning, and landing. This, in turn, requires the use of linear equations. In this chapter, you will explore linear equations, develop a strategy for solving them, and relate them to real-world situations.

Solve Equations Using the Subtraction and Addition Properties of Equality: ASE

By the end of this section, you will be able to:

- Verify a solution of an equation
- Solve equations using the Subtraction and Addition Properties of Equality
- Solve equations that require simplification
- Translate to an equation and solve
- Translate and solve applications

Verify a Solution of an Equation

Solving an equation is like discovering the answer to a puzzle. The purpose in solving an equation is to find the value or values of the variable that make each side of the equation the same – so that we end up with a true statement. Any value of the variable that makes the equation true is called a solution to the equation. It is the answer to the puzzle!

Note:

Solution of an equation

A **solution of an equation** is a value of a variable that makes a true statement when substituted into the equation.

Note:

To determine whether a number is a solution to an equation.

Substitute the number in for the variable in the equation.

Simplify the expressions on both sides of the equation.

Determine whether the resulting equation is true (the left side is equal to the right side)

- If it is true, the number is a solution.
- If it is not true, the number is not a

solution.

Example:
Exercise:

Problem:

Determine whether $x = \frac{3}{2}$ is a solution of $4x - 2 = 2x + 1$.

Solution:
Solution

Since a solution to an equation is a value of the variable that makes the equation true, begin by substituting the value of the solution for the variable.

	$4x - 2 = 2x + 1$
Substitute $\frac{3}{2}$ for x.	$4\left(\frac{3}{2}\right) - 3 \stackrel{?}{=} 2\left(\frac{3}{2}\right) + 1$
Multiply.	$6 - 2 \stackrel{?}{=} 3 + 1$
Subtract.	$4 = 4\checkmark$

Since $x = \frac{3}{2}$ results in a true equation (4 is in fact equal to 4), $\frac{3}{2}$ is a solution to the equation $4x - 2 = 2x + 1$.

Note:
Exercise:

Problem: Is $y = \frac{4}{3}$ a solution of $9y + 2 = 6y + 3$?

Solution:

no

Note:
Exercise:

Problem: Is $y = \frac{7}{5}$ a solution of $5y + 3 = 10y - 4$?

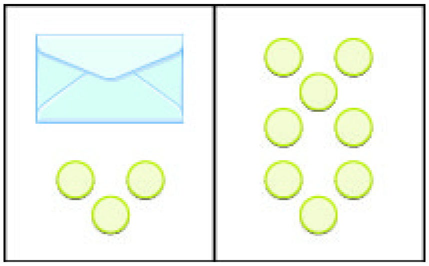
Solution:

yes

Solve Equations Using the Subtraction and Addition Properties of Equality

We are going to use a model to clarify the process of solving an equation. An envelope represents the variable – since its contents are unknown – and each counter represents one. We will set out one envelope and some counters on our workspace, as shown in [link]. Both sides of the workspace

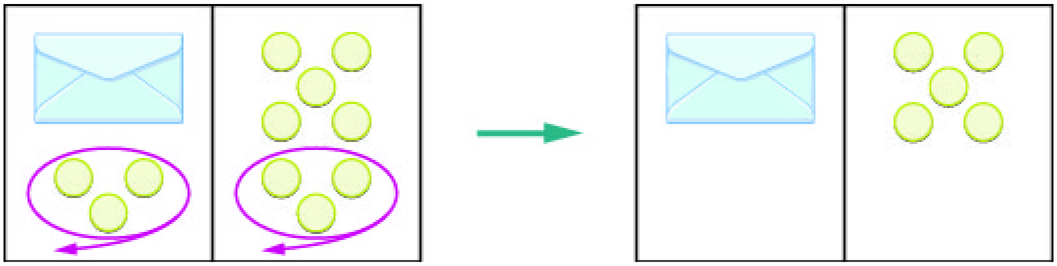
have the same number of counters, but some counters are “hidden” in the envelope. Can you tell how many counters are in the envelope?



The illustration shows a model of an equation with one variable. On the left side of the workspace is an unknown (envelope) and three counters, while on the right side of the workspace are eight counters.

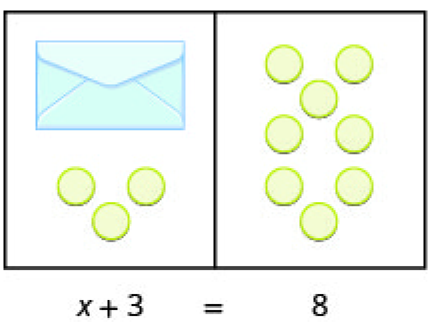
What are you thinking? What steps are you taking in your mind to figure out how many counters are in the envelope?

Perhaps you are thinking: “I need to remove the 3 counters at the bottom left to get the envelope by itself. The 3 counters on the left can be matched with 3 on the right and so I can take them away from both sides. That leaves five on the right—so there must be 5 counters in the envelope.” See [link] for an illustration of this process.



The illustration shows a model for solving an equation with one variable. On both sides of the workspace remove three counters, leaving only the unknown (envelope) and five counters on the right side. The unknown is equal to five counters.

What algebraic equation would match this situation? In [link] each side of the workspace represents an expression and the center line takes the place of the equal sign. We will call the contents of the envelope x.



The illustration shows a model for the equation $x+3=8$.

Let's write algebraically the steps we took to discover how many counters were in the envelope:

	$x+3=8$
First, we took away three from each side.	$x+3-3=8-3$
Then we were left with five.	$x=5$

Check:

Five in the envelope plus three more does equal eight!
Equation:

$$5+3=8$$

Our model has given us an idea of what we need to do to solve one kind of equation. The goal is to isolate the variable by itself on one side of the equation. To solve equations such as these mathematically, we use the **Subtraction Property of Equality**.

Note:
Subtraction Property of Equality
For any numbers *a*, *b*, and *c*,
Equation:

$$\begin{array}{llll}\text{If} & a & = & b, \\ \text{then} & a-c & = & b-c\end{array}$$

When you subtract the same quantity from both sides of an equation, you still have equality.

Let's see how to use this property to solve an equation. Remember, the goal is to isolate the variable on one side of the equation. And we check our solutions by substituting the value into the equation to make sure we have a true statement.

Example:
Exercise:

Problem: Solve: $y+37=-13$.

Solution:
Solution

To get *y* by itself, we will undo the addition of 37 by using the Subtraction Property of Equality.

		$y+37=-13$
Subtract 37 from each side to 'undo' the addition.		$y+37-37=-13-37$
Simplify.		$y=-50$
Check:	$y+37=-13$	
Substitute $y=-50$		

	$-50 + 37 = -13$	
	$-13 \stackrel{?}{=} -13\ \checkmark$	

Since $y = -50$ makes $y + 37 = -13$ a true statement, we have the solution to this equation.

Note:
Exercise:

Problem: Solve: $x + 19 = -27$.

Solution:

$x = -46$

Note:
Exercise:

Problem: Solve: $x + 16 = -34$.

Solution:

$x = -50$

What happens when an equation has a number subtracted from the variable, as in the equation $x - 5 = 8$? We use another property of equations to

solve equations where a number is subtracted from the variable. We want to isolate the variable, so to ‘undo’ the subtraction we will add the number to both sides. We use the **Addition Property of Equality**.

Note:
Addition Property of Equality
For any numbers a, b, and c,
Equation:

$$\begin{array}{lrcl} \text{If} & a & = & b, \\ \text{then} & a+c & = & b+c \end{array}$$

When you add the same quantity to both sides of an equation, you still have equality.

In [link], 37 was added to the y and so we subtracted 37 to ‘undo’ the addition. In [link], we will need to ‘undo’ subtraction by using the Addition Property of Equality.

Example:
Exercise:

Problem: Solve: $a-28=-37$.

Solution:
Solution

		$a - 28 = -37$
Add 28 to each side to ‘undo’ the subtraction.		$a - 28 + 28 = -37 + 28$
Simplify.		$a = -9$
Check:	$a - 28 = -37$	
Substitute $a = -9$	$-9 - 28 = -37$	
	$-37 \stackrel{?}{=} -37$ ✓	
		The solution to $a - 28 = -37$ is $a = -9$.

Note:
Exercise:

Problem: Solve: $n - 61 = -75$.

Solution:

$n = -14$

Note:
Exercise:

Problem: Solve: $p - 41 = -73$.

Solution:

$p = -32$

Example:
Exercise:

Problem: Solve: $x - \frac{5}{8} = \frac{3}{4}$.

Solution:
Solution

	$x - \frac{5}{8} = \frac{3}{4}$
Use the Addition Property of Equality.	$x - \frac{5}{8} + \frac{5}{8} = \frac{3}{4} + \frac{5}{8}$
Find the LCD to add the fractions on the right.	$x - \frac{5}{8} + \frac{5}{8} = \frac{6}{8} + \frac{5}{8}$

Simplify.		$x=\frac{11}{8}$
Check:	$x-\frac{5}{8}=\frac{3}{4}$	
Substitute $x=\frac{11}{8}$.	$\frac{11}{8}-\frac{5}{8}\stackrel{?}{=}\frac{3}{4}$	
Subtract.	$\frac{6}{8}\stackrel{?}{=}\frac{3}{4}$	
Simplify.	$\frac{3}{4}=\frac{3}{4}\checkmark$	
		The solution to $x-\frac{5}{8}=\frac{3}{4}$ is $x=\frac{11}{8}$.

Note:
Exercise:

Problem: Solve: $p-\frac{2}{3}=\frac{5}{6}$.

Solution:

$p=\frac{9}{6}\,p=\frac{3}{2}$

Note:
Exercise:

Problem: Solve: $q - \frac{1}{2} = \frac{5}{6}$.

Solution:

$q = \frac{4}{3}$

The next example will be an equation with decimals.

Example:
Exercise:

Problem: Solve: $n - 0.63 = -4.2$.

Solution:
Solution

	$n - 0.63 = -4.2$
Use the Addition Property of Equality.	$n - 0.63 + 0.63 = -4.2 + 0.63$
Add.	$n = -3.57$

Check:	$n = -3.57$	
Let $n = -3.57$.	$-3.57 - 0.63 \stackrel{?}{=} -4.2$	
	$-4.2 = -4.2 \checkmark$	

Note:
Exercise:

Problem: Solve: $b - 0.47 = -2.1$.

Solution:

$b = -1.63$

Note:
Exercise:

Problem: Solve: $c - 0.93 = -4.6$.

Solution:

$c = -3.67$

Solve Equations That Require Simplification

In the previous examples, we were able to isolate the variable with just one operation. Most of the equations we encounter in algebra will take more steps to solve. Usually, we will need to simplify one or both sides of an equation before using the Subtraction or Addition Properties of Equality.

You should always simplify as much as possible before you try to isolate the variable. Remember that to simplify an expression means to do all the operations in the expression. Simplify one side of the equation at a time. Note that simplification is different from the process used to solve an equation in which we apply an operation to both sides.

Example:
How to Solve Equations That Require Simplification
Exercise:

Problem: Solve: $9x - 5 - 8x - 6 = 7$.

Solution:
Solution

Step 1. Simplify the expressions on each side as much as possible.	Rearrange the terms, using the Commutative Property of Addition. Combine like terms. Notice that each side is now simplified as much as possible.	$9x - 5 - 8x - 6 = 7$ $9x - 8x - 5 - 6 = 7$ $x - 11 = 7$
Step 2. Isolate the variable.	Now isolate x. Undo subtraction by adding 11 to both sides.	$x - 11 + 11 = 7 + 11$
Step 3. Simplify the expressions on both sides of the equation.		$x = 18$

Step 4. Check the solution.		**Check:** Substitute $x = 18$. $9x - 5 - 8x - 6 = 7$ $9(18) - 5 - 8(18) - 6 \stackrel{?}{=} 7$ $162 - 5 - 144 - 6 \stackrel{?}{=} 7$ $157 - 144 - 6 \stackrel{?}{=} 7$ $13 - 6 \stackrel{?}{=} 7$ $7 = 7$ ✓ The solution to $9x - 5 - 8x - 6 = 7$ is $x = 18$.

Note:
Exercise:

Problem: Solve: $8y - 4 - 7y - 7 = 4$.

Solution:

$y = 15$

Note:
Exercise:

Problem: Solve: $6z + 5 - 5z - 4 = 3$.

Solution:

$z = 2$

Example:
Exercise:

Problem: Solve: $5(n-4)-4n=-8$.

Solution:
Solution

We simplify both sides of the equation as much as possible before we try to isolate the variable.

	$5(n-4)-4n=-8$
Distribute on the left.	$5n-20-4n=-8$
Use the Commutative Property to rearrange terms.	$5n-4n-20=-8$
Combine like terms.	$n-20=-8$
Each side is as simplified as possible. Next, isolate n.	
Undo subtraction by using the Addition Property of Equality.	$n-20+20=-8+20$
Add.	$n=12$

Check. Substitute $n = 12$. $5(n-4) - 4n = -8$ $5(12-4) - 4(12) \stackrel{?}{=} -8z$ $5(8) - 48 \stackrel{?}{=} -8$ $40 - 48 \stackrel{?}{=} -8$ $-8 = -8$ ✓	
	The solution to $5(n-4) - 4n = -8$ is $n = 12$.

Note:
Exercise:

Problem: Solve: $5(p-3) - 4p = -10$.

Solution:

$p = 5$

Note:
Exercise:

Problem: Solve: $4(q+2) - 3q = -8$.

Solution:

$q = -16$

Example:
Exercise:

Problem: Solve: $3(2y - 1) - 5y = 2\left(y + 1\right) - 2\left(y + 3\right)$.

Solution:
Solution

We simplify both sides of the equation before we isolate the variable.

	$3(2y - 1) - 5y = 2(y + 1) - 2(y + 3)$
Distribute on both sides.	$6y - 3 - 5y = 2y + 2 - 2y - 6$
Use the Commutative Property of Addition.	$6y - 5y - 3 = 2y - 2y + 2 - 6$
Combine like terms.	$y - 3 = -4$
Each side is as simplified as	

possible. Next, isolate y.	
Undo subtraction by using the Addition Property of Equality.	$y - 3 + 3 = -4 + 3$
Add.	$y = -1$
Check. Let $y = -1$. $3(2y - 1) - 5y = 2(y + 1) - 2(y + 3)$ $3(2(-1) - 1) - 5(-1) \stackrel{?}{=} 2(-1 + 1) - 2(-1 + 3)$ $3(-2 - 1) + 5 \stackrel{?}{=} 2(0) - 2(2)$ $3(-3) + 5 \stackrel{?}{=} -4$ $-9 + 5 \stackrel{?}{=} -4$ $-4 = -4 \checkmark$	
	The solution to $3(2y - 1) - 5y = 2\left(y + 1\right) - 2\left(y + 3\right)$ is $y = -1$.

Note:
Exercise:

Problem: Solve: $4(2h - 3) - 7h = 6\left(h - 2\right) - 6\left(h - 1\right)$.

Solution:

$h = 6$

Note:
Exercise:

Problem: Solve: $2(5x+2)-9x=3\left(x-2\right)-3\left(x-4\right)$.

Solution:

$x=2$

Translate to an Equation and Solve

To solve applications algebraically, we will begin by translating from English sentences into equations. Our first step is to look for the word (or words) that would translate to the equals sign. [link] shows us some of the words that are commonly used.

Equals =
is is equal to is the same as the result is gives was will be

The steps we use to translate a sentence into an equation are listed below.

Note:
Translate an English sentence to an algebraic equation.

Locate the “equals” word(s). Translate to an equals sign (=).
Translate the words to the left of the “equals” word(s) into an algebraic expression.
Translate the words to the right of the “equals” word(s) into an algebraic expression.

Example:
Exercise:

Problem: Translate and solve: Eleven more than *x* is equal to 54.

Solution:
Solution

Translate.	$\underbrace{\text{Eleven more than } x}_{x+11} \quad \underbrace{\text{is equal to}}_{=} \quad \underbrace{54}_{54}$
Subtract 11 from both sides.	$x + 11 - 11 = 54 - 11$
Simplify.	$x = 43$

Check: Is 54 eleven more than 43? $$\begin{aligned} 43 + 11 &\stackrel{?}{=} 54 \\ 54 &= 54\checkmark \end{aligned}$$	

Note:
Exercise:

Problem: Translate and solve: Ten more than *x* is equal to 41.

Solution:

$x + 10 = 41; x = 31$

Note:
Exercise:

Problem: Translate and solve: Twelve less than *x* is equal to 51.

Solution:

$y - 12 = 51; y = 63$

Example:
Exercise:

Problem: Translate and solve: The difference of $12t$ and $11t$ is -14.

Solution:
Solution

Translate.	$\underbrace{\text{The difference of } 12t \text{ and } 11t}_{12t-11t}\ \underbrace{\text{is}}_{=}\ \underbrace{-14}_{-14}$
Simplify.	$t=-14$
Check: $$\begin{aligned} 12(-14)-11(-14) &\overset{?}{=} -14 \\ -168+154 &\overset{?}{=} -14 \\ -14 &= -14\checkmark \end{aligned}$$	

Note:
Exercise:

Problem: Translate and solve: The difference of $4x$ and $3x$ is 14.

Solution:

$4x-3x=14; x=14$

Note:
Exercise:

Problem: Translate and solve: The difference of $7a$ and $6a$ is -8.

Solution:

$7a - 6a = -8; a = -8$

Translate and Solve Applications

Most of the time a question that requires an algebraic solution comes out of a real life question. To begin with that question is asked in English (or the language of the person asking) and not in math symbols. Because of this, it is an important skill to be able to translate an everyday situation into algebraic language.

We will start by restating the problem in just one sentence, assign a variable, and then translate the sentence into an equation to solve. When assigning a variable, choose a letter that reminds you of what you are looking for. For example, you might use q for the number of quarters if you were solving a problem about coins.

Example:
How to Solve Translate and Solve Applications
Exercise:

Problem:

The MacIntyre family recycled newspapers for two months. The two months of newspapers weighed a total of 57 pounds. The second month, the newspapers weighed 28 pounds. How much did the newspapers weigh the first month?

Solution:
Solution

Step 1. Read the problem. Make sure all the words and ideas are understood.	The problem is about the weight of newspapers.	
Step 2. Identify what we are asked to find.	What are we asked to find?	"How much did the newspapers weigh the 2nd month?"
Step 3. Name what we are looking for. Choose a variable to represent that quantity.	Choose a variable.	Let w = weight of the newspapers the 1st month
Step 4. Translate into an equation. It may be helpful to restate the problem in one sentence with the important information.	Restate the problem. We know the weight of the newspapers the second month is 28 pounds. Translate into an equation, using the variable w.	Weight of newspapers the 1st month plus the weight of the newspapers the 2nd month equals 57 pounds. Weight from 1st month plus 28 equals 57. $w + 28 = 57$
Step 5. Solve the equation using good algebra techniques.	Solve.	$w + 28 - 28 = 57 - 28$ $w = 29$
Step 6. Check the answer in the problem and make sure it makes sense.	Does 1st month's weight plus 2nd month's weight equal 57 pounds?	**Check:** Does 1st month's weight plus 2nd month's weight equal 57 pounds? $29 + 28 \stackrel{?}{=} 57$ $57 = 57$ ✓

Step 7. Answer the question with a complete sentence.	Write a sentence to answer "How much did the newspapers weigh the 2^{nd} month?"	The 2^{nd} month the newspapers weighed 29 pounds.

Note:
Exercise:

Problem: Translate into an algebraic equation and solve:

The Pappas family has two cats, Zeus and Athena. Together, they weigh 23 pounds. Zeus weighs 16 pounds. How much does Athena weigh?

Solution:

7 pounds

Note:
Exercise:

Problem: Translate into an algebraic equation and solve:

Sam and Henry are roommates. Together, they have 68 books. Sam has 26 books. How many books does Henry have?

Solution:

42 books

Note:
Solve an application.

Readthe problem. Make sure all the words and ideas are understood.
Identifywhat we are looking for.
Namewhat we are looking for. Choose a variable to represent that quantity.
Translateinto an equation. It may be helpful to restate the problem in one sentence with the important information.
Solvethe equation using good algebra techniques.
Checkthe answer in the problem and make sure it makes sense.
Answerthe question with a complete sentence.

Example:
Exercise:

Problem:

Randell paid $28,675 for his new car. This was $875 less than the sticker price. What was the sticker price of the car?

Solution:
Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	"What was the sticker price of the car?"
Step 3. Name what we are looking for.	Let $s =$ the sticker price of the car.

Choose a variable to represent that quantity.	
Step 4. Translate into an equation. Restate the problem in one sentence.	\$28,675 is \$875 less than the sticker price
Step 5. Solve the equation.	\$28,675 is \$875 less than s mtd mtd mtd mtd mtd
Step 6. Check the answer. Is \$875 less than \$29,550 equal to \$28,675? $29{,}550 - 875 \stackrel{?}{=} 28{,}675$ $28{,}675 = 28{,}675\checkmark$	
Step 7. Answer the question with a complete sentence.	The sticker price of the car was \$29,550.

Note:
Exercise:

Problem: Translate into an algebraic equation and solve:

Eddie paid \$19,875 for his new car. This was \$1,025 less than the sticker price. What was the sticker price of the car?

Solution:

$20,900

Note:
Exercise:

Problem: Translate into an algebraic equation and solve:

The admission price for the movies during the day is $7.75. This is $3.25 less the price at night. How much does the movie cost at night?

Solution:

$11.00

Key Concepts

- **To Determine Whether a Number is a Solution to an Equation**

  Substitute the number in for the variable in the equation.
  Simplify the expressions on both sides of the equation.
  Determine whether the resulting statement is true.
    - If it is true, the number is a solution.
    - If it is not true, the number is not a solution.
- **Addition Property of Equality**
    - For any numbers a, b, and c, if $a=b$, then $a+c=b+c$.

- **Subtraction Property of Equality**
  - For any numbers a, b, and c, if $a = b$, then $a - c = b - c$.
- **To Translate a Sentence to an Equation**

  Locate the “equals” word(s). Translate to an equal sign (=).
  Translate the words to the left of the “equals” word(s) into an algebraic expression.
  Translate the words to the right of the “equals” word(s) into an algebraic expression.
- **To Solve an Application**

  Read the problem. Make sure all the words and ideas are understood.
  Identify what we are looking for.
  Name what we are looking for. Choose a variable to represent that quantity.
  Translate into an equation. It may be helpful to restate the problem in one sentence with the important information.
  Solve the equation using good algebra techniques.
  Check the answer in the problem and make sure it makes sense.
  Answer the question with a complete sentence.

Practice Makes Perfect

Verify a Solution of an Equation

In the following exercises, determine whether the given value is a solution to the equation.
Exercise:

Problem: Is $y = \frac{5}{3}$ a solution of $6y + 10 = 12y$?

Solution:

yes

Exercise:

Problem: Is $x = \frac{9}{4}$ a solution of $4x + 9 = 8x$?

Exercise:

Problem: Is $u = -\frac{1}{2}$ a solution of $8u - 1 = 6u$?

Solution:

no

Exercise:

Problem: Is $v = -\frac{1}{3}$ a solution of $9v - 2 = 3v$?

Solve Equations using the Subtraction and Addition Properties of Equality

In the following exercises, solve each equation using the Subtraction and Addition Properties of Equality.

Exercise:

Problem: $x + 24 = 35$

Solution:

$x = 11$

Exercise:

Problem: $x + 17 = 22$

Exercise:

Problem: $y+45=-66$

Solution:

$y=-111$

Exercise:

Problem: $y+39=-83$

Exercise:

Problem: $b+\frac{1}{4}=\frac{3}{4}$

Solution:

$b=\frac{1}{2}$

Exercise:

Problem: $a+\frac{2}{5}=\frac{4}{5}$

Exercise:

Problem: $p+2.4=-9.3$

Solution:

$p=-11.7$

Exercise:

Problem: $m+7.9=11.6$

Exercise:

Problem: $a - 45 = 76$

Solution:

$a = 121$

Exercise:

Problem: $a - 30 = 57$

Exercise:

Problem: $m - 18 = -200$

Solution:

$m = -182$

Exercise:

Problem: $m - 12 = -12$

Exercise:

Problem: $x - \frac{1}{3} = 2$

Solution:

$x = \frac{7}{3}$

Exercise:

Problem: $x - \frac{1}{5} = 4$

Exercise:

Problem: $y - 3.8 = 10$

Solution:

$y = 13.8$

Exercise:

Problem: $y - 7.2 = 5$

Exercise:

Problem: $x - 165 = -420$

Solution:

$x = -255$

Exercise:

Problem: $z - 101 = -314$

Exercise:

Problem: $z + 0.52 = -8.5$

Solution:

$z = -9.02$

Exercise:

Problem: $x + 0.93 = -4.1$

Exercise:

Problem: $q + \frac{3}{4} = \frac{1}{2}$

Solution:

$q = -\frac{1}{4}$

Exercise:

Problem: $p + \frac{1}{3} = \frac{5}{6}$

Exercise:

Problem: $p - \frac{2}{5} = \frac{2}{3}$

Solution:

$p = \frac{16}{15}$

Exercise:

Problem: $y - \frac{3}{4} = \frac{3}{5}$

Solve Equations that Require Simplification

In the following exercises, solve each equation.

Exercise:

Problem: $c + 31 - 10 = 46$

Solution:

$c = 25$

Exercise:

Problem: $m + 16 - 28 = 5$

Exercise:

Problem: $9x + 5 - 8x + 14 = 20$

Solution:

$x = 1$

Exercise:

Problem: $6x + 8 - 5x + 16 = 32$

Exercise:

Problem: $-6x - 11 + 7x - 5 = -16$

Solution:

$x = 0$

Exercise:

Problem: $-8n - 17 + 9n - 4 = -41$

Exercise:

Problem: $5\left(y - 6\right) - 4y = -6$

Solution:

$y = 8y = 24$

Exercise:

Problem: $9\left(y - 2\right) - 8y = -16$

Exercise:

Problem: $8\left(u + 1.5\right) - 7u = 4.9$

Solution:

$u = -7.1$

Exercise:

Problem: $5\left(w+2.2\right)-4w=9.3$

Exercise:

Problem: $6a-5\left(a-2\right)+9=-11$

Solution:

$a=-30$

Exercise:

Problem: $8c-7\left(c-3\right)+4=-16$

Exercise:

Problem: $6\left(y-2\right)-5y=4\left(y+3\right) -4\left(y-1\right)$

Solution:

$y=28$

Exercise:

Problem: $9\left(x-1\right)-8x=-3\left(x+5\right) +3\left(x-5\right)$

Exercise:

Problem: $3(5n-1)-14n+9 =10\left(n-4\right)-6n-4\left(n+1\right)$

Solution:

$n=-50$

Exercise:

Problem: $$\begin{array}{l}2(8m+3)-15m-4\\ =9\left(m+6\right)-2\left(m-1\right)-7m\end{array}$$

Exercise:

Problem: $-\left(j+2\right)+2j-1=5$

Solution:

$j=8$

Exercise:

Problem: $-\left(k+7\right)+2k+8=7$

Exercise:

Problem: $-\left(\frac{1}{4}a-\frac{3}{4}\right)+\frac{5}{4}a=-2$

Solution:

$a=-\frac{11}{4}$

Exercise:

Problem: $-\left(\frac{2}{3}d-\frac{1}{3}\right)+\frac{5}{3}d=-4$

Exercise:

Problem: $$\begin{array}{l}8(4x+5)-5(6x)-x\\ =53-6(x+1)+3(2x+2)\end{array}$$

Solution:

$x = 13$

Exercise:

Problem: $$6(9y-1)-10(5y)-3y = 22-4\left(2y-12\right)+8\left(y-6\right)$$

Translate to an Equation and Solve

In the following exercises, translate to an equation and then solve it.

Exercise:

Problem: Nine more than x is equal to 52.

Solution:

$x+9=52; x=43$

Exercise:

Problem: The sum of x and -15 is 23.

Exercise:

Problem: Ten less than m is -14.

Solution:

$m-10=-14; m=-4$

Exercise:

Problem: Three less than y is -19.

Exercise:

Problem: The sum of y and -30 is 40.

Solution:

$y+(-30)=40; y=70$

Exercise:

Problem: Twelve more than p is equal to 67.

Exercise:

Problem: The difference of $9x$ and $8x$ is 107.

Solution:

$9x-8x=107; 107$

Exercise:

Problem: The difference of $5c$ and $4c$ is 602.

Exercise:

Problem: The difference of n and $\frac{1}{6}$ is $\frac{1}{2}$.

Solution:

$n-\frac{1}{6}=\frac{1}{2}; \frac{2}{3}$

Exercise:

Problem: The difference of f and $\frac{1}{3}$ is $\frac{1}{12}$.

Exercise:

Problem: The sum of $-4n$ and $5n$ is -82.

Solution:

$-4n + 5n = -82; -82$

Exercise:

Problem: The sum of $-9m$ and $10m$ is -95.

Translate and Solve Applications

In the following exercises, translate into an equation and solve.

Exercise:

Problem:

Distance Avril rode her bike a total of 18 miles, from home to the library and then to the beach. The distance from Avril’s house to the library is 7 miles. What is the distance from the library to the beach?

Solution:

11 miles

Exercise:

Problem:

Reading Jeff read a total of 54 pages in his History and Sociology textbooks. He read 41 pages in his History textbook. How many pages did he read in his Sociology textbook?

Exercise:

Problem:

Age Eva’s daughter is 15 years younger than her son. Eva’s son is 22 years old. How old is her daughter?

Solution:

7 years old

Exercise:

Problem:

Age Pablo's father is 3 years older than his mother. Pablo's mother is 42 years old. How old is his father?

Exercise:

Problem:

Groceries For a family birthday dinner, Celeste bought a turkey that weighed 5 pounds less than the one she bought for Thanksgiving. The birthday turkey weighed 16 pounds. How much did the Thanksgiving turkey weigh?

Solution:

21 pounds

Exercise:

Problem:

Weight Allie weighs 8 pounds less than her twin sister Lorrie. Allie weighs 124 pounds. How much does Lorrie weigh?

Exercise:

Problem:

Health Connor's temperature was 0.7 degrees higher this morning than it had been last night. His temperature this morning was 101.2 degrees. What was his temperature last night?

Solution:

100.5 degrees

Exercise:

Problem:

Health The nurse reported that Tricia’s daughter had gained 4.2 pounds since her last checkup and now weighs 31.6 pounds. How much did Tricia’s daughter weigh at her last checkup?

Exercise:

Problem:

Salary Ron’s paycheck this week was \$17.43 less than his paycheck last week. His paycheck this week was \$103.76. How much was Ron’s paycheck last week?

Solution:

\$121.19

Exercise:

Problem:

Textbooks Melissa’s math book cost \$22.85 less than her art book cost. Her math book cost \$93.75. How much did her art book cost?

Everyday Math

Exercise:

Problem:

Construction Miguel wants to drill a hole for a $\frac{5}{8}$ inch screw. The hole should be $\frac{1}{12}$ inch smaller than the screw. Let d equal the size of the hole he should drill. Solve the equation $d - \frac{1}{12} = \frac{5}{8}$ to see what size the hole should be.

Solution:

$d = \frac{17}{24}$ inch

Exercise:

Problem:

Baking Kelsey needs $\frac{2}{3}$ cup of sugar for the cookie recipe she wants to make. She only has $\frac{3}{8}$ cup of sugar and will borrow the rest from her neighbor. Let s equal the amount of sugar she will borrow. Solve the equation $\frac{3}{8} + s = \frac{2}{3}$ to find the amount of sugar she should ask to borrow.

Writing Exercises

Exercise:

Problem:

Is -8 a solution to the equation $3x = 16 - 5x$? How do you know?

Solution:

No. Justifications will vary.

Exercise:

Problem:

What is the first step in your solution to the equation $10x + 2 = 4x + 26$?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
verify a solution of an equation.			
solve equations using the subtraction and addition properties of equality.			
solve equations that require simplification.			
translate to an equation and solve.			
translate and solve applications.			

ⓑ If most of your checks were:

…confidently. Congratulations! You have achieved your goals in this section! Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific!

…with some help. This must be addressed quickly as topics you do not master become potholes in your road to success. Math is sequential - every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is critical and you must not ignore it. You need to get help immediately or you will quickly be overwhelmed. See your instructor as soon as possible to discuss your situation. Together you can come up with a plan to get you the help you need.

Glossary

solution of an equation
: A solution of an equation is a value of a variable that makes a true statement when substituted into the equation.

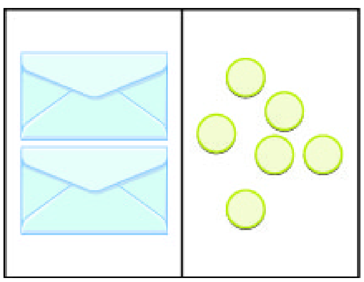
Solve Equations using the Division and Multiplication Properties of Equality: ASE
By the end of this section, you will be able to:

- Solve equations using the Division and Multiplication Properties of Equality
- Solve equations that require simplification
- Translate to an equation and solve
- Translate and solve applications

Solve Equations Using the Division and Multiplication Properties of Equality

You may have noticed that all of the equations we have solved so far have been of the form $x + a = b$ or $x - a = b$. We were able to isolate the variable by adding or subtracting the constant term on the side of the equation with the variable. Now we will see how to solve equations that have a variable multiplied by a constant and so will require division to isolate the variable.

Let's look at our puzzle again with the envelopes and counters in [link].



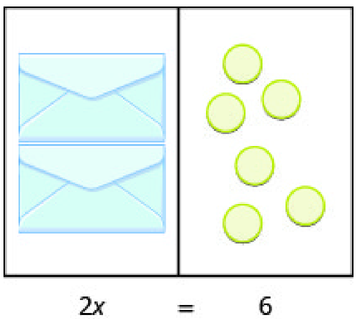
The illustration shows a model of an equation with one variable multiplied by a constant. On the left side of the workspace are two instances of the unknown (envelope), while on the right side of the workspace are six counters.

In the illustration there are two identical envelopes that contain the same number of counters. Remember, the left side of the workspace must equal the right side, but the

counters on the left side are “hidden” in the envelopes. So how many counters are in each envelope?

How do we determine the number? We have to separate the counters on the right side into two groups of the same size to correspond with the two envelopes on the left side. The 6 counters divided into 2 equal groups gives 3 counters in each group (since $6 \div 2 = 3$).

What equation models the situation shown in [link]? There are two envelopes, and each contains x counters. Together, the two envelopes must contain a total of 6 counters.



The illustration shows a model of the equation $2x = 6$.

	$2x = 6$
If we divide both sides of the equation by 2, as we did with the envelopes and counters,	$\frac{2x}{2} = \frac{6}{2}$
we get:	$x = 3$

We found that each envelope contains 3 counters. Does this check? We know $2 \cdot 3 = 6$, so it works! Three counters in each of two envelopes does equal six!

This example leads to the **Division Property of Equality**.

Note:
The Division Property of Equality
For any numbers a, b, and c, and $c \neq 0$,
Equation:

$$\begin{array}{lccc} \text{If} & a & = & b, \\ \text{then} & \frac{a}{c} & = & \frac{b}{c} \end{array}$$

When you divide both sides of an equation by any non-zero number, you still have equality.

The goal in solving an equation is to 'undo' the operation on the variable. In the next example, the variable is multiplied by 5, so we will divide both sides by 5 to 'undo' the multiplication.

Example:
Exercise:

Problem: Solve: $5x = -27$.

Solution:
Solution

To isolate x, "undo" the multiplication by 5.	$5x = -27$
Divide to 'undo' the multiplication.	$\frac{5x}{5} = -\frac{27}{5}$
Simplify.	$x = -\frac{27}{5}$

Check:	$5x = -27$	
Substitute $-\frac{27}{5}$ for x.	$5\left(-\frac{27}{5}\right) \stackrel{?}{=} -27$	
	$-27 = -27 \checkmark$	
		Since this is a true statement, $x = -\frac{27}{5}$ is the solution to $5x = -27$.

Note:
Exercise:

Problem: Solve: $3y = -41$.

Solution:

$y = \frac{-41}{3}$

Note:
Exercise:

Problem: Solve: $4z = -55$.

Solution:

$z = \frac{-55}{4}$

Consider the equation $\frac{x}{4} = 3$. We want to know what number divided by 4 gives 3. So to “undo” the division, we will need to multiply by 4. The **Multiplication Property of**

Equality will allow us to do this. This property says that if we start with two equal quantities and multiply both by the same number, the results are equal.

Note:
The Multiplication Property of Equality
For any numbers a, b, and c,
Equation:

$$\begin{array}{lrcl}\text{If} & a & = & b, \\ \text{then} & ac & = & bc\end{array}$$

If you multiply both sides of an equation by the same number, you still have equality.

Example:
Exercise:

Problem: Solve: $\frac{y}{-7} = -14$.

Solution:
Solution

Here y is divided by -7. We must multiply by -7 to isolate y.

	$\frac{y}{-7} = -14$
Multiply both sides by -7.	$-7\left(\frac{y}{-7}\right) = -7(-14)$
Multiply.	$\frac{-7y}{7} = 98$
Simplify.	

		$y=98$
Check: $\frac{y}{-7}=-14$		
Substitute $y=98$.	$\frac{98}{-7}\stackrel{?}{=}-14$	
Divide.	$-14=-14\checkmark$	

Note:
Exercise:

Problem: Solve: $\frac{a}{-7}=-42$.

Solution:

$a=294$

Note:
Exercise:

Problem: Solve: $\frac{b}{-6}=-24$.

Solution:

$b=144$

Example:
Exercise:

Problem: Solve: $-n=9$.

Solution:
Solution

	$-n = 9$
Remember $-n$ is equivalent to $-1n$.	$-1n = 9$
Divide both sides by -1.	$\frac{-1n}{-1} = \frac{9}{-1}$
Divide.	$n = -9$
Notice that there are two other ways to solve $-n = 9$. We can also solve this equation by multiplying both sides by -1 and also by taking the opposite of both sides.	

Check:	$-n = 9$	
Substitute $n = -9$.	$-(-9) \stackrel{?}{=} 9$	
Simplify.	$9 = 9\checkmark$	

Note:
Exercise:

Problem: Solve: $-k = 8$.

Solution:

$k = -8$

Note:
Exercise:

Problem: Solve: $-g = 3$.

Solution:

$g = -3$

Example:
Exercise:

Problem: Solve: $\frac{3}{4}x = 12$.

Solution:
Solution

Since the product of a number and its reciprocal is 1, our strategy will be to isolate x by multiplying by the reciprocal of $\frac{3}{4}$.

	$\frac{3}{4}x = 12$
Multiply by the reciprocal of $\frac{3}{4}$.	$\frac{4}{3} \cdot \frac{3}{4}x = \frac{4}{3} \cdot 12$
Reciprocals multiply to 1.	$1x = \frac{4}{3} \cdot \frac{12}{1}$

Multiply.		$x = 16$
Notice that we could have divided both sides of the equation $\frac{3}{4}x = 12$ by $\frac{3}{4}$ to isolate x. While this would work, most people would find multiplying by the reciprocal easier.		
Check:	$\frac{3}{4}x = 12$	
Substitute $x = 16$.	$\frac{3}{4} \cdot 16 \stackrel{?}{=} 12$	
	$12 = 12 \checkmark$	

Note:
Exercise:

Problem: Solve: $\frac{2}{5}n = 14$.

Solution:

$n = 35$

Note:
Exercise:

Problem: Solve: $\frac{5}{6}y = 15$.

Solution:

$y = 18$

In the next example, all the variable terms are on the right side of the equation. As always, our goal in solving the equation is to isolate the variable.

Example:
Exercise:

Problem: Solve: $\frac{8}{15} = -\frac{4}{5}x$.

Solution:
Solution

		$\frac{8}{15} = -\frac{4}{5}x$
Multiply by the reciprocal of $-\frac{4}{5}$.		$\left(-\frac{5}{4}\right)\left(\frac{8}{15}\right) = \left(-\frac{5}{4}\right)\left(-\frac{4}{5}x\right)$
Reciprocals multiply to 1.		$-\frac{\cancel{5} \cdot \cancel{4} \cdot 2}{\cancel{4} \cdot 3 \cdot \cancel{5}} = 1x$
Multiply.		$-\frac{2}{3} = x$
Check:	$\frac{8}{15} = -\frac{4}{5}x$	
Let $x = -\frac{2}{3}$.	$\frac{8}{15} = -\frac{4}{5}\left(-\frac{2}{3}\right)$	

	$\frac{8}{15} = \frac{8}{15}$ ✓	

Note:
Exercise:

Problem: Solve: $\frac{9}{25} = -\frac{4}{5}z$.

Solution:

$z = -\frac{9}{5}$

Note:
Exercise:

Problem: Solve: $\frac{5}{6} = -\frac{8}{3}r$.

Solution:

$r = -\frac{5}{16}$

Solve Equations That Require Simplification

Many equations start out more complicated than the ones we have been working with.

With these more complicated equations the first step is to simplify both sides of the equation as much as possible. This usually involves combining like terms or using the distributive property.

Example:
Exercise:

Problem: Solve: $14 - 23 = 12y - 4y - 5y$.

Solution:
Solution

Begin by simplifying each side of the equation.

		$14 - 23 \stackrel{?}{=} -36 + 12 + 15$
Simplify each side.		$-9 = -9$ ✓
Divide both sides by 3 to isolate y.		$14 - 23 = 12y - 4y - 5y$
Divide.		$-9 = 3y$
Check:	$14 - 23 = 12y - 4y - 5y$	
Substitute $y = -3$.	$14 - 23 \stackrel{?}{=} 12(-3) - 4(-3) - 5(-3)$	
	$14 - 23 \stackrel{?}{=} -36 + 12 + 15$	
	$-9 = -9$ ✓	

Note:
Exercise:

Problem: Solve: $18 - 27 = 15c - 9c - 3c$.

Solution:

$c = -3$

Note:
Exercise:

Problem: Solve:$18 - 22 = 12x - x - 4x$.

Solution:

$x = -\frac{4}{7}$

Example:
Exercise:

Problem: Solve: $-4(a - 3) - 7 = 25$.

Solution:
Solution

Here we will simplify each side of the equation by using the distributive property first.

	$-4(a - 3) - 7 = 25$
Distribute.	$-4a + 12 - 7 = 25$
Simplify.	$-4a + 5 = 25$

Simplify.		$-4a = 20$
Divide both sides by -4 to isolate a.		$\frac{-4a}{-4} = \frac{20}{-4}$
Divide.		$a = -5$
Check:	$-4(a - 3) - 7 = 25$	
Substitute $a = -5$.	$-4(-5 - 3) - 7 \stackrel{?}{=} 25$	
	$-4(-8) - 7 \stackrel{?}{=} 25$	
	$32 - 7 \stackrel{?}{=} 25$	
	$25 = 25$ ✓	

Note:
Exercise:

Problem: Solve: $-4(q - 2) - 8 = 24$.

Solution:

$q = -6$

Note:
Exercise:

Problem: Solve: $-6(r-2)-12=30$.

Solution:

$r=-5$

Now we have covered all four properties of equality—subtraction, addition, division, and multiplication. We'll list them all together here for easy reference.

Note:
Properties of Equality
Equation:

Subtraction Property of Equality
For any real numbers a, b, and c,

$$\begin{array}{lrcl} \text{if} & a & = & b, \\ \text{then} & a-c & = & b-c. \end{array}$$

Addition Property of Equality
For any real numbers a, b, and c,

$$\begin{array}{lrcl} \text{if} & a & = & b, \\ \text{then} & a+c & = & b+c. \end{array}$$

Division Property of Equality
For any numbers a, b, and c, and $c \neq 0$,

$$\begin{array}{lrcl} \text{if} & a & = & b, \\ \text{then} & \frac{a}{c} & = & \frac{b}{c}. \end{array}$$

Multiplication Property of Equality
For any numbers a, b, and c,

$$\begin{array}{lrcl} \text{if} & a & = & b, \\ \text{then} & ac & = & bc. \end{array}$$

When you add, subtract, multiply, or divide the same quantity from both sides of an equation, you still have equality.

Translate to an Equation and Solve

In the next few examples, we will translate sentences into equations and then solve the equations. You might want to review the translation table in the previous chapter.

Example:
Exercise:

Problem: Translate and solve: The number 143 is the product of -11 and y.

Solution:
Solution

Begin by translating the sentence into an equation.

Translate.	$\underbrace{\text{The number 143}}_{143}\ \underbrace{\text{is}}_{=}\ \underbrace{\text{the product of }-11\text{ and y.}}_{-11y}$
Divide by -11.	$\frac{143}{-11} = \frac{-11y}{-11}$
Simplify.	$-13 = y$
Check: $\begin{aligned} 143 &= -11y \\ 143 &\overset{?}{=} -11(-13) \\ 143 &= 143\checkmark \end{aligned}$	

Note:
Exercise:

Problem: Translate and solve: The number 132 is the product of -12 and y.

Solution:

$132 = -12y; y = -11$

Note:
Exercise:

Problem: Translate and solve: The number 117 is the product of -13 and z.

Solution:

$117=-13z; z=-9$

Example:
Exercise:

Problem: Translate and solve: n divided by 8 is -32.

Solution:
Solution

Begin by translating the sentence into an equation. Translate.		$\underbrace{n \text{ divided by } 8}_{\frac{n}{8}}\ \underbrace{\text{is}}_{=}\ \underbrace{-32.}_{-32}$
Multiple both sides by 8.		$8 \cdot \frac{n}{8} = 8(-32)$
Simplify.		$n=-256$
Check:	Is n divided by 8 equal to -32?	
Let $n=-256.$	Is -256 divided by 8 equal to -32?	
Translate.	$\frac{-256}{8} \stackrel{?}{=} -32$	

Simplify.	$-32 = -32\checkmark$	

Note:
Exercise:

Problem: Translate and solve: n divided by 7 is equal to -21.

Solution:

$\frac{n}{7} = -21; n = -147$

Note:
Exercise:

Problem: Translate and solve: n divided by 8 is equal to -56.

Solution:

$\frac{n}{8} = -56; n = -448$

Example:
Exercise:

Problem: Translate and solve: The quotient of y and -4 is 68.

Solution:
Solution

Begin by translating the sentence into an equation.

Translate.	

		The quotient of y and -4 is 68. $\frac{y}{-4} = 68$
Multiply both sides by -4.		$-4\left(\frac{y}{-4}\right) = -4(68)$
Simplify.		$y = -272$
Check:	Is the quotient of y and -4 equal to 68?	
Let $y = -272$.	Is the quotient of -272 and -4 equal to 68?	
Translate.	$\frac{-272}{-4} \stackrel{?}{=} 68$	
Simplify.	$68 = 68\checkmark$	

Note:
Exercise:

Problem: Translate and solve: The quotient of q and -8 is 72.

Solution:

$\frac{q}{-8} = 72; q = -576$

Note:
Exercise:

Problem: Translate and solve: The quotient of p and -9 is 81.

Solution:

$\frac{p}{-9}=81; p=-729$

Example:
Exercise:

Problem: Translate and solve: Three-fourths of p is 18.

Solution:
Solution

Begin by translating the sentence into an equation. Remember, “of” translates into multiplication.

Translate.		$\underbrace{\text{Three-fourths of } p}_{\frac{3}{4}p}\ \underbrace{\text{is}}_{=}\ \underbrace{18.}_{18}$
Multiply both sides by $\frac{4}{3}$.		$\frac{4}{3}\cdot\frac{3}{4}p=\frac{4}{3}\cdot 18$
Simplify.		$p=24$
Check:	Is three-fourths of p equal to 18?	
Let $p=24$.	Is three-fourths of 24 equal to 18?	
Translate.	$\frac{3}{4}\cdot 24\stackrel{?}{=}18$	
Simplify.	$18=18✓$	

Note:
Exercise:

Problem: Translate and solve: Two-fifths of f is 16.

Solution:

$\frac{2}{5}f = 16; f = 40$

Note:
Exercise:

Problem: Translate and solve: Three-fourths of f is 21.

Solution:

$\frac{3}{4}f = 21; f = 28$

Example:
Exercise:

Problem: Translate and solve: The sum of three-eighths and x is one-half.

Solution:
Solution

Begin by translating the sentence into an equation.

Translate.	$\underbrace{\text{The sum of three – eighths and } x}_{\frac{3}{8} + x} \underbrace{\text{is}}_{=} \underbrace{\frac{1}{2}}_{\frac{1}{2}}$

Subtract $\frac{3}{8}$ from each side.			$\frac{3}{8}-\frac{3}{8}+x=\frac{1}{2}-\frac{3}{8}$
Simplify and rewrite fractions with common denominators.			$x=\frac{4}{8}-\frac{3}{8}$
Simplify.			$x=\frac{1}{8}$
Check:		Is the sum of three-eighths and x equal to one-half?	
Let $x=\frac{1}{8}$.		Is the sum of three-eighths and one-eighth equal to one-half?	
Translate.		$\frac{3}{8}+\frac{1}{8}\stackrel{?}{=}\frac{1}{2}$	
Simplify.		$\frac{4}{8}\stackrel{?}{=}\frac{1}{2}$	
Simplify.		$\frac{1}{2}=\frac{1}{2}\checkmark$	

Note:
Exercise:

Problem: Translate and solve: The sum of five-eighths and *x* is one-fourth.

Solution:

$\frac{5}{8}+x=\frac{1}{4};x=-\frac{3}{8}$

Note:
Exercise:

Problem: Translate and solve: The sum of three-fourths and *x* is five-sixths.

Solution:

$\frac{3}{4} + x = \frac{5}{6}; x = \frac{1}{12}$

Translate and Solve Applications

To solve applications using the Division and Multiplication Properties of Equality, we will follow the same steps we used in the last section. We will restate the problem in just one sentence, assign a variable, and then translate the sentence into an equation to solve.

Example:
Exercise:

Problem:

Denae bought 6 pounds of grapes for $10.74. What was the cost of one pound of grapes?

Solution:
Solution

What are you asked to find?	The cost of 1 pound of grapes
Assign a variable.	Let c = the cost of one pound.
Write a sentence that gives the information to find it.	The cost of 6 pounds is $10.74.
Translate into an equation.	$6c = 10.74$
Solve.	$\frac{6c}{6} = \frac{10.74}{6}$ $c = 1.79$
	The grapes cost $1.79 per

	pound.
Check: If one pound costs \$1.79, do 6 pounds cost #10.74? $$\begin{array}{rcl} 6\left(1.79\right) & \stackrel{?}{=} & 10.74 \\ 10.74 & = & 10.74\checkmark \end{array}$$	

Note:
Exercise:

Problem: Translate and solve:

Arianna bought a 24-pack of water bottles for \$9.36. What was the cost of one water bottle?

Solution:

\$0.39

Note:
Exercise:

Problem: Translate and solve:

At JB’s Bowling Alley, 6 people can play on one lane for \$34.98. What is the cost for each person?

Solution:

\$5.83

Example:
Exercise:

Problem:

Andreas bought a used car for $12,000. Because the car was 4-years old, its price was $\frac{3}{4}$ of the original price, when the car was new. What was the original price of the car?

Solution:
Solution

What are you asked to find?	The original price of the car
Assign a variable.	Let p = the original price.
Write a sentence that gives the information to find it.	$12,000 is $\frac{3}{4}$ of the original price.
Translate into an equation.	$12{,}000 = \frac{3}{4}p$
Solve.	$\frac{4}{3}(12{,}000) = \frac{4}{3} \cdot \frac{3}{4}p$ $16{,}000 = p$
	The original cost of the car was $16,000.
Check: Is $\frac{3}{4}$ of $16,000 equal to $12,000? $\frac{3}{4} \cdot 16{,}000 \stackrel{?}{=} 12{,}000$ $12{,}000 = 12{,}000\checkmark$	

Note:
Exercise:

Problem: Translate and solve:

The annual property tax on the Mehta's house is $1,800, calculated as $\frac{15}{1,000}$ of the assessed value of the house. What is the assessed value of the Mehta's house?

Solution:

$120,000

Note:
Exercise:

Problem: Translate and solve:

Stella planted 14 flats of flowers in $\frac{2}{3}$ of her garden. How many flats of flowers would she need to fill the whole garden?

Solution:

21 flats

Key Concepts

- **The Division Property of Equality—**For any numbers a, b, and c, and $c \neq 0$, if $a = b$, then $\frac{a}{c} = \frac{b}{c}$.
  When you divide both sides of an equation by any non-zero number, you still have equality.
- **The Multiplication Property of Equality—**For any numbers a, b, and c, if $a = b$, then $ac = bc$.
  If you multiply both sides of an equation by the same number, you still have equality.

Practice Makes Perfect

Solve Equations Using the Division and Multiplication Properties of Equality

In the following exercises, solve each equation using the Division and Multiplication Properties of Equality and check the solution.
Exercise:

Problem: $8x = 56$

Solution:

$x = 7$

Exercise:

Problem: $7p = 63$

Exercise:

Problem: $-5c = 55$

Solution:

$c = -11$

Exercise:

Problem: $-9x = -27$

Exercise:

Problem: $-809 = 15y$

Solution:

$y = -\frac{809}{15}$

Exercise:

Problem: $-731 = 19y$

Exercise:

Problem: $-37p = -541$

Solution:

$p = -\frac{541}{37}$

Exercise:

Problem: $-19m = -586$

Exercise:

Problem: $0.25z = 3.25$

Solution:

$z = 13$

Exercise:

Problem: $0.75a = 11.25$

Exercise:

Problem: $-13x = 0$

Solution:

$x = 0$

Exercise:

Problem: $24x = 0$

Exercise:

Problem: $\frac{x}{4} = 35$

Solution:

$x = 140$

Exercise:

Problem: $\frac{z}{2} = 54$

Exercise:

Problem: $-20 = \frac{q}{-5}$

Solution:

$q = 100$

Exercise:

Problem: $\frac{c}{-3} = -12$

Exercise:

Problem: $\frac{y}{9} = -16$

Solution:

$y = -144$

Exercise:

Problem: $\frac{q}{6} = -38$

Exercise:

Problem: $\frac{m}{-12} = 45$

Solution:

$m = -540$

Exercise:

Problem: $-24 = \frac{p}{-20}$

Exercise:

Problem: $-y = 6$

Solution:

$y = -6$

Exercise:

Problem: $-u = 15$

Exercise:

Problem: $-v = -72$

Solution:

$v = 72$

Exercise:

Problem: $-x = -39$

Exercise:

Problem: $\frac{2}{3}y = 48$

Solution:

$y = 72$

Exercise:

Problem: $\frac{3}{5}r = 75$

Exercise:

Problem: $-\frac{5}{8}w = 40$

Solution:

$w = -64$

Exercise:

Problem: $24 = -\frac{3}{4}x$

Exercise:

Problem: $-\frac{2}{5} = \frac{1}{10}a$

Solution:

$a = -4$

Exercise:

Problem: $-\frac{1}{3}q = -\frac{5}{6}$

Exercise:

Problem: $-\frac{7}{10}x = -\frac{14}{3}$

Solution:

$x = \frac{20}{3}$

Exercise:

Problem: $\frac{3}{8}y = -\frac{1}{4}$

Exercise:

Problem: $\frac{7}{12} = -\frac{3}{4}p$

Solution:

$p = -\frac{7}{9}$

Exercise:

Problem: $\frac{11}{18} = -\frac{5}{6}q$

Exercise:

Problem: $-\frac{5}{18} = -\frac{10}{9}u$

Solution:

$u = \frac{1}{4}$

Exercise:

Problem: $-\frac{7}{20} = -\frac{7}{4}v$

Solve Equations That Require Simplification

In the following exercises, solve each equation requiring simplification.

Exercise:

Problem: $100 - 16 = 4p - 10p - p$

Solution:

$p = -12$

Exercise:

Problem: $-18 - 7 = 5t - 9t - 6t$

Exercise:

Problem: $\frac{7}{8}n - \frac{3}{4}n = 9 + 2$

Solution:

$n = 88$

Exercise:

Problem: $\frac{5}{12}q + \frac{1}{2}q = 25 - 3$

Exercise:

Problem: $0.25d + 0.10d = 6 - 0.75$

Solution:

$d = 15$

Exercise:

Problem: $0.05p - 0.01p = 2 + 0.24$

Exercise:

Problem: $-10(q - 4) - 57 = 93$

Solution:

$q = -11$

Exercise:

Problem: $-12(d - 5) - 29 = 43$

Exercise:

Problem: $-10(x + 4) - 19 = 85$

Solution:

$x = -\frac{72}{5}$

Exercise:

Problem: $-15(z+9)-11=75$

Mixed Practice

In the following exercises, solve each equation.

Exercise:

Problem: $\frac{9}{10}x=90$

Solution:

$x=100$

Exercise:

Problem: $\frac{5}{12}y=60$

Exercise:

Problem: $y+46=55$

Solution:

$y=9$

Exercise:

Problem: $x+33=41$

Exercise:

Problem: $\frac{w}{-2}=99$

Solution:

$w=-198$

Exercise:

Problem: $\frac{s}{-3}=-60$

Exercise:

Problem: $27=6a$

Solution:

$a = \frac{9}{2}$

Exercise:

Problem: $-a = 7$

Exercise:

Problem: $-x = 2$

Solution:

$x = -2$

Exercise:

Problem: $z - 16 = -59$

Exercise:

Problem: $m - 41 = -14$

Solution:

$m = 27$

Exercise:

Problem: $0.04r = 52.60$

Exercise:

Problem: $63.90 = 0.03p$

Solution:

$p = 2130$

Exercise:

Problem: $-15x = -120$

Exercise:

Problem: $84 = -12z$

Solution:

$y = -7$

Exercise:

Problem: $19.36 = x - 0.2x$

Exercise:

Problem: $c - 0.3c = 35.70$

Solution:

$c = 51$

Exercise:

Problem: $-y = -9$

Exercise:

Problem: $-x = -8$

Solution:

$x = 8$

Translate to an Equation and Solve

In the following exercises, translate to an equation and then solve.

Exercise:

Problem: 187 is the product of -17 and *m*.

Exercise:

Problem: 133 is the product of -19 and *n*.

Solution:

$133 = -19n; n = -7$

Exercise:

Problem: -184 is the product of 23 and *p*.

Exercise:

Problem: -152 is the product of 8 and q.

Solution:

$-152 = 8q; q = -19$

Exercise:

Problem: u divided by 7 is equal to -49.

Exercise:

Problem: r divided by 12 is equal to -48.

Solution:

$\frac{r}{12} = -48; r = -576$

Exercise:

Problem: h divided by -13 is equal to -65.

Exercise:

Problem: j divided by -20 is equal to -80.

Solution:

$\frac{j}{-20} = -80; j = 1{,}600$

Exercise:

Problem: The quotient c and -19 is 38.

Exercise:

Problem: The quotient of b and -6 is 18.

Solution:

$\frac{b}{-6} = 18; b = -108$

Exercise:

Problem: The quotient of h and 26 is -52.

Exercise:

Problem: The quotient k and 22 is -66.

Solution:

$\frac{k}{22} = -66; k = -1{,}452$

Exercise:

Problem: Five-sixths of *y* is 15.

Exercise:

Problem: Three-tenths of *x* is 15.

Solution:

$\frac{3}{10}x = 15; x = 50$

Exercise:

Problem: Four-thirds of *w* is 36.

Exercise:

Problem: Five-halves of *v* is 50.

Solution:

$\frac{5}{2}v = 50; v = 20$

Exercise:

Problem: The sum of nine-tenths and *g* is two-thirds.

Exercise:

Problem: The sum of two-fifths and *f* is one-half.

Solution:

$\frac{2}{5} + f = \frac{1}{2}; f = \frac{1}{10}$

Exercise:

Problem: The difference of p and one-sixth is two-thirds.

Exercise:

Problem: The difference of q and one-eighth is three-fourths.

Solution:

$q - \frac{1}{8} = \frac{3}{4}; q = \frac{7}{8}$

Translate and Solve Applications

In the following exercises, translate into an equation and solve.

Exercise:

Problem:

Kindergarten Connie's kindergarten class has 24 children. She wants them to get into 4 equal groups. How many children will she put in each group?

Exercise:

Problem:

Balloons Ramona bought 18 balloons for a party. She wants to make 3 equal bunches. How many balloons did she use in each bunch?

Solution:

6 balloons

Exercise:

Problem:

Tickets Mollie paid $36.25 for 5 movie tickets. What was the price of each ticket?

Exercise:

Problem:

Shopping Serena paid $12.96 for a pack of 12 pairs of sport socks. What was the price of pair of sport socks?

Solution:

$1.08

Exercise:

Problem:

Sewing Nancy used 14 yards of fabric to make flags for one-third of the drill team. How much fabric, would Nancy need to make flags for the whole team?

Exercise:

Problem:

MPG John's SUV gets 18 miles per gallon (mpg). This is half as many mpg as his wife's hybrid car. How many miles per gallon does the hybrid car get?

Solution:

36 mpg

Exercise:

Problem:

Height Aiden is 27 inches tall. He is $\frac{3}{8}$ as tall as his father. How tall is his father?

Exercise:

Problem:

Real estate Bea earned $11,700 commission for selling a house, calculated as $\frac{6}{100}$ of the selling price. What was the selling price of the house?

Solution:

$195,000

Everyday Math

Exercise:

Problem:

Commission Every week Perry gets paid $150 plus 12% of his total sales amount. Solve the equation $840 = 150 + 0.12(a - 1250)$ for a, to find the total amount Perry must sell in order to be paid $840 one week.

Exercise:

Problem:

Stamps Travis bought \$9.45 worth of 49-cent stamps and 21-cent stamps. The number of 21-cent stamps was 5 less than the number of 49-cent stamps. Solve the equation $0.49s + 0.21(s - 5) = 9.45$ for s, to find the number of 49-cent stamps Travis bought.

Solution:

15 49-cent stamps

Writing Exercises

Exercise:

Problem:

Frida started to solve the equation $-3x = 36$ by adding 3 to both sides. Explain why Frida's method will not solve the equation.

Exercise:

Problem:

Emiliano thinks $x = 40$ is the solution to the equation $\frac{1}{2}x = 80$. Explain why he is wrong.

Solution:

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve equations using the Division and Multiplication Properties of equality.			
solve equations that require simplification.			
translate to an equation and solve.			
translate and solve applications.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Solve Equations with Variables and Constants on Both Sides: ASE
By the end of this section, you will be able to:

- Solve an equation with constants on both sides
- Solve an equation with variables on both sides
- Solve an equation with variables and constants on both sides

Solve Equations with Constants on Both Sides

In all the equations we have solved so far, all the variable terms were on only one side of the equation with the constants on the other side. This does not happen all the time—so now we will learn to solve equations in which the variable terms, or constant terms, or both are on both sides of the equation.

Our strategy will involve choosing one side of the equation to be the “variable side”, and the other side of the equation to be the “constant side.” Then, we will use the Subtraction and Addition Properties of Equality to get all the variable terms together on one side of the equation and the constant terms together on the other side.

By doing this, we will transform the equation that began with variables and constants on both sides into the form $ax = b$. We already know how to solve equations of this form by using the Division or Multiplication Properties of Equality.

Example:
Exercise:

Problem: Solve: $7x + 8 = -13$.

Solution:
Solution

In this equation, the variable is found only on the left side. It makes sense to call the left side the “variable” side. Therefore, the right side will be the “constant” side. We will write the labels above the equation to help us remember what goes where.

$$\overset{\text{variable}}{7x+8} = \overset{\text{constant}}{-13}$$

Since the left side is the “x”, or variable side, the 8 is out of place. We must “undo” adding 8 by subtracting 8, and to keep the equality we must subtract 8 from both sides.

	$\overset{\text{variable}}{7x+8} = \overset{\text{constant}}{-14\ 3}$
Use the Subtraction Property of Equality.	$7x+8-8=-13-8$
Simplify.	$7x=-21$
Now all the variables are on the left and the constant on the right. The equation looks like those you learned to solve earlier.	
Use the Division Property of Equality.	$\frac{7x}{7}=\frac{-21}{7}$
Simplify.	$x=-3$

Check:	$7x + 8 = -13$	
Let $x = -3$.	$7(-3) + 8 \stackrel{?}{=} -13$	
	$-21 + 8 \stackrel{?}{=} -13$ $-13 = -13$ ✓	

Note:
Exercise:

Problem: Solve: $3x + 4 = -8$.

Solution:

$x = -4$

Note:
Exercise:

Problem: Solve: $5a + 3 = -37$.

Solution:

$a = -8$

Example:
Exercise:

Problem: Solve: $8y - 9 = 31$.

Solution:
Solution

Notice, the variable is only on the left side of the equation, so we will call this side the "variable" side, and the right side will be the "constant" side. Since the left side is the "variable" side, the 9 is out of place. It is subtracted from the $8y$, so to "undo" subtraction, add 9 to both sides. Remember, whatever you do to the left, you must do to the right.

	$\underset{\text{variable}}{8y - 9} = \underset{\text{constant}}{31}$
Add 9 to both sides.	$8y - 9 + 9 = 31 + 9$
Simplify.	$8y = 40$
	The variables are now on one side and the constants on the other. We continue from here as we did earlier.
Divide both sides by	

8.		$\frac{8y}{8} = \frac{40}{8}$
Simplify.		$y = 5$
Check:	$8y - 9 = 31$	
Let $y = 5$.	$8 \cdot 5 - 9 \stackrel{?}{=} 31$	
	$40 - 9 \stackrel{?}{=} 31$	
	$31 = 31$ ✓	

Note:
Exercise:

Problem: Solve: $5y - 9 = 16$.

Solution:

$y = 5$

Note:
Exercise:

Problem: Solve: $3m - 8 = 19$.

Solution:

$m = 9$

Solve Equations with Variables on Both Sides

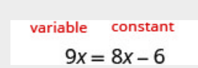
What if there are variables on both sides of the equation? For equations like this, begin as we did above—choose a “variable” side and a “constant” side, and then use the subtraction and addition properties of equality to collect all variables on one side and all constants on the other side.

Example:
Exercise:

Problem: Solve: $9x = 8x - 6$.

Solution:
Solution

Here the variable is on both sides, but the constants only appear on the right side, so let’s make the right side the “constant” side. Then the left side will be the “variable” side.



We don’t want any x’s on the right, so subtract the $8x$ from both sides.		$9x - 8x = 8x - 8x - 6$
Simplify.		$x = -6$
We succeeded in getting the variables on one side and the constants on the other, and have obtained the solution.		
Check:	$9x = 8x - 6$	
Let $x = -6$.	$9(-6) \stackrel{?}{=} 8(-6) - 6$	
	$-54 \stackrel{?}{=} -48 - 6$	
	$-54 = -54$ ✓	

Note:
Exercise:

Problem: Solve: $6n = 5n - 10$.

Solution:

$n = -10$

Note:
Exercise:

Problem: Solve: $-6c = -7c - 1$.

Solution:

$c = -1$

Example:
Exercise:

Problem: Solve: $5y - 9 = 8y$.

Solution:
Solution

The only constant is on the left and the y's are on both sides. Let's leave the constant on the left and get the variables to the right.

	constant variable $5y - 9 = 8y$
Subtract $5y$ from both sides.	$5y - 5y - 9 = 8y - 5y$
Simplify.	$-9 = 3y$

We have the y's on the right and the constants on the left. Divide both sides by 3.		$\frac{-9}{3} = \frac{3y}{3}$
Simplify.		$-3 = y$
Check:	$5y - 9 = 8y$	
Let $y = -3$.	$5(-3) - 9 \stackrel{?}{=} 8(-3)$	
	$-15 - 9 \stackrel{?}{=} -24$	
	$-24 = -24$ ✓	

Note:
Exercise:

Problem: Solve: $3p - 14 = 5p$.

Solution:

$p = -7$

Note:
Exercise:

Problem: Solve: $8m + 9 = 5m$.

Solution:

$m = -3$

Example:
Exercise:

Problem: Solve: $12x = -x + 26$.

Solution:
Solution

The only constant is on the right, so let the left side be the “variable” side.

	$\overset{\text{variable}}{12x} = \overset{\text{constant}}{-x + 26}$
Remove the $-x$ from the right side by adding x to both sides.	$12x + x = -x + x + 26$
Simplify.	$13x = 26$

All the x’s are on the left and the constants are on the right. Divide both sides by 13.	$\frac{13x}{13}=\frac{26}{13}$
Simplify.	$x=2$

Note:
Exercise:

Problem: Solve: $12j=-4j+32$.

Solution:

$j=2$

Note:
Exercise:

Problem: Solve: $8h=-4h+12$.

Solution:

$h=1$

Solve Equations with Variables and Constants on Both Sides

The next example will be the first to have variables and constants on both sides of the equation. It may take several steps to solve this equation, so we

need a clear and organized strategy.

Example:
How to Solve Equations with Variables and Constants on Both Sides
Exercise:

Problem: Solve: $7x + 5 = 6x + 2$.

Solution:
Solution

Step 1. Choose which side will be the "variable" side—the other side will be the "constant" side.	The variable terms are $7x$ and $6x$. Since 7 is greater than 6, we will make the left side the "x" side. The right side will be the "constant" side.	$\underbrace{7x + 5}_{\text{variable}} = \underbrace{6x + 2}_{\text{constant}}$
Step 2. Collect the variable terms to the "variable" side of the equation, using the addition or subtraction property of equality.	With the right side as the "constant" side, the $6x$ is out of place, so subtract $6x$ from both sides. Combine like terms. Now, the variable is only on the left side!	$7x - 6x + 5 = 6x - 6x + 2$ $x + 5 = 2$
Step 3. Collect all the constants to the other side of the equation, using the addition or subtraction property of equality.	The right side is the "constant" side, so the 5 is out of place. Subtract 5 from both sides. Simplify.	$x + 5 - 5 = 2 - 5$ $x = -3$
Step 4. Make the coefficient of the variable equal 1, using the multiplication or division property of equality.	The coefficient of x is one. The equation is solved.	

Step 5. Check.	Let $x = -3$	**Check:** $7x + 6 = 6x + 2$
	Simplify.	$(-3) + 5 = 6(-3) + 2$
	Add.	$-21 + 5 = -18 + 2$
		$-16 = -16$ ✓

Note:
Exercise:

Problem: Solve: $12x + 8 = 6x + 2$.

Solution:

$x = -1$

Note:
Exercise:

Problem: Solve: $9y + 4 = 7y + 12$.

Solution:

$y = 4$

We’ll list the steps below so you can easily refer to them. But we’ll call this the ‘Beginning Strategy’ because we’ll be adding some steps later in this chapter.

Note:
Beginning Strategy for Solving Equations with Variables and Constants on Both Sides of the Equation.

Choose which side will be the “variable” side—the other side will be the “constant” side.
Collect the variable terms to the “variable” side of the equation, using the Addition or Subtraction Property of Equality.
Collect all the constants to the other side of the equation, using the Addition or Subtraction Property of Equality.
Make the coefficient of the variable equal 1, using the Multiplication or Division Property of Equality.
Check the solution by substituting it into the original equation.

In Step 1, a helpful approach is to make the “variable” side the side that has the variable with the larger coefficient. This usually makes the arithmetic easier.

Example:
Exercise:

Problem: Solve: $8n - 4 = -2n + 6$.

Solution:
Solution

In the first step, choose the variable side by comparing the coefficients of the variables on each side.

Since $8 > -2$, make the left side the “variable” side.		$\underset{\text{variable}}{8n-4} = \underset{\text{constant}}{-2n+6}$
We don’t want variable terms on the right side—add $2n$ to both sides to leave only constants on the right.		$8n + 2n - 4 = -2n + 2n + 6$
Combine like terms.		$10n - 4 = 6$
We don’t want any constants on the left side, so add 4 to both sides.		$10n - 4 + 4 = 6 + 4$
Simplify.		$10n = 10$
The variable term is on the left and the constant term is on the right. To get the coefficient of n to be one, divide both sides by 10.		$\frac{10n}{10} = \frac{10}{10}$
Simplify.		$n = 1$
Check:	$8n - 4 = -2n + 6$	
Let $n = 1$.	$8 \cdot 1 - 4 \stackrel{?}{=} -2 \cdot 1 + 6$	
	$8 - 4 \stackrel{?}{=} -2 + 6$	
	$4 = 4 ✓$	

Note:
Exercise:

Problem: Solve: $8q - 5 = -4q + 7$.

Solution:

$q = 1$

Note:
Exercise:

Problem: Solve: $7n - 3 = n + 3$.

Solution:

$n = 1$

Example:
Exercise:

Problem: Solve: $7a - 3 = 13a + 7$.

Solution:
Solution

In the first step, choose the variable side by comparing the coefficients of the variables on each side.

Since $13 > 7$, make the right side the “variable” side and the left side the “constant” side.

		constant variable $7a - 3 = 13a + 7$
Subtract $7a$ from both sides to remove the variable term from the left.		$7a - 7a - 3 = 13a - 7a + 7$
Combine like terms.		$-3 = 6a + 7$
Subtract 7 from both sides to remove the constant from the right.		$-3 - 7 = 6a + 7 - 7$
Simplify.		$-10 = 6a$
Divide both sides by 6 to make 1 the coefficient of a.		$\frac{-10}{6} = \frac{6a}{6}$
Simplify.		$-\frac{5}{3} = a$
Check:	$7a - 3 = 13a + 7$	
Let $a = -\frac{5}{3}$.	$7\left(-\frac{5}{3}\right) - 3 \stackrel{?}{=} 13\left(-\frac{5}{3}\right) + 7$	

	$-\frac{35}{3}-\frac{9}{3}\stackrel{?}{=}-\frac{65}{3}+\frac{21}{3}$	
	$-\frac{54}{3}=-\frac{54}{3}$ ✓	

Note:
Exercise:

Problem: Solve: $2a-2=6a+18$.

Solution:

$a=-5$

Note:
Exercise:

Problem: Solve: $4k-1=7k+17$.

Solution:

$k=-6$

In the last example, we could have made the left side the “variable” side, but it would have led to a negative coefficient on the variable term. (Try it!) While we could work with the negative, there is less chance of errors when working with positives. The strategy outlined above helps avoid the negatives!

To solve an equation with fractions, we just follow the steps of our strategy to get the solution!

Example:
Exercise:

Problem: Solve: $\frac{5}{4}x + 6 = \frac{1}{4}x - 2$.

Solution:
Solution

Since $\frac{5}{4} > \frac{1}{4}$, make the left side the “variable” side and the right side the “constant” side.

	$\overset{\text{variable}}{\frac{5}{4}x + 6} = \overset{\text{constant}}{\frac{1}{4}x - 2}$
Subtract $\frac{1}{4}x$ from both sides.	$\frac{5}{4}x - \frac{1}{4}x + 6 = \frac{1}{4}x - \frac{1}{4}x - 2$
Combine like terms.	$x + 6 = -2$
Subtract 6 from both sides.	$x + 6 - 6 = -2 - 6$

Simplify.	$x = -8$

Check: $\frac{5}{4}x + 6 = \frac{1}{4}x - 2$

Let $x = -8$.

$$\begin{aligned} \frac{5}{4}x + 6 &= \frac{1}{4}x - 2 \\ \frac{5}{4}(-8) + 6 &\stackrel{?}{=} \frac{1}{4}(-8) - 2 \\ -10 + 6 &\stackrel{?}{=} -2 - 2 \\ -4 &= -4\checkmark \end{aligned}$$

Note:
Exercise:

Problem: Solve: $\frac{7}{8}x - 12 = -\frac{1}{8}x - 2.$

Solution:

$x = 10$

Note:
Exercise:

Problem: Solve: $\frac{7}{6}y + 11 = \frac{1}{6}y + 8.$

Solution:

$y = -3$

We will use the same strategy to find the solution for an equation with decimals.

Example:
Exercise:

Problem: Solve: $7.8x + 4 = 5.4x - 8$.

Solution:
Solution

Since $7.8 > 5.4$, make the left side the "variable" side and the right side the "constant" side.

	variable side constant side $7.8x + 4 = 5.4x - 8$
Subtract $5.4x$ from both sides.	$7.8x - 5.4x + 4 = 5.4x - 5.4x - 8$
Combine like terms.	$2.4x + 4 = -8$
Subtract 4 from both sides.	$2.4x + 4 - 4 = -8 - 4$
Simplify.	$2.4x = -12$

Use the Division Propery of Equality.		$\dfrac{2.4x}{2.4} = \dfrac{-12}{2.4}$
Simplify.		$x = -5$
Check:	$7.8x + 4 = 5.4x - 5$	
Let $x = -5$.	$7.8(-5) + 4 = 5.4(-5) - 8$	
	$-39 + 4 \stackrel{?}{=} -27 - 8$	
	$-35 = -35 \checkmark$	

Note:
Exercise:

Problem: Solve: $2.8x + 12 = -1.4x - 9$.

Solution:

$x = -5$

Note:
Exercise:

Problem: Solve: $3.6y + 8 = 1.2y - 4$.

Solution:

$y = -5$

Key Concepts

- **Beginning Strategy for Solving an Equation with Variables and Constants on Both Sides of the Equation**

  Choose which side will be the “variable” side—the other side will be the “constant” side.
  Collect the variable terms to the “variable” side of the equation, using the Addition or Subtraction Property of Equality.
  Collect all the constants to the other side of the equation, using the Addition or Subtraction Property of Equality.
  Make the coefficient of the variable equal 1, using the Multiplication or Division Property of Equality.
  Check the solution by substituting it into the original equation.

Practice Makes Perfect

Solve Equations with Constants on Both Sides

In the following exercises, solve the following equations with constants on both sides.

Exercise:

Problem: $9x - 3 = 60$

Exercise:

Problem: $12x - 8 = 64$

Solution:

$x = 6$

Exercise:

Problem: $14w + 5 = 117$

Exercise:

Problem: $15y + 7 = 97$

Solution:

$y = 6$

Exercise:

Problem: $2a + 8 = -28$

Exercise:

Problem: $3m + 9 = -15$

Solution:

$m = -8$

Exercise:

Problem: $-62 = 8n - 6$

Exercise:

Problem: $-77 = 9b - 5$

Solution:

$b = -8$

Exercise:

Problem: $35 = -13y + 9$

Exercise:

Problem: $60 = -21x - 24$

Solution:

$x = -4$

Exercise:

Problem: $-12p - 9 = 9$

Exercise:

Problem: $-14q - 2 = 16$

Solution:

$q = -\frac{9}{7}$

Solve Equations with Variables on Both Sides

In the following exercises, solve the following equations with variables on both sides.

Exercise:

Problem: $19z = 18z - 7$

Exercise:

Problem: $21k = 20k - 11$

Solution:

$k = -11$

Exercise:

Problem: $9x + 36 = 15x$

Exercise:

Problem: $8x + 27 = 11x$

Solution:

$x = 9$

Exercise:

Problem: $c = -3c - 20$

Exercise:

Problem: $b = -4b - 15$

Solution:

$b = -3$

Exercise:

Problem: $9q = 44 - 2q$

Exercise:

Problem: $5z = 39 - 8z$

Solution:

$z = 3$

Exercise:

Problem: $6y + \frac{1}{2} = 5y$

Exercise:

Problem: $4x + \frac{3}{4} = 3x$

Solution:

$x = -\frac{3}{4}$

Exercise:

Problem: $-18a - 8 = -22a$

Exercise:

Problem: $-11r - 8 = -7r$

Solution:

$r = -2$

Solve Equations with Variables and Constants on Both Sides

In the following exercises, solve the following equations with variables and constants on both sides.

Exercise:

Problem: $8x - 15 = 7x + 3$

Exercise:

Problem: $6x - 17 = 5x + 2$

Solution:

$x = 19$

Exercise:

Problem: $26 + 13d = 14d + 11$

Exercise:

Problem: $21 + 18f = 19f + 14$

Solution:

$f = 7$

Exercise:

Problem: $2p - 1 = 4p - 33$

Exercise:

Problem: $12q - 5 = 9q - 20$

Solution:

$q = -5$

Exercise:

Problem: $4a + 5 = -a - 40$

Exercise:

Problem: $8c + 7 = -3c - 37$

Solution:

$c = -4$

Exercise:

Problem: $5y - 30 = -5y + 30$

Exercise:

Problem: $7x - 17 = -8x + 13$

Solution:

$x = 2$

Exercise:

Problem: $7s + 12 = 5 + 4s$

Exercise:

Problem: $9p + 14 = 6 + 4p$

Solution:

$p = -\frac{8}{5}$

Exercise:

Problem: $2z - 6 = 23 - z$

Exercise:

Problem: $3y - 4 = 12 - y$

Solution:

$y = 4$

Exercise:

Problem: $\frac{5}{3}c - 3 = \frac{2}{3}c - 16$

Exercise:

Problem: $\frac{7}{4}m - 7 = \frac{3}{4}m - 13$

Solution:

$m = -6$

Exercise:

Problem: $8 - \frac{2}{5}q = \frac{3}{5}q + 6$

Exercise:

Problem: $11 - \frac{1}{5}a = \frac{4}{5}a + 4$

Solution:

$a = 7$

Exercise:

Problem: $\frac{4}{3}n + 9 = \frac{1}{3}n - 9$

Exercise:

Problem: $\frac{5}{4}a + 15 = \frac{3}{4}a - 5$

Solution:

$a = -40$

Exercise:

Problem: $\frac{1}{4}y + 7 = \frac{3}{4}y - 3$

Exercise:

Problem: $\frac{3}{5}p + 2 = \frac{4}{5}p - 1$

Solution:

$p = 15$

Exercise:

Problem: $14n + 8.25 = 9n + 19.60$

Exercise:

Problem: $13z + 6.45 = 8z + 23.75$

Solution:

$z = 3.46$

Exercise:

Problem: $2.4w - 100 = 0.8w + 28$

Exercise:

Problem: $2.7w - 80 = 1.2w + 10$

Solution:

$w = 60$

Exercise:

Problem: $5.6r + 13.1 = 3.5r + 57.2$

Exercise:

Problem: $6.6x - 18.9 = 3.4x + 54.7$

Solution:

$x = 23$

Everyday Math

Exercise:

Problem:

Concert tickets At a school concert the total value of tickets sold was \$1506. Student tickets sold for \$6 and adult tickets sold for \$9. The number of adult tickets sold was 5 less than 3 times the number of student tickets. Find the number of student tickets sold, *s*, by solving the equation $6s + 27s - 45 = 1506$.

Exercise:

Problem:

Making a fence Jovani has 150 feet of fencing to make a rectangular garden in his backyard. He wants the length to be 15 feet more than the width. Find the width, *w*, by solving the equation $150 = 2w + 30 + 2w$.

Solution:

30 feet

Writing Exercises

Exercise:

Problem:

Solve the equation $\frac{6}{5}y - 8 = \frac{1}{5}y + 7$ explaining all the steps of your solution as in the examples in this section.

Exercise:

Problem:

Solve the equation $10x + 14 = -2x + 38$ explaining all the steps of your solution as in the examples in this section.

Solution:

$x = 2$ Justifications will vary.

Exercise:

Problem:

When solving an equation with variables on both sides, why is it usually better to choose the side with the larger coefficient of x to be the “variable” side?

Exercise:

Problem:

Is $x = -2$ a solution to the equation $5 - 2x = -4x + 1$? How do you know?

Solution:

Yes. Justifications will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve an equation with constants on both sides.			
solve an equation with variables on both sides.			
solve an equation with variables and constants on both sides.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Use a General Strategy to Solve Linear Equations: ASE
By the end of this section, you will be able to:

- Solve equations using a general strategy
- Classify equations

Solve Equations Using the General Strategy

Until now we have dealt with solving one specific form of a linear equation. It is time now to lay out one overall strategy that can be used to solve any linear equation. Some equations we solve will not require all these steps to solve, but many will.

Beginning by simplifying each side of the equation makes the remaining steps easier.

Example:
How to Solve Linear Equations Using the General Strategy
Exercise:

Problem: Solve: $-6\left(x+3\right)=24$.

Solution:
Solution

Step 1. Simplify each side of the equation as much as possible.	Use the Distributive Property. Notice that each side of the equation is simplified as much as possible.	$-6(x+3)=24$ $-6x-18=24$
Step 2. Collect all variable terms on one side of the equation.	Nothing to do – all x's are on the left side.	

Step 3. Collect constant terms on the other side of the equation.	To get constants only on the right, add 18 to each side. Simplify.	$-6x - 18 + 18 = 24 + 18$ $-6x = 42$
Step 4. Make the coefficient of the variable term to equal to 1.	Divide each side by −6. Simplify.	$\frac{-6x}{-6} = \frac{42}{-6}$ $x = -7$
Step 5. Check the solution.	Let $x = -7$ Simplify. Multiply.	**Check:** $-6(x + 3) = 24$ $-6(-7 + 3) \stackrel{?}{=} 24$ $-6(-4) \stackrel{?}{=} 24$ $24 = 24\checkmark$

Note:
Exercise:

Problem: Solve: $5\left(x+3\right)=35.$

Solution:

$x=4$

Note:
Exercise:

Problem: Solve: $6\left(y-4\right)=-18.$

Solution:

$y=1$

Note:
General strategy for solving linear equations.

Simplify each side of the equation as much as possible.	Use the Distributive Property to remove any parentheses.	Combine like terms.
Collect all the variable terms on one side of the equation.	Use the Addition or Subtraction Property of Equality.	
Collect all the constant terms on the other side of the equation.	Use the Addition or Subtraction Property of Equality.	
Make the coefficient of the variable term to equal to 1.	Use the Multiplication or Division Property of Equality.	State the solution to the equation.
Check the solution.	Substitute the solution into the original equation to make sure the result is a true statement.	

Example:
Exercise:

Problem: Solve: $-(y+9)=8$.

Solution:
Solution

	$-(y+9)=8$

Simplify each side of the equation as much as possible by distributing.		$-y-9=8$
The only y term is on the left side, so all variable terms are on the left side of the equation.		
Add 9 to both sides to get all constant terms on the right side of the equation.		$-y-9+9=8+9$
Simplify.		$-y=17$
Rewrite $-y$ as $-1y$.		$-1y=17$
Make the coefficient of the variable term to equal to 1 by dividing both sides by -1.		$\frac{-1y}{-1}=\frac{17}{-1}$
Simplify.		$y=-17$
Check:	$-(y+9)=8$	
Let $y=-17$.	$-(-17+9)\stackrel{?}{=}8$	
	$-(-8)\stackrel{?}{=}8$	
	$8=8$ ✓	

Note:
Exercise:

Problem: Solve: $-\left(y+8\right)=-2.$

Solution:

$y=-6$

Note:
Exercise:

Problem: Solve: $-\left(z+4\right)=-12.$

Solution:

$z=8$

Example:
Exercise:

Problem: Solve: $5\left(a-3\right)+5=-10.$

Solution:
Solution

	$5(a-3)+5=-10$

Simplify each side of the equation as much as possible.		
Distribute.		$5a - 15 + 5 = -10$
Combine like terms.		$5a - 10 = -10$
The only a term is on the left side, so all variable terms are on one side of the equation.		
Add 10 to both sides to get all constant terms on the other side of the equation.		$5a - 10 + 10 = -10 + 10$
Simplify.		$5a = 0$
Make the coefficient of the variable term to equal to 1 by dividing both sides by 5.		$\frac{5a}{5} = \frac{0}{5}$
Simplify.		$a = 0$
Check:	$5(a - 3) + 5 = -10$	
Let $a = 0$.	$5(0 - 3) + 5 \stackrel{?}{=} -10$	
	$5(-3) + 5 \stackrel{?}{=} -10$	

	$-15 + 5 \stackrel{?}{=} -10$	
	$-10 = -10$ ✓	

Note:
Exercise:

Problem: Solve: $2\left(m-4\right)+3=-1.$

Solution:

$m=2$

Note:
Exercise:

Problem: Solve: $7\left(n-3\right)-8=-15.$

Solution:

$n=2$

Example:
Exercise:

Problem: Solve: $\frac{2}{3}\left(6m-3\right)=8-m.$

Solution:
Solution

	$\frac{2}{3}(6m-3)=8-m$
Distribute.	$4m-2=8-m$
Add m to get the variables only to the left.	$4m+m-2=8-m+m$
Simplify.	$5m-2=8$
Add 2 to get constants only on the right.	$5m-2+2=8+2$
Simplify.	$5m=10$
Divide by 5.	$\frac{5m}{5}=\frac{10}{5}$
Simplify.	$m=2$
Check:	

	$\frac{2}{3}(6m-3)=8-m$	
Let $m=2$.	$\frac{2}{3}(6\cdot 2-3)\stackrel{?}{=}8-2$	
	$\frac{2}{3}(12-3)\stackrel{?}{=}6$	
	$\frac{2}{3}(9)\stackrel{?}{=}6$	
	$6=6\checkmark$	

Note:
Exercise:

Problem: Solve: $\frac{1}{3}(6u+3)=7-u$.

Solution:

$u=2$

Note:
Exercise:

Problem: Solve: $\frac{2}{3}(9x - 12) = 8 + 2x$.

Solution:

$x = 4$

Example:
Exercise:

Problem: Solve: $8 - 2(3y + 5) = 0$.

Solution:
Solution

	$8 - 2(3y + 5) = 0$
Simplify—use the Distributive Property.	$8 - 6y - 10 = 0$
Combine like terms.	$-6y - 2 = 0$
Add 2 to both sides to collect constants on the right.	$-6y - 2 + 2 = 0 + 2$
Simplify.	$-6y = 2$

Divide both sides by -6.	$\frac{-6y}{-6} = \frac{2}{-6}$
Simplify.	$y = -\frac{1}{3}$
Check: Let $y = -\frac{1}{3}$. $$\begin{aligned} 8 - 2(3y + 5) &= 0 \\ 8 - 2\left[3\left(-\frac{1}{3}\right) + 5\right] &= 0 \\ 8 - 2(-1 + 5) &\stackrel{?}{=} 0 \\ 8 - 2(4) &\stackrel{?}{=} 0 \\ 8 - 8 &\stackrel{?}{=} 0 \\ 0 &= 0 \checkmark \end{aligned}$$	

Note:
Exercise:

Problem: Solve: $12 - 3\left(4j + 3\right) = -17$.

Solution:

$j = \frac{5}{3}$

Note:
Exercise:

Problem: Solve: $-6-8\left(k-2\right)=-10$.

Solution:

$k=\frac{5}{2}$

Example:
Exercise:

Problem: Solve: $4\left(x-1\right)-2=5\left(2x+3\right)+6$.

Solution:
Solution

	$4(x-1)-2=5(2x+3)+6$
Distribute.	$4x-4-2=10x+15+6$
Combine like terms.	$4x-6=10x+21$
Subtract $4x$ to get the variables only on the right side since $10>4$.	$4x-4x-6=10x-4x+21$

Simplify.		$-6 = 6x + 21$
Subtract 21 to get the constants on left.		$-6 - 21 = 6x + 21 - 21$
Simplify.		$-27 = 6x$
Divide by 6.		$\frac{-27}{6} = \frac{6x}{6}$
Simplify.		$-\frac{9}{2} = x$
Check:	$4(x - 1) - 2 = 5(2x + 3) + 6$	
Let $x = -\frac{9}{2}$.	$4\left(-\frac{9}{2} - 1\right) - 2 \stackrel{?}{=} 5\left[2\left(-\frac{9}{2}\right) + 3\right] + 6$	
	$4\left(-\frac{11}{2}\right) - 2 \stackrel{?}{=} 5(-9 + 3) + 6$	
	$-22 - 2 \stackrel{?}{=} 5(-6) + 6$	
	$-24 \stackrel{?}{=} -30 + 6$	
	$-24 = -24\ \checkmark$	

Note:
Exercise:

Problem: Solve: $6\left(p-3\right)-7=5\left(4p+3\right)-12.$

Solution:

$p=-2$

Note:
Exercise:

Problem: Solve: $8\left(q+1\right)-5=3\left(2q-4\right)-1.$

Solution:

$q=-8$

Example:
Exercise:

Problem: Solve: $10\left[3-8\left(2s-5\right)\right]=15\left(40-5s\right).$

Solution:
Solution

	$10[3 - 8(2s - 5)] = 15(40 - 5s)$
Simplify from the innermost parentheses first.	$10[3 - 16s + 40] = 15(40 - 5s)$
Combine like terms in the brackets.	$10[43 - 16s] = 15(40 - 5s)$
Distribute.	$430 - 160s = 600 - 75s$
Add 160s to get the s's to the right.	$430 - 160s + 160s = 600 - 75s + 160s$
Simplify.	$430 = 600 + 85s$
Subtract 600 to get the constants to the left.	$430 - 600 = 600 + 85s - 600$
Simplify.	$-170 = 85s$
Divide.	$\frac{-170}{85} = \frac{85s}{85}$
Simplify.	$-2 = s$

Check:	$10[3 - 8(2s - 5)] = 15(40 - 5s)$	
Substitute	$10[3 - 8(2(-2) - 5)] \stackrel{?}{=} 15(40 - 5(-2))$	

$s = -2.$		
	$10[3 - 8(-4 - 5)] \stackrel{?}{=} 15(40 + 10)$	
	$10[3 - 8(-9)] \stackrel{?}{=} 15(50)$	
	$10[3 + 72] \stackrel{?}{=} 750$	
	$10[75] \stackrel{?}{=} 750$	
	$750 = 750$ ✓	

Note:
Exercise:

Problem: Solve: $6\left[4 - 2\left(7y - 1\right)\right] = 8\left(13 - 8y\right)$.

Solution:

$y = -\frac{17}{5}$

Note:
Exercise:

Problem: Solve: $12\left[1-5\left(4z-1\right)\right]=3\left(24+11z\right)$.

Solution:

$z=0$

Example:
Exercise:

Problem: Solve: $0.36(100n+5)=0.6(30n+15)$.

Solution:
Solution

	$0.36(100n+5)=0.6(30n+15)$
Distribute.	$36n+1.8=18n+9$
Subtract $18n$ to get the variables to the left.	$36n-18n+1.8=18n-18n+9$
Simplify.	$18n+1.8=9$
Subtract 1.8 to get the constants to the right.	$18n+1.8-1.8=9-1.8$

Simplify.		$18n = 7.2$
Divide.		$\frac{18n}{18} = \frac{7.2}{18}$
Simplify.		$n = 0.4$
Check:	$0.36(100n + 5) = 0.6(30n + 15)$	
Let $n = 0.4$.	$0.36(100(0.4) + 5) \stackrel{?}{=} 0.6(30(0.4) + 15)$	
	$0.36(40 + 5) \stackrel{?}{=} 0.6(12 + 15)$	
	$0.36(45) \stackrel{?}{=} 0.6(27)$	
	$16.2 = 16.2 \checkmark$	

Note:
Exercise:

Problem: Solve: $0.55(100n + 8) = 0.6(85n + 14)$.

Solution:

$n = 1$

Note:
Exercise:

Problem: Solve: $0.15(40m - 120) = 0.5(60m + 12)$.

Solution:

$m = -1$

Classify Equations

Consider the equation we solved at the start of the last section, $7x + 8 = -13$. The solution we found was $x = -3$. This means the equation $7x + 8 = -13$ is true when we replace the variable, *x*, with the value -3. We showed this when we checked the solution $x = -3$ and evaluated $7x + 8 = -13$ for $x = -3$.

$$7(-3) + 8 \stackrel{?}{=} -13$$
$$-21 + 8 \stackrel{?}{=} -13$$
$$-13 = -13 \checkmark$$

If we evaluate $7x + 8$ for a different value of *x*, the left side will not be -13.

The equation $7x + 8 = -13$ is true when we replace the variable, *x*, with the value -3, but not true when we replace *x* with any other value. Whether or not the equation $7x + 8 = -13$ is true depends on the value of the variable. Equations like this are called conditional equations.

All the equations we have solved so far are conditional equations.

Note:
Conditional equation
An equation that is true for one or more values of the variable and false for all other values of the variable is a **conditional equation**.

Now let’s consider the equation $2y + 6 = 2\left(y + 3\right)$. Do you recognize that the left side and the right side are equivalent? Let’s see what happens when we solve for *y*.

	$2y + 6 = 2(y + 3)$
Distribute.	$2y + 6 = 2y + 6$
Subtract $2y$ to get the y’s to one side.	$2y - 2y + 6 = 2y - 2y + 6$
Simplify—the y’s are gone!	$6 = 6$

But $6 = 6$ is true.

This means that the equation $2y + 6 = 2\left(y + 3\right)$ is true for any value of *y*. We say the solution to the equation is all of the real numbers. An equation that is true for any value of the variable like this is called an identity.

Note:
Identity
An equation that is true for any value of the variable is called an **identity**. The solution of an identity is all real numbers.

What happens when we solve the equation $5z = 5z - 1$?

	$5z = 5z - 1$
Subtract $5z$ to get the constant alone on the right.	$5z - 5z = 5z - 5z - 1$
Simplify—the z's are gone!	$0 \neq -1$

But $0 \neq -1$.

Solving the equation $5z = 5z - 1$ led to the false statement $0 = -1$. The equation $5z = 5z - 1$ will not be true for any value of z. It has no solution. An equation that has no solution, or that is false for all values of the variable, is called a contradiction.

Note:
Contradiction
An equation that is false for all values of the variable is called a **contradiction**.

A contradiction has no solution.

Example:
Exercise:

Problem:

Classify the equation as a conditional equation, an identity, or a contradiction. Then state the solution.

$6(2n-1)+3=2n-8+5(2n+1)$

Solution:
Solution

	$6(2n-1)+3=2n-8+5(2n+1)$
Distribute.	$12n-6+3=2n-8+10n+5$
Combine like terms.	$12n-3=12n-3$
Subtract $12n$ to get the n's to one side.	$12n-12n-3=12n-12n-3$
Simplify.	$-3=-3$

This is a true statement.	The equation is an identity. The solution is all real numbers.

Note:
Exercise:

Problem:

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:

$$4 + 9(3x - 7) = -42x - 13 + 23(3x - 2)$$

Solution:

identity; all real numbers

Note:
Exercise:

Problem:

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:

$$8(1 - 3x) + 15(2x + 7) = 2(x + 50) + 4(x + 3) + 1$$

Solution:

identity; all real numbers

Example:
Exercise:

Problem:

Classify as a conditional equation, an identity, or a contradiction. Then state the solution.

$10+4(p-5)=0$

Solution:
Solution

	$10+4(p-5)=0$
Distribute.	$10+4p-20=0$
Combine like terms.	$4p-10=0$
Add 10 to both sides.	$4p-10+10=0+10$
Simplify.	$4p=10$
Divide.	$\frac{4p}{4}=\frac{10}{4}$

Simplify.	$p = \frac{5}{2}$
The equation is true when $p = \frac{5}{2}$.	This is a conditional equation. The solution is $p = \frac{5}{2}$.

Note:
Exercise:

Problem:

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $11(q+3)-5=19$

Solution:

conditional equation; $q = \frac{9}{11}$

Note:
Exercise:

Problem:

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $6+14(k-8)=95$

Solution:

conditional equation; $k = \frac{193}{14}$

Example:
Exercise:

Problem:

Classify the equation as a conditional equation, an identity, or a contradiction. Then state the solution.

$$5m + 3\left(9 + 3m\right) = 2\left(7m - 11\right)$$

Solution:
Solution

	$5m + 3(9 + 3m) = 2(7m - 11)$
Distribute.	$5m + 27 + 9m = 14m - 22$
Combine like terms.	$14m + 27 = 14m - 22$
Subtract $14m$ from both sides.	$14m + 27 - 14m = 14m - 22 - 14m$
Simplify.	$27 \neq -22$

But $27 \neq -22$.	The equation is a contradiction. It has no solution.

Note:
Exercise:

Problem:

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:

$12c + 5(5 + 3c) = 3(9c - 4)$

Solution:

contradiction; no solution

Note:
Exercise:

Problem:

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:

$4(7d + 18) = 13(3d - 2) - 11d$

Solution:

contradiction; no solution

Type of equation	What happens when you solve it?	Solution
Conditional Equation	True for one or more values of the variables and false for all other values	One or more values
Identity	**True** for any value of the variable	All real numbers
Contradiction	**False** for all values of the variable	No solution

Key Concepts

- **General Strategy for Solving Linear Equations**

Simplify each side of the equation as much as possible.	Use the Distributive Property to remove any parentheses.	Combine like terms.
Collect all the variable terms on one side of the equation.	Use the Addition or Subtraction Property of Equality.	
Collect all the constant terms on the other side of the equation.	Use the Addition or Subtraction Property of Equality.	
Make the coefficient of the variable term to equal to 1.	Use the Multiplication or Division Property of Equality.	State the solution to the equation.
Check the solution.	Substitute the solution into the original equation.	

Practice Makes Perfect

Solve Equations Using the General Strategy for Solving Linear Equations

In the following exercises, solve each linear equation.

Exercise:

Problem: $15\left(y-9\right)=-60$

Exercise:

Problem: $21\left(y-5\right)=-42$

Solution:

$y=3$

Exercise:

Problem: $-9\left(2n+1\right)=36$

Exercise:

Problem: $-16\left(3n+4\right)=32$

Solution:

$n=-2$

Exercise:

Problem: $8\left(22+11r\right)=0$

Exercise:

Problem: $5\left(8+6p\right)=0$

Solution:

$p=-\frac{4}{3}$

Exercise:

Problem: $-\left(w-12\right)=30$

Exercise:

Problem: $-\left(t-19\right)=28$

Solution:

$t=-9$

Exercise:

Problem: $9\left(6a+8\right)+9=81$

Exercise:

Problem: $8\left(9b-4\right)-12=100$

Solution:

$b=2$

Exercise:

Problem: $32+3(z+4)=41$

Exercise:

Problem: $21+2(m-4)=25$

Solution:

$m=6$

Exercise:

Problem: $51+5(4-q)=56$

Exercise:

Problem: $-6+6(5-k)=15$

Solution:

$k=\frac{3}{2}$

Exercise:

Problem: $2(9s-6)-62=16$

Exercise:

Problem: $8(6t-5)-35=-27$

Solution:

$t=1$

Exercise:

Problem: $3(10-2x)+54=0$

Exercise:

Problem: $-2(11-7x)+54=4$

Solution:

$x=-2$

Exercise:

Problem: $\frac{2}{3}(9c-3)=22$

Exercise:

Problem: $\frac{3}{5}(10x - 5) = 27$

Solution:

$x = 5$

Exercise:

Problem: $\frac{1}{5}(15c + 10) = c + 7$

Exercise:

Problem: $\frac{1}{4}(20d + 12) = d + 7$

Solution:

$d = 1$

Exercise:

Problem: $18 - (9r + 7) = -16$

Exercise:

Problem: $15 - (3r + 8) = 28$

Solution:

$r = -7$

Exercise:

Problem: $5 - (n - 1) = 19$

Exercise:

Problem: $-3 - (m - 1) = 13$

Solution:

$m=-15$

Exercise:

Problem: $11-4(y-8)=43$

Exercise:

Problem: $18-2(y-3)=32$

Solution:

$y=-4$

Exercise:

Problem: $24-8(3v+6)=0$

Exercise:

Problem: $35-5(2w+8)=-10$

Solution:

$w=\frac{1}{2}$

Exercise:

Problem: $4(a-12)=3\left(a+5\right)$

Exercise:

Problem: $-2(a-6)=4\left(a-3\right)$

Solution:

$a = 4$

Exercise:

Problem: $2(5 - u) = -3\left(2u + 6\right)$

Exercise:

Problem: $5(8 - r) = -2\left(2r - 16\right)$

Solution:

$r = 8$

Exercise:

Problem: $3(4n - 1) - 2 = 8n + 3$

Exercise:

Problem: $9(2m - 3) - 8 = 4m + 7$

Solution:

$m = 3$

Exercise:

Problem: $12 + 2(5 - 3y) = -9\left(y - 1\right) - 2$

Exercise:

Problem: $-15 + 4(2 - 5y) = -7\left(y - 4\right) + 4$

Solution:

$y = -3$

Exercise:

Problem: $8\left(x-4\right)-7x=14$

Exercise:

Problem: $5\left(x-4\right)-4x=14$

Solution:

$x=34$

Exercise:

Problem: $5+6(3s-5)=-3+2\left(8s-1\right)$

Exercise:

Problem: $-12+8(x-5)=-4+3\left(5x-2\right)$

Solution:

$x=-6$

Exercise:

Problem: $4\left(u-1\right)-8=6\left(3u-2\right)-7$

Exercise:

Problem: $7\left(2n-5\right)=8\left(4n-1\right)-9$

Solution:

$n=-1$

Exercise:

Problem: $4\left(p-4\right)-\left(p+7\right)=5\left(p-3\right)$

Exercise:

Problem: $3\left(a-2\right)-\left(a+6\right)=4\left(a-1\right)$

Solution:

$a=-4$

Exercise:

Problem: $-(9y+5)-(3y-7)$
$=16-(4y-2)$

Exercise:

Problem: $-\left(7m+4\right)-\left(2m-5\right)$
$=14-\left(5m-3\right)$

Solution:

$m=-4$

Exercise:

Problem: $4\left[5-8\left(4c-3\right)\right]$
$=12\left(1-13c\right)-8$

Exercise:

Problem: $5\left[9-2\left(6d-1\right)\right]$
$=11\left(4-10d\right)-139$

Solution:

$d=-3$

Exercise:

Problem: $3\left[-9+8\left(4h-3\right)\right] = 2\left(5-12h\right)-19$

Exercise:

Problem: $3\left[-14+2\left(15k-6\right)\right] = 8\left(3-5k\right)-24$

Solution:

$k=\frac{3}{5}$

Exercise:

Problem: $5\left[2\left(m+4\right)+8\left(m-7\right)\right] = 2\left[3\left(5+m\right)-\left(21-3m\right)\right]$

Exercise:

Problem: $10\left[5\left(n+1\right)+4\left(n-1\right)\right] = 11[7\left(5+n\right)-\left(25-3n\right)]$

Solution:

$n=-5$

Exercise:

Problem: $5\left(1.2u-4.8\right)=-12$

Exercise:

Problem: $4\left(2.5v-0.6\right)=7.6$

Solution:

$v=1$

Exercise:

Problem: $0.25(q-6)=0.1\,(q+18)$

Exercise:

Problem: $0.2(p-6)=0.4\,(p+14)$

Solution:

$p=-34$

Exercise:

Problem: $0.2\,(30n+50)=28$

Exercise:

Problem: $0.5\,(16m+34)=-15$

Solution:

$m=-4$

Classify Equations

In the following exercises, classify each equation as a conditional equation, an identity, or a contradiction and then state the solution.

Exercise:

Problem: $23z+19=3\,(5z-9)+8z+46$

Exercise:

Problem: $15y+32=2\,(10y-7)-5y+46$

Solution:

identity; all real numbers

Exercise:

Problem: $5\left(b-9\right)+4\left(3b+9\right)=6\left(4b-5\right)-7b+21$

Exercise:

Problem: $9\left(a-4\right)+3\left(2a+5\right)=7\left(3a-4\right)-6a+7$

Solution:

identity; all real numbers

Exercise:

Problem: $18(5j-1)+29=47$

Exercise:

Problem: $24(3d-4)+100=52$

Solution:

conditional equation; $d=\frac{2}{3}$

Exercise:

Problem: $22(3m-4)=8\left(2m+9\right)$

Exercise:

Problem: $30(2n-1)=5\left(10n+8\right)$

Solution:

conditional equation; $n=7$

Exercise:

Problem: $7v + 42 = 11\left(3v + 8\right) - 2\left(13v - 1\right)$

Exercise:

Problem: $18u - 51 = 9\left(4u + 5\right) - 6\left(3u - 10\right)$

Solution:

contradiction; no solution

Exercise:

Problem: $3\left(6q - 9\right) + 7\left(q + 4\right) = 5\left(6q + 8\right) - 5\left(q + 1\right)$

Exercise:

Problem: $5\left(p + 4\right) + 8\left(2p - 1\right) = 9\left(3p - 5\right) - 6\left(p - 2\right)$

Solution:

contradiction; no solution

Exercise:

Problem: $12(6h - 1) = 8\left(8h + 5\right) - 4$

Exercise:

Problem: $9(4k - 7) = 11\left(3k + 1\right) + 4$

Solution:

conditional equation; $k = 26$

Exercise:

Problem: $45(3y - 2) = 9\left(15y - 6\right)$

Exercise:

Problem: $60(2x - 1) = 15\,(8x + 5)$

Solution:

contradiction; no solution

Exercise:

Problem: $16(6n + 15) = 48\,(2n + 5)$

Exercise:

Problem: $36(4m + 5) = 12\,(12m + 15)$

Solution:

identity; all real numbers

Exercise:

Problem: $9(14d + 9) + 4d = 13\,(10d + 6) + 3$

Exercise:

Problem: $11(8c + 5) - 8c = 2\,(40c + 25) + 5$

Solution:

identity; all real numbers

Everyday Math

Exercise:

Problem:

Fencing Micah has 44 feet of fencing to make a dog run in his yard. He wants the length to be 2.5 feet more than the width. Find the length, L, by solving the equation $2L + 2(L - 2.5) = 44$.

Exercise:

Problem:

Coins Rhonda has $1.90 in nickels and dimes. The number of dimes is one less than twice the number of nickels. Find the number of nickels, n, by solving the equation $0.05n + 0.10(2n - 1) = 1.90$.

Solution:

8 nickels

Writing Exercises

Exercise:

Problem:

Using your own words, list the steps in the general strategy for solving linear equations.

Exercise:

Problem:

Explain why you should simplify both sides of an equation as much as possible before collecting the variable terms to one side and the constant terms to the other side.

Solution:

Answers will vary.

Exercise:

Problem:

What is the first step you take when solving the equation $3 - 7(y - 4) = 38$? Why is this your first step?

Exercise:

Problem:

Solve the equation $\frac{1}{4}(8x + 20) = 3x - 4$ explaining all the steps of your solution as in the examples in this section.

Solution:

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objective of this section.

I can...	Confidently	With some help	No-I don't get it!
solve equations using the general strategy for solving linear equations.			
classify equations.			

ⓑ On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Glossary

conditional equation
: An equation that is true for one or more values of the variable and false for all other values of the variable is a conditional equation.

contradiction

An equation that is false for all values of the variable is called a contradiction. A contradiction has no solution.

identity

An equation that is true for any value of the variable is called an identity. The solution of an identity is all real numbers.

Solve Equations with Fractions or Decimals: ASE
By the end of this section, you will be able to:

- Solve equations with fraction coefficients
- Solve equations with decimal coefficients

Solve Equations with Fraction Coefficients

Let’s use the general strategy for solving linear equations introduced earlier to solve the equation, $\frac{1}{8}x + \frac{1}{2} = \frac{1}{4}$.

	$\frac{1}{8}x + \frac{1}{2} = \frac{1}{4}$
To isolate the x term, subtract $\frac{1}{2}$ from both sides.	$\frac{1}{8}x + \frac{1}{2} - \frac{1}{2} = \frac{1}{4} - \frac{1}{2}$
Simplify the left side.	$\frac{1}{8}x = \frac{1}{4} - \frac{1}{2}$
Change the constants to equivalent fractions with the LCD.	$\frac{1}{8}x = \frac{1}{4} - \frac{2}{4}$
Subtract.	$\frac{1}{8}x = -\frac{1}{4}$
Multiply both sides by the reciprocal of $\frac{1}{8}$.	

	$\frac{8}{1} \cdot \frac{1}{8}x = \frac{8}{1}\left(-\frac{1}{4}\right)$
Simplify.	$x = -2$

This method worked fine, but many students do not feel very confident when they see all those fractions. So, we are going to show an alternate method to solve equations with fractions. This alternate method eliminates the fractions.

We will apply the Multiplication Property of Equality and multiply both sides of an equation by the least common denominator of all the fractions in the equation. The result of this operation will be a new equation, equivalent to the first, but without fractions. This process is called "clearing" the equation of fractions.

Let's solve a similar equation, but this time use the method that eliminates the fractions.

Example:
How to Solve Equations with Fraction Coefficients
Exercise:

Problem: Solve: $\frac{1}{6}y - \frac{1}{3} = \frac{5}{6}$.

Solution:
Solution

Step 1. Find the least common denominator of *all* the fractions in the equation.	What is the LCD of $\frac{1}{6}$, $\frac{1}{3}$, and $\frac{5}{6}$?	$\frac{1}{6}y - \frac{1}{3} = \frac{5}{6}$ LCD = 6

Step 2. Multiply both sides of the equation by that LCD. This clears the fractions.	Multiply both sides of the equation by the LCD 6.	$6\left(\frac{1}{6}y - \frac{1}{3}\right) = 6\left(\frac{5}{6}\right)$
	Use the Distributive Property.	$6 \cdot \frac{1}{6}y - 6 \cdot \frac{1}{3} = 6 \cdot \frac{5}{6}$
	Simplify – and notice, no more fractions!	$y - 2 = 5$

Step 3. Solve using the General Strategy for Solving Linear Equations.	To isolate the "*y*" term, add 2.	$y - 2 + 2 = 5 + 2$
	Simplify.	$y = 7$

Note:
Exercise:

Problem: Solve: $\frac{1}{4}x + \frac{1}{2} = \frac{5}{8}$.

Solution:

$x = \frac{1}{2}$

Note:
Exercise:

Problem: Solve: $\frac{1}{8}x + \frac{1}{2} = \frac{1}{4}$.

Solution:

$x = -2$

Notice in [link], once we cleared the equation of fractions, the equation was like those we solved earlier in this chapter. We changed the problem to one we already knew how to solve! We then used the General Strategy for Solving Linear Equations.

Note:
Strategy to solve equations with fraction coefficients.

Find the least common denominator of *all* the fractions in the equation.
Multiply both sides of the equation by that LCD. This clears the fractions.
Solve using the General Strategy for Solving Linear Equations.

Example:
Exercise:

Problem: Solve: $6=\frac{1}{2}v+\frac{2}{5}v-\frac{3}{4}v.$

Solution:
Solution

We want to clear the fractions by multiplying both sides of the equation by the LCD of all the fractions in the equation.

Find the LCD of all fractions in the equation.	$6=\frac{1}{2}v+\frac{2}{5}v-\frac{3}{4}v$
The LCD is 20.	

Multiply both sides of the equation by 20.		$20(6) = 20 \cdot \left(\frac{1}{2}v + \frac{2}{5}v - \frac{3}{4}v\right)$
Distribute.		$20(6) = 20 \cdot \frac{1}{2}v + 20 \cdot \frac{2}{5}v - 20 \cdot \frac{3}{4}v$
Simplify—notice, no more fractions!		$120 = 10v + 8v - 15v$
Combine like terms.		$120 = 3v$
Divide by 3.		$\frac{120}{3} = \frac{3v}{3}$
Simplify.		$40 = v$
Check:	$6 = \frac{1}{2}v + \frac{2}{5}v - \frac{3}{4}v$	
Let $v = 40$.	$6 \stackrel{?}{=} \frac{1}{2}(40) + \frac{2}{5}(40) - \frac{3}{4}(40)$	
	$6 \stackrel{?}{=} 20 + 16 - 30$	
	$6 = 6\checkmark$	

Note:
Exercise:

Problem: Solve: $7=\frac{1}{2}x+\frac{3}{4}x-\frac{2}{3}x.$

Solution:

$x=12$

Note:
Exercise:

Problem: Solve: $-1=\frac{1}{2}u+\frac{1}{4}u-\frac{2}{3}u.$

Solution:

$u=-12$

In the next example, we again have variables on both sides of the equation.

Example:
Exercise:

Problem: Solve: $a+\frac{3}{4}=\frac{3}{8}a-\frac{1}{2}.$

Solution:
Solution

	$a+\frac{3}{4}=\frac{3}{8}a-\frac{1}{2}$
Find the LCD of all fractions in the equation. The LCD is 8.	
Multiply both sides by the LCD.	$8\left(a+\frac{3}{4}\right)=8\left(\frac{3}{8}a-\frac{1}{2}\right)$
Distribute.	$8\cdot a+8\cdot\frac{3}{4}=8\cdot\frac{3}{8}a-8\cdot\frac{1}{2}$
Simplify—no more fractions.	$8a+6=3a-4$
Subtract $3a$ from both sides.	$8a-3a+6=3a-3a-4$
Simplify.	$5a+6=-4$
Subtract 6 from both sides.	$5a+6-6=-4-6$
Simplify.	$5a=-10$
Divide by 5.	$\frac{5a}{5}=\frac{-10}{5}$
Simplify.	

		$a = -2$
Check:	$a + \frac{3}{4} = \frac{3}{8}a - \frac{1}{2}$	
Let $a = -2$.	$-2 + \frac{3}{4} \stackrel{?}{=} \frac{3}{8}(-2) - \frac{1}{2}$	
	$-\frac{8}{4} + \frac{3}{4} \stackrel{?}{=} -\frac{16}{8} - \frac{4}{8}$	
	$-\frac{5}{4} = -\frac{10}{8}$	
	$-\frac{5}{4} = -\frac{5}{4} \checkmark$	

Note:
Exercise:

Problem: Solve: $x + \frac{1}{3} = \frac{1}{6}x - \frac{1}{2}$.

Solution:

$x = -1$

Note:
Exercise:

Problem: Solve: $c+\frac{3}{4}=\frac{1}{2}c-\frac{1}{4}$.

Solution:

$c=-2$

In the next example, we start by using the Distributive Property. This step clears the fractions right away.

Example:
Exercise:

Problem: Solve: $-5=\frac{1}{4}(8x+4)$.

Solution:
Solution

	$-5=\frac{1}{4}(8x+4)$
Distribute.	$-5=\frac{1}{4}\cdot 8x+\frac{1}{4}\cdot 4$

Simplify. Now there are no fractions.		$-5 = 2x + 1$
Subtract 1 from both sides.		$-5 - 1 = 2x + 1 - 1$
Simplify.		$-6 = 2x$
Divide by 2.		$\frac{-6}{2} = \frac{2x}{2}$
Simplify.		$-3 = x$
Check:	$-5 = \frac{1}{4}(8x + 4)$	
Let $x = -3$.	$-5 \stackrel{?}{=} \frac{1}{2}(4(-3) + 2)$	
	$-5 \stackrel{?}{=} \frac{1}{2}(-12 + 2)$	
	$-5 \stackrel{?}{=} \frac{1}{2}(-10)$	
	$-5 = -5$ ✓	

Note:
Exercise:

Problem: Solve: $-11 = \frac{1}{2}(6p + 2)$.

Solution:

$p = -4$

Note:
Exercise:

Problem: Solve: $8 = \frac{1}{3}(9q + 6)$.

Solution:

$q = 2$

In the next example, even after distributing, we still have fractions to clear.

Example:
Exercise:

Problem: Solve: $\frac{1}{2}(y - 5) = \frac{1}{4}(y - 1)$.

Solution:

Solution

	$\frac{1}{2}(y-5)=\frac{1}{4}(y-1)$
Distribute.	$\frac{1}{2}\cdot y-\frac{1}{2}\cdot 5=\frac{1}{4}\cdot y-\frac{1}{4}\cdot 1$
Simplify.	$\frac{1}{2}y-\frac{5}{2}=\frac{1}{4}y-\frac{1}{4}$
Multiply by the LCD, 4.	$4\left(\frac{1}{2}y-\frac{5}{2}\right)=4\left(\frac{1}{4}y-\frac{1}{4}\right)$
Distribute.	$4\cdot\frac{1}{2}y-4\cdot\frac{5}{2}=4\cdot\frac{1}{4}y-4\cdot\frac{1}{4}$
Simplify.	$2y-10=y-1$
Collect the variables to the left.	$2y-y-10=y-y-1$
Simplify.	$y-10=-1$
Collect the constants to the right.	$y-10+10=-1+10$

Simplify.		$y=9$
Check:	$\frac{1}{2}(y-5)=\frac{1}{4}(y-1)$	
Let $y=9$.	$\frac{1}{2}(9-5)\stackrel{?}{=}\frac{1}{4}(9-1)$	
Finish the check on your own.		

Note:
Exercise:

Problem: Solve: $\frac{1}{5}(n+3)=\frac{1}{4}(n+2)$.

Solution:

$n=2$

Note:
Exercise:

Problem: Solve: $\frac{1}{2}(m-3)=\frac{1}{4}(m-7)$.

Solution:

$m=-1$

Example:
Exercise:

Problem: Solve: $\frac{5x-3}{4}=\frac{x}{2}$.

Solution:
Solution

	$\frac{5x-3}{4}=\frac{x}{2}$
Multiply by the LCD, 4.	$4\left(\frac{5x-3}{4}\right)=4\left(\frac{x}{2}\right)$
Simplify.	$5x-3=2x$
Collect the variables to the right.	$5x-5x-3=2x-5x$
Simplify.	$-3=-3x$
Divide.	$\frac{-3}{-3}=\frac{-3x}{-3}$

Simplify.		$1 = x$
Check:	$\frac{5x-3}{4} = \frac{x}{2}$	
Let $x = 1$.	$\frac{5(1)-3}{4} \stackrel{?}{=} \frac{1}{2}$	
	$\frac{2}{4} \stackrel{?}{=} \frac{1}{2}$	
	$\frac{1}{2} = \frac{1}{2}\checkmark$	

Note:
Exercise:

Problem: Solve: $\frac{4y-7}{3} = \frac{y}{6}$.

Solution:

$y = 2$

Note:

Exercise:

Problem: Solve: $\frac{-2z-5}{4} = \frac{z}{8}$.

Solution:

$z = -2$

Example:
Exercise:

Problem: Solve: $\frac{a}{6} + 2 = \frac{a}{4} + 3$.

Solution:
Solution

	$\frac{a}{6} + 2 = \frac{a}{4} + 3$
Multiply by the LCD, 12.	$12\left(\frac{a}{6} + 2\right) = 12\left(\frac{a}{4} + 3\right)$
Distribute.	$12 \cdot \frac{a}{6} + 12 \cdot 2 = 12 \cdot \frac{a}{4} + 12 \cdot 3$
Simplify.	$2a + 24 = 3a + 36$

Collect the variables to the right.		$2a - 2a + 24 = 3a - 2a + 36$
Simplify.		$24 = a + 36$
Collect the constants to the left.		$24 - 36 = a + 36 - 36$
Simplify.		$a = -12$
Check:	$\frac{a}{6} + 2 = \frac{a}{4} + 3$	
Let $a = -12$.	$\frac{-12}{6} + 2 \stackrel{?}{=} \frac{-12}{4} + 3$	
	$-2 + 2 \stackrel{?}{=} -3 + 3$	
	$0 = 0✓$	

Note:
Exercise:

Problem: Solve: $\frac{b}{10} + 2 = \frac{b}{4} + 5$.

Solution:

$b=-20$

Note:
Exercise:

Problem: Solve: $\frac{c}{6}+3=\frac{c}{3}+4$.

Solution:

$c=-6$

Example:
Exercise:

Problem: Solve: $\frac{4q+3}{2}+6=\frac{3q+5}{4}$.

Solution:
Solution

	$\frac{4q+3}{2}+6=\frac{3q+5}{4}$
Multiply by the LCD, 4.	

	$4\left(\frac{4q+3}{2}+6\right)=4\left(\frac{3q+5}{4}\right)$
Distribute.	$4\left(\frac{4q+3}{2}\right)+4\cdot 6=4\cdot\left(\frac{3q+5}{4}\right)$
Simplify.	$2(4q+3)+24=3q+5$ $8q+6+24=3q+5$ $8q+30=3q+5$
Collect the variables to the left.	$8q-3q+30=3q-3q+5$
Simplify.	$5q+30=5$
Collect the constants to the right.	$5q+30-30=5-30$
Simplify.	$5q\ =-25$
Divide by 5.	$\frac{5q}{5}=\frac{-25}{5}$
Simplify.	$q=-5$

Check:	$\frac{4q+3}{2}+6=\frac{3q+5}{4}$	
Let $q=-5$.	$\frac{4(-5)+3}{2}+6\stackrel{?}{=}\frac{3(-5)+5}{4}$	
Finish the check on your own.		

Note:
Exercise:

Problem: Solve: $\frac{3r+5}{6}+1=\frac{4r+3}{3}$.

Solution:

$r=1$

Note:
Exercise:

Problem: Solve: $\frac{2s+3}{2}+1=\frac{3s+2}{4}$.

Solution:

$s=-8$

Solve Equations with Decimal Coefficients

Some equations have decimals in them. This kind of equation will occur when we solve problems dealing with money or percentages. But decimals can also be expressed as fractions. For example, $0.3 = \frac{3}{10}$ and $0.17 = \frac{17}{100}$. So, with an equation with decimals, we can use the same method we used to clear fractions—multiply both sides of the equation by the least common denominator.

Example:
Exercise:

Problem: Solve: $0.06x + 0.02 = 0.25x - 1.5$.

Solution:
Solution

Look at the decimals and think of the equivalent fractions.
Equation:

$$0.06 = \frac{6}{100} \qquad 0.02 = \frac{2}{100} \qquad 0.25 = \frac{25}{100} \qquad 1.5 = 1\frac{5}{10}$$

Notice, the LCD is 100.

By multiplying by the LCD, we will clear the decimals from the equation.

	$0.06x + 0.02 = 0.25x - 1.5$
Multiply both sides by 100.	

	$100(0.06x + 0.02) = 100(0.25x - 1.5)$
Distribute.	$100(0.06x) + 100(0.02) = 100(0.25x) - 100(1.5)$
Multiply, and now we have no more decimals.	$6x + 2 = 25x - 150$
Collect the variables to the right.	$6x - 6x + 2 = 25x - 6x - 150$
Simplify.	$2 = 19x - 150$
Collect the constants to the left.	$2 + 150 = 19x - 150 + 150$
Simplify.	$152 = 19x$
Divide by 19.	$\frac{152}{19} = \frac{19x}{19}$
Simplify.	$8 = x$
Check: Let $x = 8$. $0.06(8) + 0.02 \stackrel{?}{=} 0.25(8) - 1.5$ $0.48 + 0.02 \stackrel{?}{=} 2.00 - 1.5$ $0.50 = 0.50 \checkmark$	

Note:
Exercise:

Problem: Solve: $0.14h + 0.12 = 0.35h - 2.4$.

Solution:

$h = 12$

Note:
Exercise:

Problem: Solve: $0.65k - 0.1 = 0.4k - 0.35$.

Solution:

$k = -1$

The next example uses an equation that is typical of the money applications in the next chapter. Notice that we distribute the decimal before we clear all the decimals.

Example:
Exercise:

Problem: Solve: $0.25x + 0.05(x + 3) = 2.85$.

Solution:
Solution

	$0.25x + 0.05(x + 3) = 2.85$
Distribute first.	$0.25x + 0.05x + 0.15 = 2.85$
Combine like terms.	$0.30x + 0.15 = 2.85$
To clear decimals, multiply by 100.	$100(0.30x + 0.15) = 100(2.85)$
Distribute.	$30x + 15 = 285$
Subtract 15 from both sides.	$30x + 15 - 15 = 285 - 15$
Simplify.	$30x = 270$
Divide by 30.	$\frac{30x}{30} = \frac{270}{30}$
Simplify.	$x = 9$
Check it yourself by substituting $x = 9$ into the original equation.	

Note:

Exercise:

Problem: Solve: $0.25n + 0.05(n + 5) = 2.95$.

Solution:

$n = 9$

Note:
Exercise:

Problem: Solve: $0.10d + 0.05(d - 5) = 2.15$.

Solution:

$d = 16$

Key Concepts

- **Strategy to Solve an Equation with Fraction Coefficients**

  Find the least common denominator of all the fractions in the equation.
  Multiply both sides of the equation by that LCD. This clears the fractions.
  Solve using the General Strategy for Solving Linear Equations.

Practice Makes Perfect

Solve Equations with Fraction Coefficients

In the following exercises, solve each equation with fraction coefficients.

Exercise:

Problem: $\frac{1}{4}x - \frac{1}{2} = -\frac{3}{4}$

Exercise:

Problem: $\frac{3}{4}x - \frac{1}{2} = \frac{1}{4}$

Solution:

$x = 1$

Exercise:

Problem: $\frac{5}{6}y - \frac{2}{3} = -\frac{3}{2}$

Exercise:

Problem: $\frac{5}{6}y - \frac{1}{3} = -\frac{7}{6}$

Solution:

$y = -1$

Exercise:

Problem: $\frac{1}{2}a + \frac{3}{8} = \frac{3}{4}$

Exercise:

Problem: $\frac{5}{8}b + \frac{1}{2} = -\frac{3}{4}$

Solution:

$b = -2$

Exercise:

Problem: $2 = \frac{1}{3}x - \frac{1}{2}x + \frac{2}{3}x$

Exercise:

Problem: $2 = \frac{3}{5}x - \frac{1}{3}x + \frac{2}{5}x$

Solution:

$x = 3$

Exercise:

Problem: $\frac{1}{4}m - \frac{4}{5}m + \frac{1}{2}m = -1$

Exercise:

Problem: $\frac{5}{6}n - \frac{1}{4}n - \frac{1}{2}n = -2$

Solution:

$n = -24$

Exercise:

Problem: $x + \frac{1}{2} = \frac{2}{3}x - \frac{1}{2}$

Exercise:

Problem: $x + \frac{3}{4} = \frac{1}{2}x - \frac{5}{4}$

Solution:

$x = -4$

Exercise:

Problem: $\frac{1}{3}w+\frac{5}{4}=w-\frac{1}{4}$

Exercise:

Problem: $\frac{3}{2}z+\frac{1}{3}=z-\frac{2}{3}$

Solution:

$z=-2$

Exercise:

Problem: $\frac{1}{2}x-\frac{1}{4}=\frac{1}{12}x+\frac{1}{6}$

Exercise:

Problem: $\frac{1}{2}a-\frac{1}{4}=\frac{1}{6}a+\frac{1}{12}$

Solution:

$a=1$

Exercise:

Problem: $\frac{1}{3}b+\frac{1}{5}=\frac{2}{5}b-\frac{3}{5}$

Exercise:

Problem: $\frac{1}{3}x+\frac{2}{5}=\frac{1}{5}x-\frac{2}{5}$

Solution:

$x=-6$

Exercise:

Problem: $1 = \frac{1}{6}(12x - 6)$

Exercise:

Problem: $1 = \frac{1}{5}(15x - 10)$

Solution:

$x = 1$

Exercise:

Problem: $\frac{1}{4}(p - 7) = \frac{1}{3}(p + 5)$

Exercise:

Problem: $\frac{1}{5}(q + 3) = \frac{1}{2}(q - 3)$

Solution:

$q = 7$

Exercise:

Problem: $\frac{1}{2}(x + 4) = \frac{3}{4}$

Exercise:

Problem: $\frac{1}{3}(x + 5) = \frac{5}{6}$

Solution:

$x = -\frac{5}{2}$

Exercise:

Problem: $\frac{5q-8}{5} = \frac{2q}{10}$

Exercise:

Problem: $\frac{4m+2}{6} = \frac{m}{3}$

Solution:

$m = -1$

Exercise:

Problem: $\frac{4n+8}{4} = \frac{n}{3}$

Exercise:

Problem: $\frac{3p+6}{3} = \frac{p}{2}$

Solution:

$p = -4$

Exercise:

Problem: $\frac{u}{3} - 4 = \frac{u}{2} - 3$

Exercise:

Problem: $\frac{v}{10} + 1 = \frac{v}{4} - 2$

Solution:

$v = 20$

Exercise:

Problem: $\frac{c}{15}+1=\frac{c}{10}-1$

Exercise:

Problem: $\frac{d}{6}+3=\frac{d}{8}+2$

Solution:

$d=-24$

Exercise:

Problem: $\frac{3x+4}{2}+1=\frac{5x+10}{8}$

Exercise:

Problem: $\frac{10y-2}{3}+3=\frac{10y+1}{9}$

Solution:

$y=-1$

Exercise:

Problem: $\frac{7u-1}{4}-1=\frac{4u+8}{5}$

Exercise:

Problem: $\frac{3v-6}{2}+5=\frac{11v-4}{5}$

Solution:

$v=4$

Solve Equations with Decimal Coefficients

In the following exercises, solve each equation with decimal coefficients.

Exercise:

Problem: $0.6y + 3 = 9$

Exercise:

Problem: $0.4y - 4 = 2$

Solution:

$y = 15$

Exercise:

Problem: $3.6j - 2 = 5.2$

Exercise:

Problem: $2.1k + 3 = 7.2$

Solution:

$k = 2$

Exercise:

Problem: $0.4x + 0.6 = 0.5x - 1.2$

Exercise:

Problem: $0.7x + 0.4 = 0.6x + 2.4$

Solution:

$x = 20$

Exercise:

Problem: $0.23x + 1.47 = 0.37x - 1.05$

Exercise:

Problem: $0.48x + 1.56 = 0.58x - 0.64$

Solution:

$x = 22$

Exercise:

Problem: $0.9x - 1.25 = 0.75x + 1.75$

Exercise:

Problem: $1.2x - 0.91 = 0.8x + 2.29$

Solution:

$x = 8$

Exercise:

Problem: $0.05n + 0.10\,(n + 8) = 2.15$

Exercise:

Problem: $0.05n + 0.10\,(n + 7) = 3.55$

Solution:

$n = 19$

Exercise:

Problem: $0.10d + 0.25\,(d + 5) = 4.05$

Exercise:

Problem: $0.10d + 0.25\left(d + 7\right) = 5.25$

Solution:

$d = 10$

Exercise:

Problem: $0.05\left(q - 5\right) + 0.25q = 3.05$

Exercise:

Problem: $0.05\left(q - 8\right) + 0.25q = 4.10$

Solution:

$q = 15$

Everyday Math

Exercise:

Problem:

Coins Taylor has $2.00 in dimes and pennies. The number of pennies is 2 more than the number of dimes. Solve the equation $0.10d + 0.01(d + 2) = 2$ for d, the number of dimes.

Exercise:

Problem:

Stamps Paula bought $22.82 worth of 49-cent stamps and 21-cent stamps. The number of 21-cent stamps was 8 less than the number of 49-cent stamps. Solve the equation $0.49s+0.21(s-8)=22.82$ for *s*, to find the number of 49-cent stamps Paula bought.

Solution:

$s=35$

Writing Exercises

Exercise:

Problem:

Explain how you find the least common denominator of $\frac{3}{8}$, $\frac{1}{6}$, and $\frac{2}{3}$.

Exercise:

Problem:

If an equation has several fractions, how does multiplying both sides by the LCD make it easier to solve?

Solution:

Answers will vary.

Exercise:

Problem:

If an equation has fractions only on one side, why do you have to multiply both sides of the equation by the LCD?

Exercise:

Problem:

In the equation $0.35x + 2.1 = 3.85$ what is the LCD? How do you know?

Solution:

100. Justifications will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve equations with fraction coefficients.			
solve equations with decimal coefficients.			

ⓑ Overall, after looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

Solve a Formula for a Specific Variable: ASE
By the end of this section, you will be able to:

- Use the Distance, Rate, and Time formula
- Solve a formula for a specific variable

Use the Distance, Rate, and Time Formula

One formula you will use often in algebra and in everyday life is the formula for distance traveled by an object moving at a constant rate. Rate is an equivalent word for “speed.” The basic idea of rate may already familiar to you. Do you know what distance you travel if you drive at a steady rate of 60 miles per hour for 2 hours? (This might happen if you use your car’s cruise control while driving on the highway.) If you said 120 miles, you already know how to use this formula!

Note:
Distance, Rate, and Time
For an object moving at a uniform (constant) rate, the distance traveled, the elapsed time, and the rate are related by the formula:
Equation:

$$d = rt \qquad \text{where} \qquad \begin{aligned} d &= \text{distance} \\ r &= \text{rate} \\ t &= \text{time} \end{aligned}$$

We will use the Strategy for Solving Applications that we used earlier in this chapter. When our problem requires a formula, we change Step 4. In place of writing a sentence, we write the appropriate formula. We write the revised steps here for reference.

Note:
Solve an application (with a formula).

Readthe problem. Make sure all the words and ideas are understood.
Identifywhat we are looking for.
Namewhat we are looking for. Choose a variable to represent that quantity.
Translateinto an equation. Write the appropriate formula for the situation. Substitute in the given information.
Solvethe equation using good algebra techniques.
Checkthe answer in the problem and make sure it makes sense.
Answerthe question with a complete sentence.

You may want to create a mini-chart to summarize the information in the problem. See the chart in this first example.

Example:
Exercise:

Problem:

Jamal rides his bike at a uniform rate of 12 miles per hour for $3\frac{1}{2}$ hours. What distance has he traveled?

Solution:
Solution

Step 1. Read the problem.	

Step 2. Identify what you are looking for.	distance traveled
Step 3. Name. Choose a variable to represent it.	Let d = distance.
Step 4. Translate: Write the appropriate formula.	$d = rt$
	$d = ?$ $r = 12$ mph $t = 3\frac{1}{2}$ hours
Substitute in the given information.	$d = 12 \cdot 3\frac{1}{2}$
Step 5. Solve the equation.	$d = 42$ miles
Step 6. Check	
Does 42 miles make sense?	
Jamal rides:	
12 miles in 1 hour, 24 miles in 2 hours, 36 miles in 3 hours, 42 miles in $3\frac{1}{2}$ hours is reasonable 48 miles in 4 hours.	
Step 7. Answer the question with a complete sentence.	Jamal rode 42 miles.

Note:
Exercise:

Problem:

Lindsay drove for $5\frac{1}{2}$ hours at 60 miles per hour. How much distance did she travel?

Solution:

330 miles

Note:
Exercise:

Problem:

Trinh walked for $2\frac{1}{3}$ hours at 3 miles per hour. How far did she walk?

Solution:

7 miles

Example:
Exercise:

Problem:

Rey is planning to drive from his house in San Diego to visit his grandmother in Sacramento, a distance of 520 miles. If he can drive at a steady rate of 65 miles per hour, how many hours will the trip take?

Solution:
Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	How many hours (time)
Step 3. Name. Choose a variable to represent it.	Let *t* = time.
	$d = 520$ miles $r = 65$ mph $t = ?$ hours
Step 4. Translate. Write the appropriate formula.	$d = rt$
Substitute in the given information.	$520 = 65t$
Step 5. Solve the equation.	$t = 8$
Step 6. Check. Substitute the numbers into the formula and make sure the result is a true statement.	
$\begin{aligned} d &= rt \\ 520 &\stackrel{?}{=} 65 \cdot 8 \\ 520 &= 520 \checkmark \end{aligned}$	
Step 7. Answer the question with a complete sentence. Rey's trip will take 8 hours.	

Note:
Exercise:

Problem:

Lee wants to drive from Phoenix to his brother's apartment in San Francisco, a distance of 770 miles. If he drives at a steady rate of 70 miles per hour, how many hours will the trip take?

Solution:

11 hours

Note:
Exercise:

Problem:

Yesenia is 168 miles from Chicago. If she needs to be in Chicago in 3 hours, at what rate does she need to drive?

Solution:

56 mph

Solve a Formula for a Specific Variable

You are probably familiar with some geometry formulas. A formula is a mathematical description of the relationship between variables. Formulas are also used in the sciences, such as chemistry, physics, and biology. In medicine they are used for calculations for dispensing medicine or determining body mass index. Spreadsheet programs rely on formulas to make calculations. It is important to be familiar with formulas and be able to manipulate them easily.

In [link] and [link], we used the formula $d = rt$. This formula gives the value of d, distance, when you substitute in the values of r and t, the rate and time. But in [link], we had to find the value of t. We substituted in values of d and r and then used algebra to solve for t. If you had to do this often, you might wonder why there is not a formula that gives the value of t when you substitute in the values of d and r. We can make a formula like this by solving the formula $d = rt$ for t.

To solve a formula for a specific variable means to isolate that variable on one side of the equals sign with a coefficient of 1. All other variables and constants are on the other side of the equals sign. To see how to solve a formula for a specific variable, we will start with the distance, rate and time formula.

Example:
Exercise:

Problem: Solve the formula $d = rt$ for t:

ⓐ when $d = 520$ and $r = 65$
ⓑ in general

Solution:
Solution

We will write the solutions side-by-side to demonstrate that solving a formula in general uses the same steps as when we have numbers to substitute.

ⓐ when $d = 520$ and			ⓑ in general

$r = 65$					
Write the formula.	$d = rt$			Write the formula.	$d = rt$
Substitute.	$520 = 65t$				
Divide, to isolate t.	$\frac{520}{65} = \frac{65t}{65}$			Divide, to isolate t.	$\frac{d}{r} = \frac{rt}{r}$
Simplify.	$8 = t$			Simplify.	$\frac{d}{r} = t$

We say the formula $t = \frac{d}{r}$ is solved for t.

Note:
Exercise:

Problem: Solve the formula $d = rt$ for r:

ⓐ when $d = 180$ and $t = 4$ ⓑ in general

Solution:

ⓐ $r = 45$ ⓑ $r = \frac{d}{t}$

Note:
Exercise:

Problem: Solve the formula $d = rt$ for r:

ⓐ when $d = 780$ and $t = 12$ ⓑ in general

Solution:

ⓐ $r = 65$ ⓑ $r = \frac{d}{t}$

Example:
Exercise:

Problem: Solve the formula $A = \frac{1}{2}bh$ for h:

ⓐ when $A = 90$ and $b = 15$ ⓑ in general

Solution:
Solution

ⓐ when $A = 90$ and $b = 15$				ⓑ in general	
Write the formula.	$A = \frac{1}{2}bh$			Write the formula.	$A = \frac{1}{2}bh$
Substitute.	$90 = \frac{1}{2} \cdot 15 \cdot h$				
Clear the fractions.	$2 \cdot 90 = 2 \cdot \frac{1}{2}15h$			Clear the fractions.	$2 \cdot A = 2 \cdot \frac{1}{2}bh$

Simplify.	$180 = 15h$			Simplify.	$2A = bh$
Solve for h.	$12 = h$			Solve for h.	$\frac{2A}{b} = h$

We can now find the height of a triangle, if we know the area and the base, by using the formula $h = \frac{2A}{b}$.

Note:
Exercise:

Problem: Use the formula $A = \frac{1}{2}bh$ to solve for h:

ⓐ when $A = 170$ and $b = 17$ ⓑ in general

Solution:

ⓐ $h = 20$ ⓑ $h = \frac{2A}{b}$

Note:
Exercise:

Problem: Use the formula $A = \frac{1}{2}bh$ to solve for b:

ⓐ when $A = 62$ and $h = 31$ ⓑ in general

Solution:

ⓐ $b = 4$ ⓑ $b = \frac{2A}{h}$

The formula $I = Prt$ is used to calculate simple interest, I, for a principal, P, invested at rate, r, for t years.

Example:
Exercise:

Problem: Solve the formula $I = Prt$ to find the principal, P:

ⓐ when $I = \$5{,}600, r = 4\%, t = 7$ years ⓑ in general

Solution:
Solution

ⓐ $I = \$5{,}600$, $r = 4\%$, $t = 7$ years				ⓑ in general	
Write the formula.	$I = Prt$			Write the formula.	$I = Prt$
Substitute.	$5600 = P(0.04)(7)$				
Simplify.	$5600 = P(0.28)$			Simplify.	$I = P(rt)$
Divide, to isolate P.	$\frac{5600}{0.28} = \frac{P(0.28)}{0.28}$			Divide, to isolate P.	$\frac{I}{rt} = \frac{P(rt)}{rt}$

Simplify.	$20{,}000 = P$			Simplify.	$\frac{I}{rt} = P$
The principal is	$20,000				$P = \frac{I}{rt}$

Note:
Exercise:

Problem: Use the formula $I = Prt$ to find the principal, P:

ⓐ when $I = \$2{,}160, r = 6\%, t = 3$ years ⓑ in general

Solution:

ⓐ $12,000 ⓑ $P = \frac{I}{rt}$

Note:
Exercise:

Problem: Use the formula $I = Prt$ to find the principal,P:

ⓐ when $I = \$5{,}400, r = 12\%, t = 5$ years ⓑ in general

Solution:

ⓐ $9,000 ⓑ $P = \frac{I}{rt}$

Later in this class, and in future algebra classes, you'll encounter equations that relate two variables, usually *x* and *y*. You might be given an equation that is solved for *y* and need to solve it for *x*, or vice versa. In the following example, we're given an equation with both *x* and *y* on the same side and we'll solve it for *y*.

Example:
Exercise:

Problem: Solve the formula $3x + 2y = 18$ for *y*:

ⓐ when $x = 4$ ⓑ in general

Solution:
Solution

ⓐ when $x = 4$				ⓑ in general	
	$3x + 2y = 18$				$3x + 2y = 18$
Substitute.	$3(4) + 2y = 18$				
Subtract to isolate the y-term.	$12 - 12 + 2y = 18 - 12$			Subtract to isolate the y-term.	$3x - 3x + 2y = 18 - 3x$
Divide.	$\frac{2y}{2} = \frac{6}{2}$			Divide.	$\frac{2y}{2} = \frac{18}{2} - \frac{3x}{2}$

Simplify.	$y=3$			Simplify.	$y=-\frac{3x}{2}+9$

Note:
Exercise:

Problem: Solve the formula $3x+4y=10$ for *y*:

ⓐ when $x=\frac{14}{3}$ ⓑ in general

Solution:

ⓐ $y=1$ ⓑ $y=\frac{10-3x}{4}$

Note:
Exercise:

Problem: Solve the formula $5x+2y=18$ for *y:*

ⓐ when $x=4$ ⓑ in general

Solution:

ⓐ $y=-1$ ⓑ $y=\frac{18-5x}{2}$

In Examples 1.60 through 1.64 we used the numbers in part ⓐ as a guide to solving in general in part ⓑ. Now we will solve a formula in general

without using numbers as a guide.

Example:
Exercise:

Problem: Solve the formula $P = a + b + c$ for a.

Solution:
Solution

We will isolate a on one side of the equation.	$P = a + b + c$
Both b and c are added to a, so we subtract them from both sides of the equation.	$P - b - c = a + b + c - b - c$
Simplify.	$P - b - c = a$ $a = P - b - c$

Note:
Exercise:

Problem: Solve the formula $P = a + b + c$ for b.

Solution:

$b = P - a - c$

Note:
Exercise:

Problem: Solve the formula $P = a + b + c$ for c.

Solution:

$c = P - a - b$

Example:
Exercise:

Problem: Solve the formula $6x + 5y = 13$ for y.

Solution:
Solution

	$6x + 5y = 13$
Subtract $6x$ from both sides to isolate the term with y.	$6x - 6x + 5y = 13 - 6x$

Simplify.	$5y = 13 - 6x$
Divide by 5 to make the coefficient 1.	$\frac{5y}{5} = \frac{13 - 6x}{5}$
Simplify.	$y = \frac{13 - 6x}{5}$

The fraction is simplified. We cannot divide $13 - 6x$ by 5.

Note:
Exercise:

Problem: Solve the formula $4x + 7y = 9$ for *y*.

Solution:

$y = \frac{9 - 4x}{7}$

Note:
Exercise:

Problem: Solve the formula $5x + 8y = 1$ for *y*.

Solution:

$y = \frac{1 - 5x}{8}$

Key Concepts

- **To Solve an Application (with a formula)**

  Readthe problem. Make sure all the words and ideas are understood.
  Identifywhat we are looking for.
  Namewhat we are looking for. Choose a variable to represent that quantity.
  Translateinto an equation. Write the appropriate formula for the situation. Substitute in the given information.
  Solvethe equation using good algebra techniques.
  Checkthe answer in the problem and make sure it makes sense.
  Answerthe question with a complete sentence.

- **Distance, Rate and Time**
  For an object moving at a uniform (constant) rate, the distance traveled, the elapsed time, and the rate are related by the formula:
  $d = rt$ where d = distance, r = rate, t = time.
- **To solve a formula for a specific variable** means to get that variable by itself with a coefficient of 1 on one side of the equation and all other variables and constants on the other side.

Practice Makes Perfect

Use the Distance, Rate, and Time Formula

In the following exercises, solve.

Exercise:

Problem:

Steve drove for $8\frac{1}{2}$ hours at 72 miles per hour. How much distance did he travel?

Exercise:

Problem:

Socorro drove for $4\frac{5}{6}$ hours at 60 miles per hour. How much distance did she travel?

Solution:

290 miles

Exercise:

Problem:

Yuki walked for $1\frac{3}{4}$ hours at 4 miles per hour. How far did she walk?

Exercise:

Problem:

Francie rode her bike for $2\frac{1}{2}$ hours at 12 miles per hour. How far did she ride?

Solution:

30 miles

Exercise:

Problem:

Connor wants to drive from Tucson to the Grand Canyon, a distance of 338 miles. If he drives at a steady rate of 52 miles per hour, how many hours will the trip take?

Exercise:

Problem:

Megan is taking the bus from New York City to Montreal. The distance is 380 miles and the bus travels at a steady rate of 76 miles per hour. How long will the bus ride be?

Solution:

5 hours

Exercise:

Problem:

Aurelia is driving from Miami to Orlando at a rate of 65 miles per hour. The distance is 235 miles. To the nearest tenth of an hour, how long will the trip take?

Exercise:

Problem:

Kareem wants to ride his bike from St. Louis to Champaign, Illinois. The distance is 180 miles. If he rides at a steady rate of 16 miles per hour, how many hours will the trip take?

Solution:

11.25 hours

Exercise:

Problem:

Javier is driving to Bangor, 240 miles away. If he needs to be in Bangor in 4 hours, at what rate does he need to drive?

Exercise:

Problem:

Alejandra is driving to Cincinnati, 450 miles away. If she wants to be there in 6 hours, at what rate does she need to drive?

Solution:

75 mph

Exercise:

Problem:

Aisha took the train from Spokane to Seattle. The distance is 280 miles and the trip took 3.5 hours. What was the speed of the train?

Exercise:

Problem:

Philip got a ride with a friend from Denver to Las Vegas, a distance of 750 miles. If the trip took 10 hours, how fast was the friend driving?

Solution:

75 mph

Solve a Formula for a Specific Variable

In the following exercises, use the formula $d = rt$.

Exercise:

Problem: Solve for t
ⓐ when $d = 350$ and $r = 70$
ⓑ in general

Exercise:

Problem: Solve for t
ⓐ when $d = 240$ and $r = 60$
ⓑ in general

Solution:

ⓐ $t = 4$ ⓑ $t = \frac{d}{r}$

Exercise:

Solve for t
ⓐ when $d = 510$ and $r = 60$
Problem: ⓑ in general

Exercise:

Solve for t
ⓐ when $d = 175$ and $r = 50$
Problem: ⓑ in general

Solution:

ⓐ $t = 3.5$ ⓑ $t = \frac{d}{r}$

Exercise:

Solve for r
ⓐ when $d = 204$ and $t = 3$
Problem: ⓑ in general

Exercise:

Solve for r
ⓐ when $d = 420$ and $t = 6$
Problem: ⓑ in general

Solution:

ⓐ $r = 70$ ⓑ $r = \frac{d}{t}$

Exercise:

Solve for r
ⓐ when $d = 160$ and $t = 2.5$
Problem: ⓑ in general

Exercise:

Problem: Solve for r
ⓐ when $d = 180$ and $t = 4.5$
ⓑ in general

Solution:

ⓐ $r = 40$ ⓑ $r = \frac{d}{t}$

In the following exercises, use the formula $A = \frac{1}{2}bh$.

Exercise:

Problem: Solve for b
ⓐ when $A = 126$ and $h = 18$
ⓑ in general

Exercise:

Problem: Solve for h
ⓐ when $A = 176$ and $b = 22$
ⓑ in general

Solution:

ⓐ $h = 16$ ⓑ $h = \frac{2A}{b}$

Exercise:

Problem: Solve for h
ⓐ when $A = 375$ and $b = 25$
ⓑ in general

Exercise:

Problem: Solve for b
ⓐ when $A = 65$ and $h = 13$
ⓑ in general

Solution:

ⓐ $b = 10$ ⓑ $b = \frac{2A}{h}$

In the following exercises, use the formula *I* = *Prt*.

Exercise:

Problem: Solve for the principal, *P* for
ⓐ $I = \$5{,}480, r = 4\%,$
$t = 7$ years
ⓑ in general

Exercise:

Problem: Solve for the principal, *P* for
ⓐ $I = \$3{,}950, r = 6\%,$
$t = 5$ years
ⓑ in general

Solution:

ⓐ $P = \$\ 13,166.67$ ⓑ $P = \frac{I}{rt}$

Exercise:

Problem: Solve for the time, *t* for
ⓐ $I = \$2{,}376, P = \$9{,}000,$
$r = 4.4\%$
ⓑ in general

Exercise:

Problem: Solve for the time, *t* for
ⓐ $I = \$624, P = \$6{,}000,$
$r = 5.2\%$
ⓑ in general

Solution:

ⓐ $t=2$ years ⓑ $t=\frac{I}{\mathrm{Pr}}$

In the following exercises, solve.

Exercise:

Problem: Solve the formula $2x+3y=12$ for *y*
ⓐ when $x=3$
ⓑ in general

Exercise:

Problem: Solve the formula $5x+2y=10$ for *y*
ⓐ when $x=4$
ⓑ in general

Solution:

ⓐ $y=-5$ ⓑ $y=\frac{10-5x}{2}$

Exercise:

Problem: Solve the formula $3x-y=7$ for *y*
ⓐ when $x=-2$
ⓑ in general

Exercise:

Problem: Solve the formula $4x+y=5$ for *y*
ⓐ when $x=-3$
ⓑ in general

Solution:

ⓐ $y=17$ ⓑ $y=5-4x$

Exercise:

Problem: Solve $a + b = 90$ for b.

Exercise:

Problem: Solve $a + b = 90$ for a.

Solution:

$a = 90 - b$

Exercise:

Problem: Solve $180 = a + b + c$ for a.

Exercise:

Problem: Solve $180 = a + b + c$ for c.

Solution:

$c = 180 - a - b$

Exercise:

Problem: Solve the formula $8x + y = 15$ for *y*.

Exercise:

Problem: Solve the formula $9x + y = 13$ for *y*.

Solution:

$y = 13 - 9x$

Exercise:

Problem: Solve the formula $-4x + y = -6$ for *y*.

Exercise:

Problem: Solve the formula $-5x + y = -1$ for *y*.

Solution:

$y = -1 + 5x$

Exercise:

Problem: Solve the formula $4x + 3y = 7$ for *y*.

Exercise:

Problem: Solve the formula $3x + 2y = 11$ for *y*.

Solution:

$y = \frac{11-3x}{4}$

Exercise:

Problem: Solve the formula $x - y = -4$ for *y*.

Exercise:

Problem: Solve the formula $x - y = -3$ for *y*.

Solution:

$y = 3 + x$

Exercise:

Problem: Solve the formula $P = 2L + 2W$ for L.

Exercise:

Problem: Solve the formula $P = 2L + 2W$ for W.

Solution:

$W = \frac{P-2L}{2}$

Exercise:

Problem: Solve the formula $C = \pi d$ for d.

Exercise:

Problem: Solve the formula $C = \pi d$ for π.

Solution:

$\pi = \frac{C}{d}$

Exercise:

Problem: Solve the formula $V = LWH$ for L.

Exercise:

Problem: Solve the formula $V = LWH$ for H.

Solution:

$H = \frac{V}{LW}$

Everyday Math

Exercise:

Problem:

Converting temperature While on a tour in Greece, Tatyana saw that the temperature was 40^{o} Celsius. Solve for F in the formula $C=\frac{5}{9}(F-32)$ to find the Fahrenheit temperature.

Exercise:

Problem:

Converting temperature Yon was visiting the United States and he saw that the temperature in Seattle one day was 50^{o} Fahrenheit. Solve for C in the formula $F=\frac{9}{5}C+32$ to find the Celsius temperature.

Solution:

10°C

Writing Exercises

Exercise:

Problem: Solve the equation $2x+3y=6$ for y
ⓐ when $x=-3$
ⓑ in general
ⓒ Which solution is easier for you, ⓐ or ⓑ? Why?

Exercise:

Problem: Solve the equation $5x-2y=10$ for x
ⓐ when $y=10$
ⓑ in general
ⓒ Which solution is easier for you, ⓐ or ⓑ? Why?

Solution:

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
use the distance, rate, and time formula.			
solve a formula for a specific variable.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Solve Linear Inequalities: ASE
By the end of this section, you will be able to:

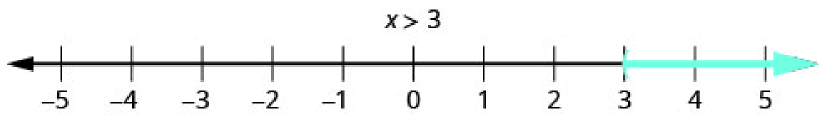
- Graph inequalities on the number line
- Solve inequalities using the Subtraction and Addition Properties of inequality
- Solve inequalities using the Division and Multiplication Properties of inequality
- Solve inequalities that require simplification
- Translate to an inequality and solve

Graph Inequalities on the Number Line

Do you remember what it means for a number to be a solution to an equation? A solution of an equation is a value of a variable that makes a true statement when substituted into the equation.

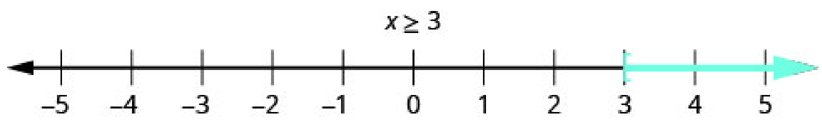
What about the solution of an inequality? What number would make the inequality $x > 3$ true? Are you thinking, '*x* could be 4'? That's correct, but *x* could be 5 too, or 20, or even 3.001. Any number greater than 3 is a solution to the inequality $x > 3$.

We show the solutions to the inequality $x > 3$ on the number line by shading in all the numbers to the right of 3, to show that all numbers greater than 3 are solutions. Because the number 3 itself is not a solution, we put an open parenthesis at 3. The graph of $x > 3$ is shown in [link]. Please note that the following convention is used: light blue arrows point in the positive direction and dark blue arrows point in the negative direction.



The inequality $x > 3$ is graphed on this number line.

The graph of the inequality $x \geq 3$ is very much like the graph of $x > 3$, but now we need to show that 3 is a solution, too. We do that by putting a bracket at $x = 3$, as shown in [link].



The inequality $x \geq 3$ is graphed on this number line.

Notice that the open parentheses symbol, (, shows that the endpoint of the inequality is not included. The open bracket symbol, [, shows that the endpoint is included.

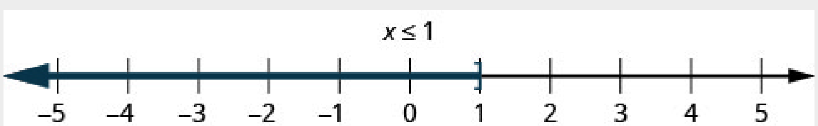
Example:
Exercise:

Problem: Graph on the number line:

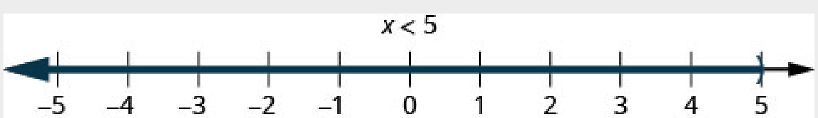
ⓐ $x \leq 1$ ⓑ $x < 5$ ⓒ $x > -1$

Solution:
Solution

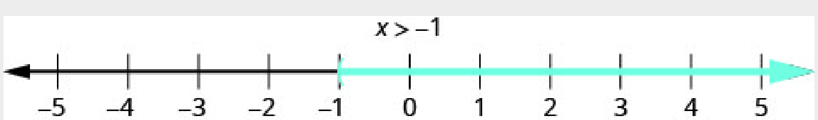
ⓐ $x \leq 1$
This means all numbers less than or equal to 1. We shade in all the numbers on the number line to the left of 1 and put a bracket at $x = 1$ to show that it is included.



ⓑ $x < 5$
This means all numbers less than 5, but not including 5. We shade in all the numbers on the number line to the left of 5 and put a parenthesis at $x = 5$ to show it is not included.



ⓒ $x > -1$
This means all numbers greater than -1, but not including -1. We shade in all the numbers on the number line to the right of -1, then put a parenthesis at $x = -1$ to show it is not included.

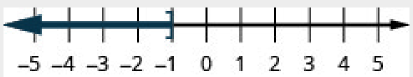


Note:
Exercise:

Problem: Graph on the number line: ⓐ $x \le -1$ ⓑ $x > 2$ ⓒ $x < 3$

Solution:

ⓐ



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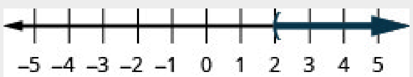

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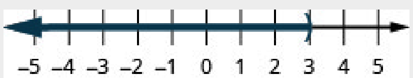
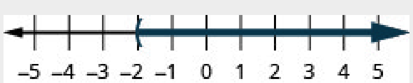


Note:
Exercise:

Problem: Graph on the number line: ⓐ $x > -2$ ⓑ $x < -3$ ⓒ $x \ge -1$

Solution:

ⓐ



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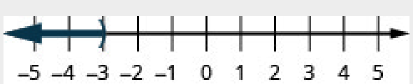
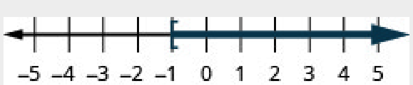
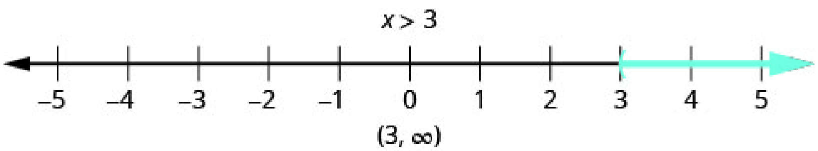


ⓒ

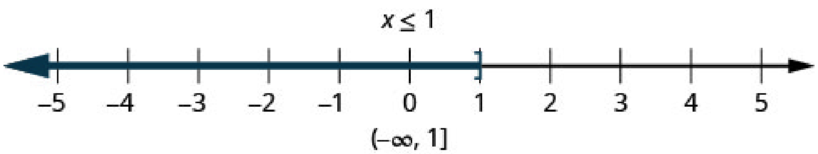


We can also represent inequalities using *interval notation.* As we saw above, the inequality $x > 3$ means all numbers greater than 3. There is no upper end to the solution to this inequality. In interval notation, we express $x > 3$ as $(3, \infty)$. The symbol ∞ is read as 'infinity'. It is not an actual number. [link] shows both the number line and the interval notation.



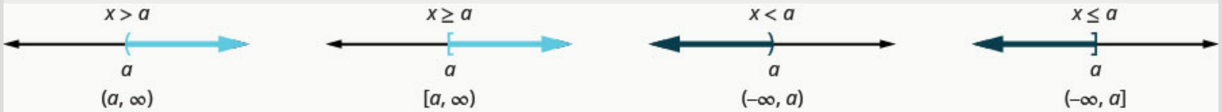
The inequality $x > 3$ is graphed on this number line and written in interval notation.

The inequality $x \leq 1$ means all numbers less than or equal to 1. There is no lower end to those numbers. We write $x \leq 1$ in interval notation as $(-\infty, 1]$. The symbol $-\infty$ is read as 'negative infinity'. [link] shows both the number line and interval notation.



The inequality $x \leq 1$ is graphed on this number line and written in interval notation.

Note:
Inequalities, Number Lines, and Interval Notation



Did you notice how the parenthesis or bracket in the interval notation matches the symbol at the endpoint of the arrow? These relationships are shown in [link].

The notation for inequalities on a number line and in interval notation use similar symbols to express the endpoints of intervals.

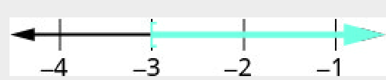
Example:
Exercise:

Problem: Graph on the number line and write in interval notation.

ⓐ $x \geq -3$ ⓑ $x < 2.5$ ⓒ $x \leq -\frac{3}{5}$

Solution:
Solution

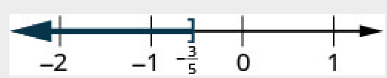
ⓐ

	$x \geq -3$
Shade to the right of -3, and put a bracket at -3.	
Write in interval notation.	$[-3, \infty)$

ⓑ

	$x < 2.5$
Shade to the left of 2.5, and put a parenthesis at 2.5.	
Write in interval notation.	$(-\infty, 2.5)$

ⓒ

	$x \leq -\frac{3}{5}$
Shade to the left of $-\frac{3}{5}$, and put a bracket at $-\frac{3}{5}$.	
Write in interval notation.	$\left(-\infty, -\frac{3}{5}\right]$

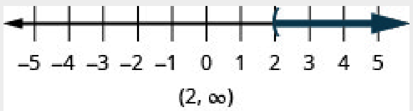
Note:
Exercise:

Problem: Graph on the number line and write in interval notation:

ⓐ $x > 2$ ⓑ $x \leq -1.5$ ⓒ $x \geq \frac{3}{4}$

Solution:

ⓐ



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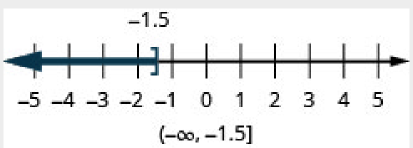

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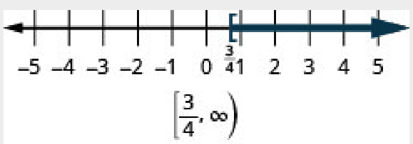


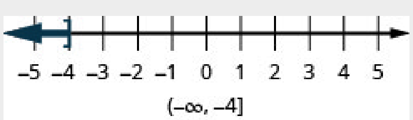
Note:
Exercise:

Problem: Graph on the number line and write in interval notation:

ⓐ $x \leq -4$ ⓑ $x \geq 0.5$ ⓒ $x < -\frac{2}{3}$

Solution:

ⓐ



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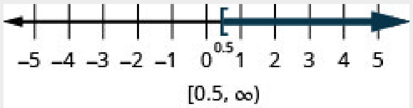

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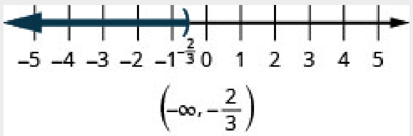


Solve Inequalities using the Subtraction and Addition Properties of Inequality

The Subtraction and Addition Properties of Equality state that if two quantities are equal, when we add or subtract the same amount from both quantities, the results will be equal.

Note:
Properties of Equality
Equation:

$$\begin{array}{llll} \textbf{Subtraction Property of Equality} & & & \\ \text{For any numbers } a, b, \text{ and } c, & & & \\ \text{if} & a & = & b, \\ \text{then} & a - c & = & b - c. \end{array} \qquad \begin{array}{llll} \textbf{Addition Property of Equality} & & & \\ \text{For any numbers } a, b, \text{ and } c, & & & \\ \text{if} & a & = & b, \\ \text{then} & a + c & = & b + c. \end{array}$$

Similar properties hold true for inequalities.

For example, we know that −4 is less than 2.	$-4 < 2$

If we subtract 5 from both quantities, is the left side still less than the right side?	$-4-5\ ?\ 2-5$
We get −9 on the left and −3 on the right.	$-9\ ?\ -3$
And we know −9 is less than −3.	$-9<-3$
	The inequality sign stayed the same.

Similarly we could show that the inequality also stays the same for addition.

This leads us to the Subtraction and Addition Properties of Inequality.

Note:
Properties of Inequality
Equation:

Subtraction Property of Inequality

For any numbers a, b, and c,

$$\begin{array}{lccc} \text{if} & a & < & b \\ \text{then} & a-c & < & b-c. \end{array}$$

$$\begin{array}{lccc} \text{if} & a & > & b \\ \text{then} & a-c & > & b-c. \end{array}$$

Addition Property of Inequality

For any numbers a, b, and c,

$$\begin{array}{lccc} \text{if} & a & < & b \\ \text{then} & a+c & < & b+c. \end{array}$$

$$\begin{array}{lccc} \text{if} & a & > & b \\ \text{then} & a+c & > & b+c. \end{array}$$

We use these properties to solve inequalities, taking the same steps we used to solve equations. Solving the inequality $x+5>9$, the steps would look like this:

	$x+5>9$

Subtract 5 from both sides to isolate x.	$x+5-5>9-5$
Simplify.	$x>4$

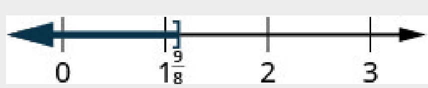
Any number greater than 4 is a solution to this inequality.

Example:
Exercise:

Problem:

Solve the inequality $n-\frac{1}{2}\le\frac{5}{8}$, graph the solution on the number line, and write the solution in interval notation.

Solution:
Solution

	$n-\frac{1}{2}\le\frac{5}{8}$
Add $\frac{1}{2}$ to both sides of the inequality.	$n-\frac{1}{2}+\frac{1}{2}\le\frac{5}{8}+\frac{1}{2}$
Simplify.	$n\le\frac{9}{8}$
Graph the solution on the number line.	
Write the solution in interval notation.	[missing_resource: CNX_ElemAlg_Figure_02_07_026e_img_new.jpg]

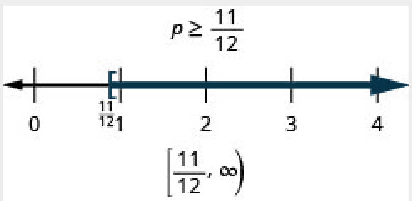
Note:
Exercise:

Problem:

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$p - \frac{3}{4} \geq \frac{1}{6}$

Solution:



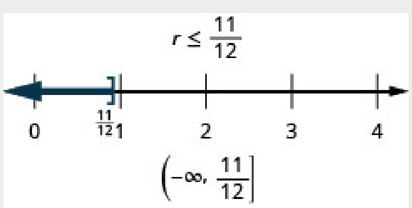
Note:
Exercise:

Problem:

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$r - \frac{1}{3} \leq \frac{7}{12}$

Solution:



Solve Inequalities using the Division and Multiplication Properties of Inequality

The Division and Multiplication Properties of Equality state that if two quantities are equal, when we divide or multiply both quantities by the same amount, the results will also be equal (provided we don't divide by 0).

Note:
Properties of Equality
Equation:

$$\begin{array}{ll} \textbf{Division Property of Equality} & \textbf{Multiplication Property of Equality} \\ \text{For any numbers } a, b, c, \text{and } c \neq 0, & \text{For any real numbers } a, b, c, \\ \quad \begin{array}{lrcl} \text{if} & a & = & b, \\ \text{then} & \frac{a}{c} & = & \frac{b}{c}. \end{array} & \quad \begin{array}{lrcl} \text{if} & a & = & b, \\ \text{then} & ac & = & bc. \end{array} \end{array}$$

Are there similar properties for inequalities? What happens to an inequality when we divide or multiply both sides by a constant?

Consider some numerical examples.

	$10 < 15$		$10 < 15$
Divide both sides by 5.	$\frac{10}{5} \; ? \; \frac{15}{5}$	Multiply both sides by 5.	$10(5) \; ? \; 15(5)$
Simplify.	$2 \; ? \; 3$		$50 \; ? \; 75$
Fill in the inequality signs.	$2 < 3$		$50 < 75$

Equation:

$$\textbf{The inequality signs stayed the same.}$$

Does the inequality stay the same when we divide or multiply by a negative number?

	$10 < 15$		$10 < 15$
Divide both sides by -5.	$\frac{10}{-5} \; ? \; \frac{15}{-5}$	Multiply both sides by -5.	$10(-5) \; ? \; 15(-5)$
Simplify.	$-2 \; ? \; -3$		$-50 \; ? \; -75$
Fill in the inequality signs.	$-2 > -3$		$-50 > -75$

Equation:

$$\textbf{The inequality signs reversed their direction.}$$

When we divide or multiply an inequality by a positive number, the inequality sign stays the same. When we divide or multiply an inequality by a negative number, the inequality sign reverses.

Note: We just observed the reversing of the inequality sign when multiplying or dividing by a negative number. This wasn't just a timely observation but logically necessary.

Consider $a < b$ if we multiply by $-c$ where $c > 0$.
On the right hand side we get $-a \cdot c$ and on the left hand side $-b \cdot c$ How do we know the direction of the inequality?
First multiply by c. Since $c > 0$ we know we get $a \cdot c < b \cdot c$
Subtract $a \cdot c$ from both sides. We get $0 < b \cdot c - a \cdot c$
Subtract $b \cdot c$ from both sides. We get $-b \cdot c < -a \cdot c$.
This is what we wanted to show, just written in the opposite direction: $-a \cdot c > -b \cdot c$.

Here are the Division and Multiplication Properties of Inequality for easy reference.

Note:
Division and Multiplication Properties of Inequality
Equation:

$$\begin{array}{c} \text{For any real numbers } a, b, c \\ \\ \begin{array}{ll} \text{if } a < b \text{ and } c > 0\text{, then} & \frac{a}{c} < \frac{b}{c} \text{ and } ac < bc. \\ \text{if } a > b \text{ and } c > 0\text{, then} & \frac{a}{c} > \frac{b}{c} \text{ and } ac > bc. \\ \text{if } a < b \text{ and } c < 0\text{, then} & \frac{a}{c} > \frac{b}{c} \text{ and } ac > bc. \\ \text{if } a > b \text{ and } c < 0\text{, then} & \frac{a}{c} < \frac{b}{c} \text{ and } ac < bc. \end{array} \end{array}$$

When we **divide or multiply** an inequality by a:

- **positive** number, the inequality stays the **same**.
- **negative** number, the inequality **reverses**.

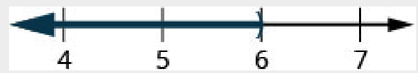
Example:
Exercise:

Problem:

Solve the inequality $7y < 42$, graph the solution on the number line, and write the solution in interval notation.

Solution:
Solution

	$7y < 42$

Divide both sides of the inequality by 7. Since $7>0$, the inequality stays the same.	$\frac{7y}{7}<\frac{42}{7}$
Simplify.	$y<6$
Graph the solution on the number line.	
Write the solution in interval notation.	$(-\infty, 6)$

Note:
Exercise:

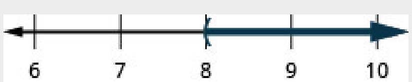
Problem:

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$9c>72$

Solution:

$c>8$



Note:
Exercise:

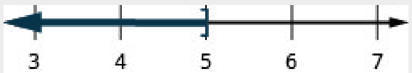
Problem:

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$12d \leq 60$

Solution:

$d \leq 5$



Example:
Exercise:

Problem:

Solve the inequality $-10a \geq 50$, graph the solution on the number line, and write the solution in interval notation.

Solution:
Solution

	$-10a \geq 50$
Divide both sides of the inequality by -10. Since $-10 < 0$, the inequality reverses.	$\frac{-10a}{-10} \leq \frac{50}{-10}$
Simplify.	$a \leq -5$
Graph the solution on the number line.	
Write the solution in interval notation.	$(-\infty, -5]$

Note:
Exercise:

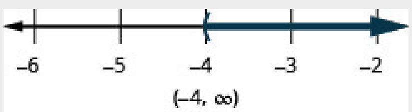
Problem:

Solve each inequality, graph the solution on the number line, and write the solution in interval notation.

$-8q < 32$

Solution:

$q > -4$



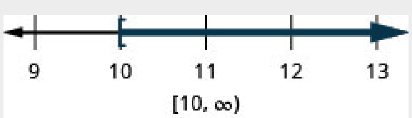
Note:
Exercise:

Problem:

Solve each inequality, graph the solution on the number line, and write the solution in interval notation.

$-7r \leq \ -70$

Solution:



Note:
Solving Inequalities

Sometimes when solving an inequality, the variable ends up on the right. We can rewrite the inequality in reverse to get the variable to the left.

Equation:

$$x > a \text{ has the same meaning as } a < x$$

Think about it as “If Xavier is taller than Alex, then Alex is shorter than Xavier.”

Example:

Exercise:

Problem:

Solve the inequality $-20 < \frac{4}{5}u$, graph the solution on the number line, and write the solution in interval notation.

Solution:

Solution

	$-20 < \frac{4}{5}u$
Multiply both sides of the inequality by $\frac{5}{4}$. Since $\frac{5}{4} > 0$, the inequality stays the same.	$\frac{5}{4}(-20) < \frac{5}{4}\left(\frac{4}{5}u\right)$
Simplify.	$-25 < u$
Rewrite the variable on the left.	$u > -25$
Graph the solution on the number line.	

Write the solution in interval notation.

(–25, ∞)

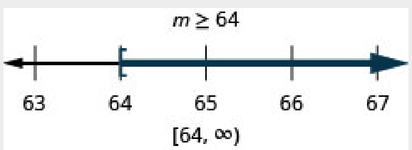
Note:
Exercise:

Problem:

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$24 \leq \frac{3}{8}m$

Solution:



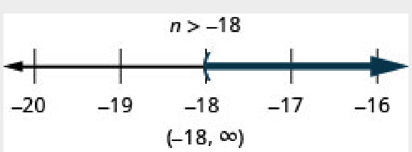
Note:
Exercise:

Problem:

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$-24 < \frac{4}{3}n$

Solution:

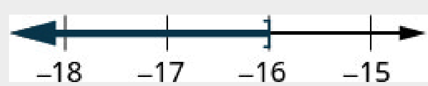


Example:
Exercise:

Problem:

Solve the inequality $\frac{t}{-2} \geq 8$, graph the solution on the number line, and write the solution in interval notation.

Solution:
Solution

	$\frac{t}{-2} \geq 8$
Multiply both sides of the inequality by -2 . Since $-2 < 0$, the inequality reverses.	$-2\left(\frac{t}{-2}\right) \leq -2(8)$
Simplify.	$t \leq -16$
Graph the solution on the number line.	
Write the solution in interval notation.	$(-\infty, -16]$

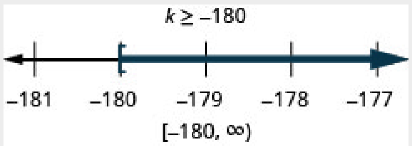
Note:
Exercise:

Problem:

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$\frac{k}{-12} \leq 15$

Solution:



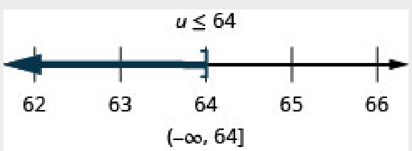
Note:
Exercise:

Problem:

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$\frac{u}{-4} \geq -16$

Solution:



Solve Inequalities That Require Simplification

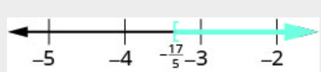
Most inequalities will take more than one step to solve. We follow the same steps we used in the general strategy for solving linear equations, but be sure to pay close attention during multiplication or division.

Example:
Exercise:

Problem:

Solve the inequality $4m \leq 9m + 17$, graph the solution on the number line, and write the solution in interval notation.

Solution:
Solution

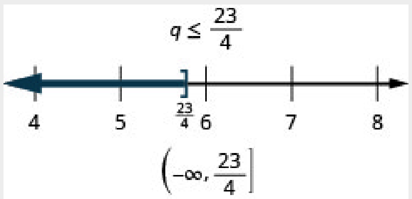
	$4m \leq 9m + 17$
Subtract $9m$ from both sides to collect the variables on the left.	$4m - 9m \leq 9m - 9m + 17$
Simplify.	$-5m \leq 17$
Divide both sides of the inequality by -5, and reverse the inequality.	$\frac{-5m}{-5} \geq \frac{17}{-5}$
Simplify.	$m \geq -\frac{17}{5}$
Graph the solution on the number line.	
Write the solution in interval notation.	$\left[-\frac{17}{5}, \infty\right)$

Note:
Exercise:

Problem:

Solve the inequality $3q \geq 7q - 23$, graph the solution on the number line, and write the solution in interval notation.

Solution:

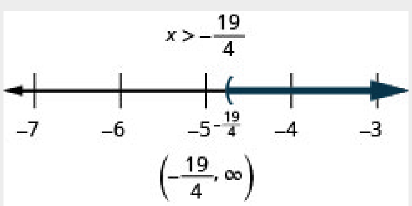


Note:
Exercise:

Problem:

Solve the inequality $6x < 10x + 19$, graph the solution on the number line, and write the solution in interval notation.

Solution:



Example:
Exercise:

Problem:

Solve the inequality $8p + 3(p - 12) > 7p - 28$, graph the solution on the number line, and write the solution in interval notation.

Solution:
Solution

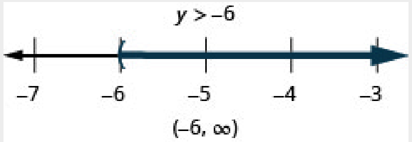
Simplify each side as much as possible.	$8p + 3(p - 12) > 7p - 28$
Distribute.	$8p + 3p - 36 > 7p - 28$
Combine like terms.	$11p - 36 > 7p - 28$
Subtract $7p$ from both sides to collect the variables on the left.	$11p - 36 - 7p > 7p - 28 - 7p$
Simplify.	$4p - 36 > -28$
Add 36 to both sides to collect the constants on the right.	$4p - 36 + 36 > -28 + 36$
Simplify.	$4p > 8$
Divide both sides of the inequality by 4; the inequality stays the same.	$\frac{4p}{4} > \frac{8}{4}$
Simplify.	$p > 2$
Graph the solution on the number line.	
Write the solution in interal notation.	$(2, \infty)$

Note:
Exercise:

Problem:

Solve the inequality $9y+2(y+6)>5y-24$, graph the solution on the number line, and write the solution in interval notation.

Solution:

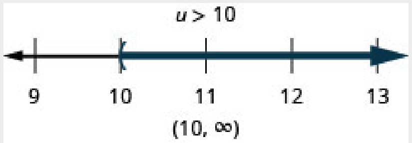


Note:
Exercise:

Problem:

Solve the inequality $6u+8(u-1)>10u+32$, graph the solution on the number line, and write the solution in interval notation.

Solution:



Just like some equations are identities and some are contradictions, inequalities may be identities or contradictions, too. We recognize these forms when we are left with only constants as we solve the inequality. If the result is a true statement, we have an identity. If the result is a false statement, we have a contradiction.

Example:
Exercise:

Problem:

Solve the inequality $8x - 2\,(5 - x) < 4\,(x + 9) + 6x$, graph the solution on the number line, and write the solution in interval notation.

Solution:
Solution

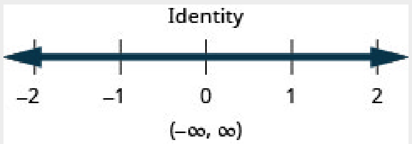
Simplify each side as much as possible.	$8x - 2(5 - x) < 4(x + 9) + 6x$
Distribute.	$8x - 10 + 2x < 4x + 36 + 6x$
Combine like terms.	$10x - 10 < 10x + 36$
Subtract $10x$ from both sides to collect the variables on the left.	$10x - 10 - 10x < 10x + 36 - 10x$
Simplify.	$-10 < 36$
The x's are gone, and we have a true statement.	The inequality is an identity. The solution is all real numbers.
Graph the solution on the number line.	
Write the solution in interval notation.	$(-\infty, \infty)$

Note:
Exercise:

Problem:

Solve the inequality $4b - 3(3 - b) > 5(b - 6) + 2b$, graph the solution on the number line, and write the solution in interval notation.

Solution:

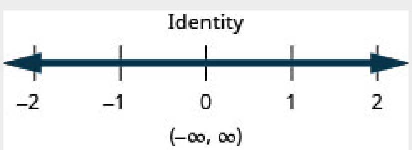


Note:
Exercise:

Problem:

Solve the inequality $9h - 7\left(2 - h\right) < 8\left(h + 11\right) + 8h$, graph the solution on the number line, and write the solution in interval notation.

Solution:



Example:
Exercise:

Problem:

Solve the inequality $\frac{1}{3}a - \frac{1}{8}a > \frac{5}{24}a + \frac{3}{4}$, graph the solution on the number line, and write the solution in interval notation.

Solution:
Solution

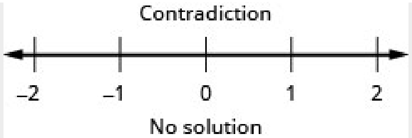
	$\frac{1}{3}a - \frac{1}{8}a > \frac{5}{24}a + \frac{3}{4}$
Multiply both sides by the LCD, 24, to clear the fractions.	$24\left(\frac{1}{3}a - \frac{1}{8}a\right) > 24\left(\frac{5}{24}a + \frac{3}{4}\right)$
Simplify.	$8a - 3a > 5a + 18$
Combine like terms.	$5a > 5a + 18$
Subtract $5a$ from both sides to collect the variables on the left.	$5a - 5a > 5a - 5a + 18$
Simplify.	$0 > 18$
The statement is false!	The inequality is a contradiction.
	There is no solution.
Graph the solution on the number line.	
Write the solution in interval notation.	There is no solution.

Note:
Exercise:

Problem:

Solve the inequality $\frac{1}{4}x - \frac{1}{12}x > \frac{1}{6}x + \frac{7}{8}$, graph the solution on the number line, and write the solution in interval notation.

Solution:

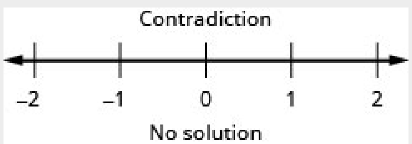


Note:
Exercise:

Problem:

Solve the inequality $\frac{2}{5}z - \frac{1}{3}z < \frac{1}{15}z - \frac{3}{5}$, graph the solution on the number line, and write the solution in interval notation.

Solution:



Translate to an Inequality and Solve

To translate English sentences into inequalities, we need to recognize the phrases that indicate the inequality. Some words are easy, like ‘more than’ and ‘less than’. But others are not as obvious.

Think about the phrase ‘at least’ – what does it mean to be ‘at least 21 years old’? It means 21 or more. The phrase ‘at least’ is the same as ‘greater than or equal to’.

[link] shows some common phrases that indicate inequalities.

$>$	$\geq$	$<$	$\leq$
is greater than	is greater than or equal to	is less than	is less than or equal to

$>$	$\geq$	$<$	$\leq$
is more than	is at least	is smaller than	is at most
is larger than	is no less than	has fewer than	is no more than
exceeds	is the minimum	is lower than	is the maximum

Example:
Exercise:

Problem:

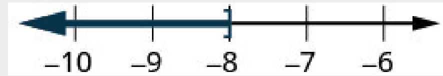
Translate and solve. Then write the solution in interval notation and graph on the number line.

Twelve times *c* is no more than 96.

Solution:
Solution

Translate.	Twelve times c is no more than 96 $12c \leq 96$
Solve—divide both sides by 12.	$\frac{12c}{12} \leq \frac{96}{12}$
Simplify.	$c \leq 8$
Write in interval notation.	$(-\infty, -8]$

Graph on the number line.



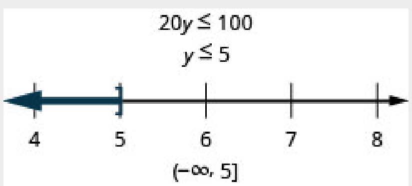
Note:
Exercise:

Problem:

Translate and solve. Then write the solution in interval notation and graph on the number line.

Twenty times y is at most 100

Solution:



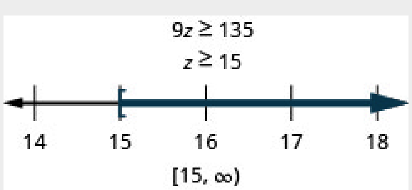
Note:
Exercise:

Problem:

Translate and solve. Then write the solution in interval notation and graph on the number line.

Nine times z is no less than 135

Solution:



Example:
Exercise:

Problem:

Translate and solve. Then write the solution in interval notation and graph on the number line.

Thirty less than x is at least 45.

Solution:
Solution

Translate.	Thirty less than x $\underbrace{\text{is at least}}$ 45. $x - 30 \geq 45$
Solve—add 30 to both sides.	$x - 30 + 30 \geq 45 + 30$
Simplify.	$x \geq 75$
Write in interval notation.	$[75, \infty)$
Graph on the number line.	

Note:

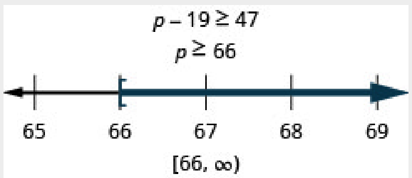
Exercise:

Problem:

Translate and solve. Then write the solution in interval notation and graph on the number line.

Nineteen less than p is no less than 47

Solution:



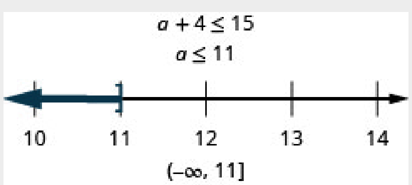
Note:
Exercise:

Problem:

Translate and solve. Then write the solution in interval notation and graph on the number line.

Four more than a is at most 15.

Solution:



Key Concepts

- **Subtraction Property of Inequality**
  For any numbers a, b, and c,
  if $a < b$ then $a - c < b - c$ and
  if $a > b$ then $a - c > b - c$.

- **Addition Property of Inequality**
  For any numbers a, b, and c,
  if $a < b$ then $a + c < b + c$ and
  if $a > b$ then $a + c > b + c$.
- **Division and Multiplication Properties of Inequality**
  For any numbers a, b, and c,
  if $a < b$ and $c > 0$, then $\frac{a}{c} < \frac{b}{c}$ and $ac > bc$.
  if $a > b$ and $c > 0$, then $\frac{a}{c} > \frac{b}{c}$ and $ac > bc$.
  if $a < b$ and $c < 0$, then $\frac{a}{c} > \frac{b}{c}$ and $ac > bc$.
  if $a > b$ and $c < 0$, then $\frac{a}{c} < \frac{b}{c}$ and $ac < bc$.
- When we **divide or multiply** an inequality by a:
  - **positive** number, the inequality stays the **same**.
  - **negative** number, the inequality **reverses**.

Section Exercises

Practice Makes Perfect

Graph Inequalities on the Number Line

In the following exercises, graph each inequality on the number line.
Exercise:

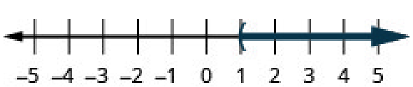
Problem: ⓐ $x \leq 2$
ⓑ $x > -1$
ⓒ $x < 0$

Exercise:

Problem: ⓐ $x > 1$
ⓑ $x < -2$
ⓒ $x \geq -3$

Solution:

ⓐ



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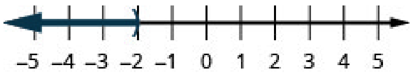

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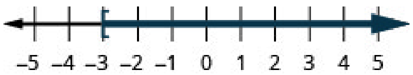


Exercise:

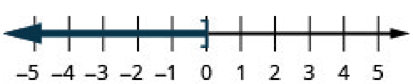
Problem:
ⓐ $x \geq -3$
ⓑ $x < 4$
ⓒ $x \leq -2$

Exercise:

Problem:
ⓐ $x \leq 0$
ⓑ $x > -4$
ⓒ $x \geq -1$

Solution:

ⓐ



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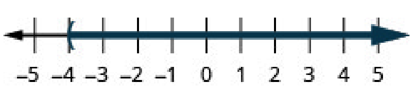

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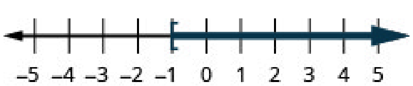


In the following exercises, graph each inequality on the number line and write in interval notation.

Exercise:

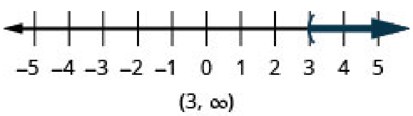
ⓐ $x < -2$
ⓑ $x \geq -3.5$
Problem: ⓒ $x \leq \frac{2}{3}$

Exercise:

ⓐ $x > 3$
ⓑ $x \leq -0.5$
Problem: ⓒ $x \geq \frac{1}{3}$

Solution:

ⓐ



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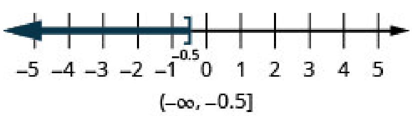

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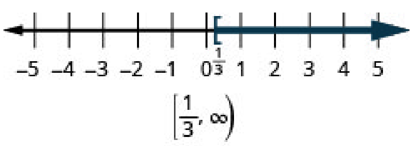


Exercise:

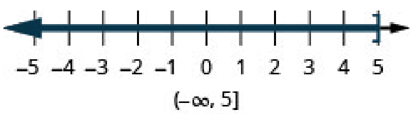
ⓐ $x \geq -4$
ⓑ $x < 2.5$
Problem: ⓒ $x > -\frac{3}{2}$

Exercise:

ⓐ $x \leq 5$
ⓑ $x \geq -1.5$
Problem: ⓒ $x < -\frac{7}{3}$

Solution:

ⓐ



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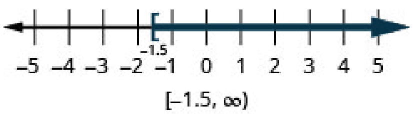

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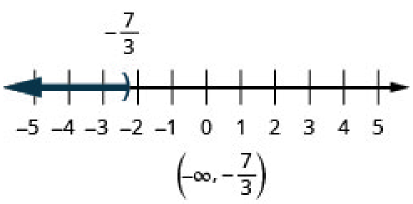


Solve Inequalities using the Subtraction and Addition Properties of Inequality

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

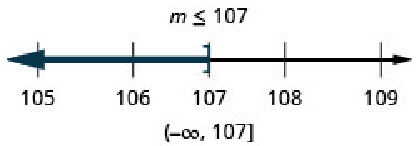
Exercise:

Problem: $n - 11 < 33$

Exercise:

Problem: $m - 45 \leq 62$

Solution:



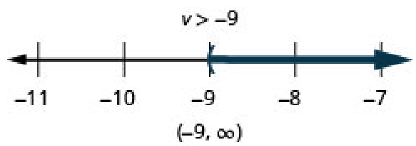
Exercise:

Problem: $u + 25 > 21$

Exercise:

Problem: $v + 12 > 3$

Solution:



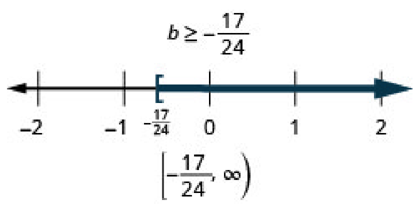
Exercise:

Problem: $a + \frac{3}{4} \geq \frac{7}{10}$

Exercise:

Problem: $b + \frac{7}{8} \geq \frac{1}{6}$

Solution:



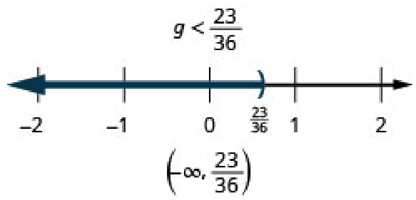
Exercise:

Problem: $f - \frac{13}{20} < -\frac{5}{12}$

Exercise:

Problem: $g - \frac{11}{12} < -\frac{5}{18}$

Solution:



Solve Inequalities using the Division and Multiplication Properties of Inequality

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

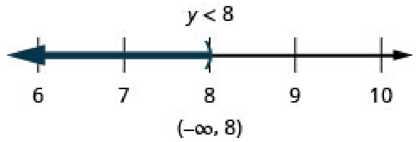
Exercise:

Problem: $8x > 72$

Exercise:

Problem: $6y < 48$

Solution:



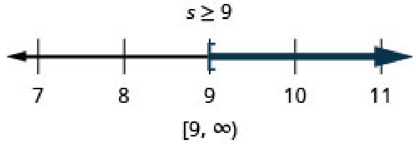
Exercise:

Problem: $7r \leq 56$

Exercise:

Problem: $9s \geq 81$

Solution:



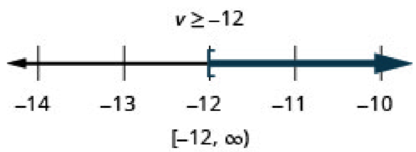
Exercise:

Problem: $-5u \geq 65$

Exercise:

Problem: $-8v \leq 96$

Solution:



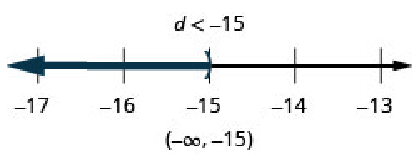
Exercise:

Problem: $-9c < 126$

Exercise:

Problem: $-7d > 105$

Solution:



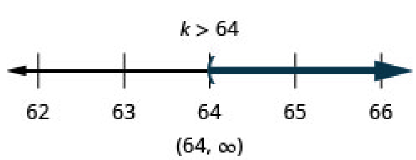
Exercise:

Problem: $20 > \frac{2}{5}h$

Exercise:

Problem: $40 < \frac{5}{8}k$

Solution:



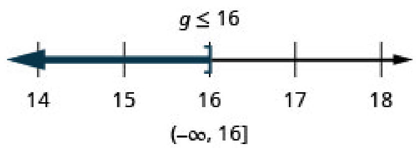
Exercise:

Problem: $\frac{7}{6}j \geq 42$

Exercise:

Problem: $\frac{9}{4}g \leq 36$

Solution:



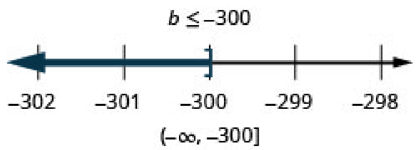
Exercise:

Problem: $\frac{a}{-3} \leq 9$

Exercise:

Problem: $\frac{b}{-10} \geq 30$

Solution:



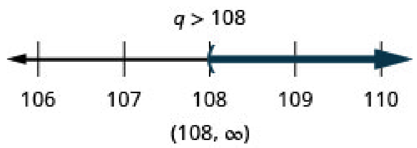
Exercise:

Problem: $-25 < \frac{p}{-5}$

Exercise:

Problem: $-18 > \frac{q}{-6}$

Solution:



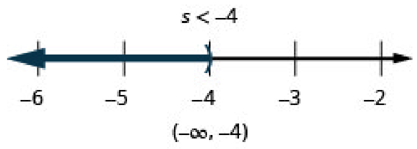
Exercise:

Problem: $9t \geq -27$

Exercise:

Problem: $7s < -28$

Solution:



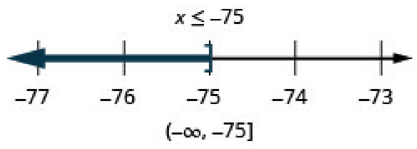
Exercise:

Problem: $\frac{2}{3}y > -36$

Exercise:

Problem: $\frac{3}{5}x \leq -45$

Solution:



Solve Inequalities That Require Simplification

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

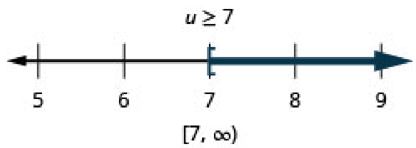
Exercise:

Problem: $4v \geq 9v - 40$

Exercise:

Problem: $5u \leq 8u - 21$

Solution:



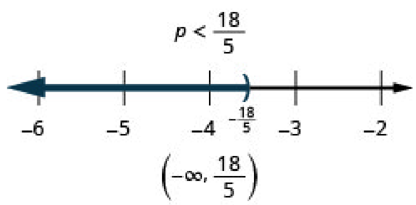
Exercise:

Problem: $13q < 7q - 29$

Exercise:

Problem: $9p > 14p - 18$

Solution:



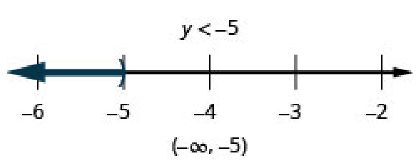
Exercise:

Problem: $12x + 3(x + 7) > 10x - 24$

Exercise:

Problem: $9y + 5(y + 3) < 4y - 35$

Solution:



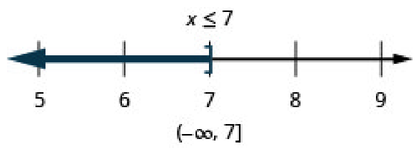
Exercise:

Problem: $6h - 4(h - 1) \leq 7h - 11$

Exercise:

Problem: $4k - (k - 2) \geq 7k - 26$

Solution:



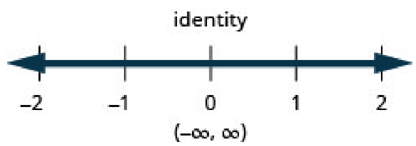
Exercise:

Problem: $8m - 2\left(14 - m\right) \geq 7\left(m - 4\right) + 3m$

Exercise:

Problem: $6n - 12\left(3 - n\right) \leq 9\left(n - 4\right) + 9n$

Solution:



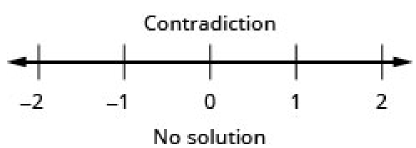
Exercise:

Problem: $\frac{3}{4}b - \frac{1}{3}b < \frac{5}{12}b - \frac{1}{2}$

Exercise:

Problem: $9u + 5(2u - 5) \geq 12(u - 1) + 7u$

Solution:



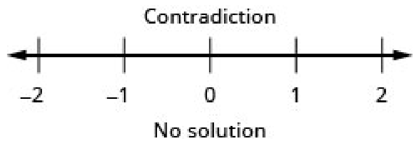
Exercise:

Problem: $\frac{2}{3}g - \frac{1}{2}(g - 14) \leq \frac{1}{6}(g + 42)$

Exercise:

Problem: $\frac{5}{6}a - \frac{1}{4}a > \frac{7}{12}a + \frac{2}{3}$

Solution:



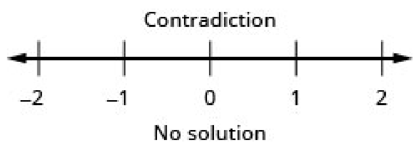
Exercise:

Problem: $\frac{4}{5}h-\frac{2}{3}(h-9)\geq\frac{1}{15}(2h+90)$

Exercise:

Problem: $12v+3(4v-1)\leq 19(v-2)+5v$

Solution:



Mixed practice

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

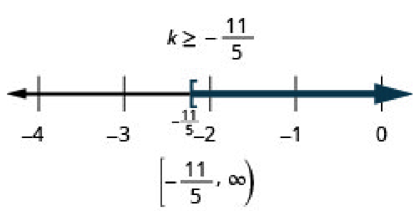
Exercise:

Problem: $15k\leq -40$

Exercise:

Problem: $35k\geq -77$

Solution:



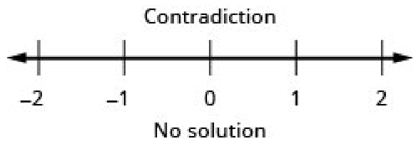
Exercise:

Problem: $23p-2(6-5p)>3(11p-4)$

Exercise:

Problem: $18q-4(10-3q)<5(6q-8)$

Solution:



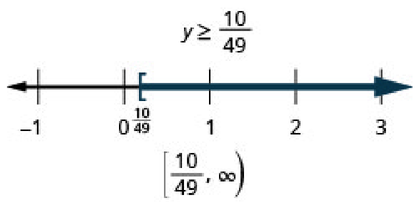
Exercise:

Problem: $-\frac{9}{4}x \geq -\frac{5}{12}$

Exercise:

Problem: $-\frac{21}{8}y \leq -\frac{15}{28}$

Solution:



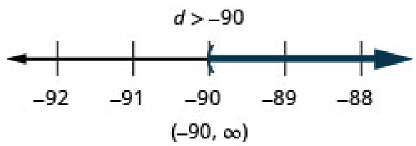
Exercise:

Problem: $c+34<-99$

Exercise:

Problem: $d+29>-61$

Solution:



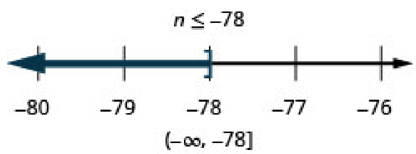
Exercise:

Problem: $\frac{m}{18} \geq -4$

Exercise:

Problem: $\frac{n}{13} \leq -6$

Solution:



Translate to an Inequality and Solve

In the following exercises, translate and solve .Then write the solution in interval notation and graph on the number line.

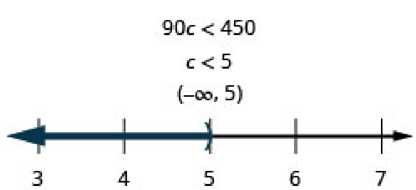
Exercise:

Problem: Fourteen times *d* is greater than 56.

Exercise:

Problem: Ninety times *c* is less than 450.

Solution:



Exercise:

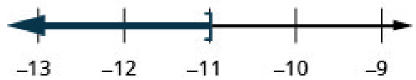
Problem: Eight times *z* is smaller than -40.

Exercise:

Problem: Ten times *y* is at most -110.

Solution:

$$-10y \leq -110$$
$$y \leq -11$$
$$(-\infty, -11]$$



Exercise:

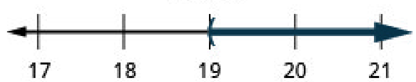
Problem: Three more than *h* is no less than 25.

Exercise:

Problem: Six more than *k* exceeds 25.

Solution:

$$k + 6 > 25$$
$$k > 19$$
$$(19, \infty)$$



Exercise:

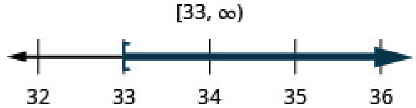
Problem: Ten less than *w* is at least 39.

Exercise:

Problem: Twelve less than *x* is no less than 21.

Solution:

$$x - 12 \geq 21$$
$$x \geq 33$$
$$[33, \infty)$$



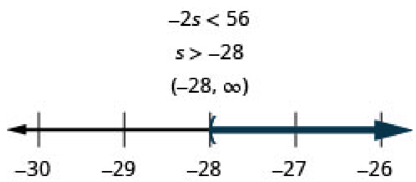
Exercise:

Problem: Negative five times *r* is no more than 95.

Exercise:

Problem: Negative two times s is lower than 56.

Solution:



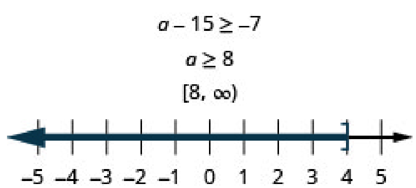
Exercise:

Problem: Nineteen less than b is at most -22.

Exercise:

Problem: Fifteen less than a is at least -7.

Solution:



Everyday Math

Exercise:

Problem:

Safety A child’s height, h, must be at least 57 inches for the child to safely ride in the front seat of a car. Write this as an inequality.

Exercise:

Problem:

Fighter pilots The maximum height, h, of a fighter pilot is 77 inches. Write this as an inequality.

Solution:

$h \leq 77$

Exercise:

Problem:

Elevators The total weight, *w*, of an elevator's passengers can be no more than 1,200 pounds. Write this as an inequality.

Exercise:

Problem:

Shopping The number of items, *n*, a shopper can have in the express check-out lane is at most 8. Write this as an inequality.

Solution:

$n \leq 8$

Writing Exercises

Exercise:

Problem: Give an example from your life using the phrase 'at least'.

Exercise:

Problem: Give an example from your life using the phrase 'at most'.

Solution:

Answers will vary.

Exercise:

Problem:

Explain why it is necessary to reverse the inequality when solving $-5x > 10$.

Exercise:

Problem:

Explain why it is necessary to reverse the inequality when solving $\frac{n}{-3} < 12$.

Solution:

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
graph inequalities on the number line.			
solve inequalities using the Subtraction and Addition Properties of Inequality.			
solve inequalities using the Division and Multiplication Properties of Inequality.			
solve inequalities that require simplification.			
translate to an inequality and solve.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Chapter Review Exercises

Solve Equations using the Subtraction and Addition Properties of Equality

Verify a Solution of an Equation

In the following exercises, determine whether each number is a solution to the equation.
Exercise:

Problem: $10x - 1 = 5x; x = \frac{1}{5}$

Exercise:

Problem: $w + 2 = \frac{5}{8}; w = \frac{3}{8}$

Solution:

no

Exercise:

Problem: $-12n + 5 = 8n; n = -\frac{5}{4}$

Exercise:

Problem: $6a - 3 = -7a, a = \frac{3}{13}$

Solution:

yes

Solve Equations using the Subtraction and Addition Properties of Equality

In the following exercises, solve each equation using the Subtraction Property of Equality.

Exercise:

Problem: $x + 7 = 19$

Exercise:

Problem: $y + 2 = -6$

Solution:

$y = -8$

Exercise:

Problem: $a + \frac{1}{3} = \frac{5}{3}$

Exercise:

Problem: $n + 3.6 = 5.1$

Solution:

$n = 1.5$

In the following exercises, solve each equation using the Addition Property of Equality.

Exercise:

Problem: $u - 7 = 10$

Exercise:

Problem: $x - 9 = -4$

Solution:

$x = 5$

Exercise:

Problem: $c - \frac{3}{11} = \frac{9}{11}$

Exercise:

Problem: $p - 4.8 = 14$

Solution:

$p = 18.8$

In the following exercises, solve each equation.

Exercise:

Problem: $n - 12 = 32$

Exercise:

Problem: $y + 16 = -9$

Solution:

$y = -25$

Exercise:

Problem: $f + \frac{2}{3} = 4$

Exercise:

Problem: $d - 3.9 = 8.2$

Solution:

$d = 12.1$

Solve Equations That Require Simplification

In the following exercises, solve each equation.

Exercise:

Problem: $y+8-15=-3$

Exercise:

Problem: $7x+10-6x+3=5$

Solution:

$x=-8$

Exercise:

Problem: $6(n-1)-5n=-14$

Exercise:

Problem: $8(3p+5)-23(p-1)=35$

Solution:

$p=-28$

Translate to an Equation and Solve

In the following exercises, translate each English sentence into an algebraic equation and then solve it.

Exercise:

Problem: The sum of -6 and m is 25.

Exercise:

Problem: Four less than n is 13.

Solution:

$n-4=13; n=17$

Translate and Solve Applications

In the following exercises, translate into an algebraic equation and solve.

Exercise:

Problem:

Rochelle’s daughter is 11 years old. Her son is 3 years younger. How old is her son?

Exercise:

Problem:

Tan weighs 146 pounds. Minh weighs 15 pounds more than Tan. How much does Minh weigh?

Solution:

161 pounds

Exercise:

Problem:

Peter paid $9.75 to go to the movies, which was $46.25 less than he paid to go to a concert. How much did he pay for the concert?

Exercise:

Problem:

Elissa earned $152.84 this week, which was $21.65 more than she earned last week. How much did she earn last week?

Solution:

$131.19

Solve Equations using the Division and Multiplication Properties of Equality

Solve Equations Using the Division and Multiplication Properties of Equality

In the following exercises, solve each equation using the division and multiplication properties of equality and check the solution.

Exercise:

Problem: $8x = 72$

Exercise:

Problem: $13a = -65$

Solution:

$a = -5$

Exercise:

Problem: $0.25p = 5.25$

Exercise:

Problem: $-y = 4$

Solution:

$y = -4$

Exercise:

Problem: $\frac{n}{6} = 18$

Exercise:

Problem: $\frac{y}{-10} = 30$

Solution:

$y = -300$

Exercise:

Problem: $36 = \frac{3}{4}x$

Exercise:

Problem: $\frac{5}{8}u = \frac{15}{16}$

Solution:

$u = \frac{3}{2}$

Exercise:

Problem: $-18m = -72$

Exercise:

Problem: $\frac{c}{9} = 36$

Solution:

$c = 324$

Exercise:

Problem: $0.45x = 6.75$

Exercise:

Problem: $\frac{11}{12} = \frac{2}{3}y$

Solution:

$y = \frac{11}{8}$

Solve Equations That Require Simplification

In the following exercises, solve each equation requiring simplification.

Exercise:

Problem: $5r - 3r + 9r = 35 - 2$

Exercise:

Problem: $24x + 8x - 11x = -7 - 14$

Solution:

$x = -1$

Exercise:

Problem: $\frac{11}{12}n - \frac{5}{6}n = 9 - 5$

Exercise:

Problem: $-9(d - 2) - 15 = -24$

Solution:

$d = 3$

Translate to an Equation and Solve

In the following exercises, translate to an equation and then solve.

Exercise:

Problem: 143 is the product of -11 and *y*.

Exercise:

Problem: The quotient of *b* and and 9 is -27.

Solution:

$\frac{b}{9} = -27; b = -243$

Exercise:

Problem: The sum of *q* and one-fourth is one.

Exercise:

Problem: The difference of *s* and one-twelfth is one fourth.

Solution:

$s - \frac{1}{12} = \frac{1}{4}; s = \frac{1}{3}$

Translate and Solve Applications

In the following exercises, translate into an equation and solve.

Exercise:

Problem:

Ray paid $21 for 12 tickets at the county fair. What was the price of each ticket?

Exercise:

Problem:

Janet gets paid $24 per hour. She heard that this is $\frac{3}{4}$ of what Adam is paid. How much is Adam paid per hour?

Solution:

$32

Solve Equations with Variables and Constants on Both Sides

Solve an Equation with Constants on Both Sides

In the following exercises, solve the following equations with constants on both sides.
Exercise:

Problem: $8p + 7 = 47$

Exercise:

Problem: $10w - 5 = 65$

Solution:

$w = 7$

Exercise:

Problem: $3x + 19 = -47$

Exercise:

Problem: $32 = -4 - 9n$

Solution:

$n = -4$

Solve an Equation with Variables on Both Sides

In the following exercises, solve the following equations with variables on both sides.
Exercise:

Problem: $7y = 6y - 13$

Exercise:

Problem: $5a + 21 = 2a$

Solution:

$a = -7$

Exercise:

Problem: $k = -6k - 35$

Exercise:

Problem: $4x - \frac{3}{8} = 3x$

Solution:

$x = \frac{3}{8}$

Solve an Equation with Variables and Constants on Both Sides

In the following exercises, solve the following equations with variables and constants on both sides.

Exercise:

Problem: $12x - 9 = 3x + 45$

Exercise:

Problem: $5n - 20 = -7n - 80$

Solution:

$n = -5$

Exercise:

Problem: $4u + 16 = -19 - u$

Exercise:

Problem: $\frac{5}{8}c - 4 = \frac{3}{8}c + 4$

Solution:

$c = 32$

[Use a General Strategy for Solving Linear Equations]

Solve Equations Using the General Strategy for Solving Linear Equations

In the following exercises, solve each linear equation.

Exercise:

Problem: $6(x + 6) = 24$

Exercise:

Problem: $9(2p - 5) = 72$

Solution:

$p = \frac{13}{2}$

Exercise:

Problem: $-(s + 4) = 18$

Exercise:

Problem: $8 + 3(n - 9) = 17$

Solution:

$n = 12$

Exercise:

Problem: $23 - 3(y - 7) = 8$

Exercise:

Problem: $\frac{1}{3}(6m + 21) = m - 7$

Solution:

$m = -14$

Exercise:

Problem: $4(3.5y + 0.25) = 365$

Exercise:

Problem: $0.25(q - 8) = 0.1(q + 7)$

Solution:

$q = 18$

Exercise:

Problem: $8(r - 2) = 6(r + 10)$

Exercise:

Problem: $5 + 7(2 - 5x) = 2(9x + 1) -(13x - 57)$

Solution:

$x = -1$

Exercise:

Problem: $(9n + 5) - (3n - 7) = 20 - (4n - 2)$

Exercise:

Problem: $2[-16 + 5(8k - 6)] = 8(3 - 4k) - 32$

Solution:

$k = \frac{3}{4}$

Classify Equations

In the following exercises, classify each equation as a conditional equation, an identity, or a contradiction and then state the solution.

Exercise:

Problem: $17y - 3(4 - 2y) = 11(y - 1) + 12y - 1$

Exercise:

Problem: $9u + 32 = 15(u - 4) - 3(2u + 21)$

Solution:

contradiction; no solution

Exercise:

Problem: $-8(7m+4)=-6(8m+9)$

Exercise:

Problem: $21(c-1)-19(c+1) = 2(c-20)$

Solution:

identity; all real numbers

Solve Equations with Fractions and Decimals

Solve Equations with Fraction Coefficients

In the following exercises, solve each equation with fraction coefficients.

Exercise:

Problem: $\frac{2}{5}n-\frac{1}{10}=\frac{7}{10}$

Exercise:

Problem: $\frac{1}{3}x+\frac{1}{5}x=8$

Solution:

$x=15$

Exercise:

Problem: $\frac{3}{4}a-\frac{1}{3}=\frac{1}{2}a-\frac{5}{6}$

Exercise:

Problem: $\frac{1}{2}(k-3)=\frac{1}{3}(k+16)$

Solution:

$k=41$

Exercise:

Problem: $\frac{3x-2}{5}=\frac{3x+4}{8}$

Exercise:

Problem: $\frac{5y-1}{3}+4=\frac{-8y+4}{6}$

Solution:

$y=-1$

Solve Equations with Decimal Coefficients

In the following exercises, solve each equation with decimal coefficients.
Exercise:

Problem: $0.8x-0.3=0.7x+0.2$

Exercise:

Problem: $0.36u+2.55=0.41u+6.8$

Solution:

$u=-85$

Exercise:

Problem: $0.6p-1.9=0.78p+1.7$

Exercise:

Problem: $0.6p-1.9=0.78p+1.7$

Solution:

$d=-20$

[Solve a Formula for a Specific Variable]

Use the Distance, Rate, and Time Formula

In the following exercises, solve.

Exercise:

Problem:

Natalie drove for $7\frac{1}{2}$ hours at 60 miles per hour. How much distance did she travel?

Exercise:

Problem:

Mallory is taking the bus from St. Louis to Chicago. The distance is 300 miles and the bus travels at a steady rate of 60 miles per hour. How long will the bus ride be?

Solution:

5 hours

Exercise:

Problem:

Aaron's friend drove him from Buffalo to Cleveland. The distance is 187 miles and the trip took 2.75 hours. How fast was Aaron's friend driving?

Exercise:

Problem:

Link rode his bike at a steady rate of 15 miles per hour for $2\frac{1}{2}$ hours. How much distance did he travel?

Solution:

37.5 miles

Solve a Formula for a Specific Variable

In the following exercises, solve.

Exercise:

Problem: Use the formula. $d = rt$ to solve for t
ⓐ when $d = 510$ and $r = 60$
ⓑ in general

Exercise:

Problem: Use the formula. $d=rt$ to solve for r
ⓐ when when $d=451$ and $t=5.5$
ⓑ in general

Solution:

ⓐ $r=82\,\text{mph}$; ⓑ $r=\frac{D}{t}$

Exercise:

Problem: Use the formula $A=\frac{1}{2}bh$ to solve for b
ⓐ when $A=390$ and $h=26$
ⓑ in general

Exercise:

Problem: Use the formula $A=\frac{1}{2}bh$ to solve for h
ⓐ when $A=153$ and $b=18$
ⓑ in general

Solution:

ⓐ $h=17$ ⓑ $h=\frac{2A}{b}$

Exercise:

Problem: Use the formula $I=Prt$ to solve for the principal, P for
ⓐ $I=\$2,501, r=4.1\%,$
$t=5$ years
ⓑ in general

Exercise:

Problem: Solve the formula $4x+3y=6$ for y
ⓐ when $x=-2$
ⓑ in general

Solution:

ⓐ $y=\frac{14}{3}$ ⓑ $y=\frac{6-4x}{3}$

Exercise:

Problem: Solve $180=a+b+c$ for c.

Exercise:

Problem: Solve the formula $V = LWH$ for H.

Solution:

$H = \frac{V}{LW}$

Solve Linear Inequalities

Graph Inequalities on the Number Line

In the following exercises, graph each inequality on the number line.

Exercise:

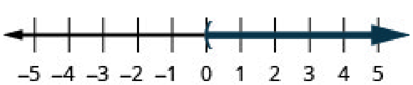
Problem: ⓐ $x \leq 4$ ⓑ $x > -2$ ⓒ $x < 1$

Exercise:

Problem: ⓐ $x > 0$ ⓑ $x < -3$ ⓒ $x \geq -1$

Solution:

ⓐ



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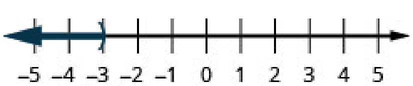

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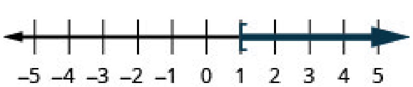


In the following exercises, graph each inequality on the number line and write in interval notation.

Exercise:

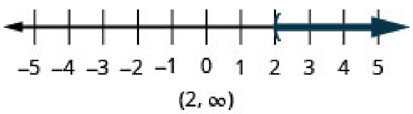
Problem: ⓐ $x < -1$
ⓑ $x \geq -2.5$
ⓒ $x \leq \frac{5}{4}$

Exercise:

Problem: ⓐ $x > 2$
ⓑ $x \leq -1.5$
ⓒ $x \geq \frac{5}{3}$

Solution:

ⓐ



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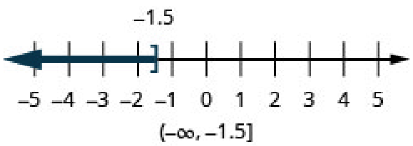

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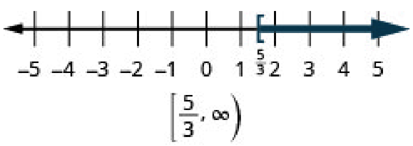


Solve Inequalities using the Subtraction and Addition Properties of Inequality

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

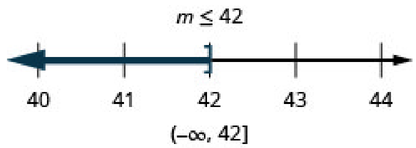
Exercise:

Problem: $n - 12 \leq 23$

Exercise:

Problem: $m + 14 \leq 56$

Solution:



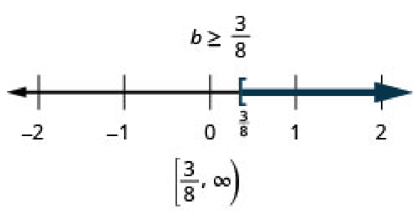
Exercise:

Problem: $a + \frac{2}{3} \geq \frac{7}{12}$

Exercise:

Problem: $b - \frac{7}{8} \geq -\frac{1}{2}$

Solution:



Solve Inequalities using the Division and Multiplication Properties of Inequality

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

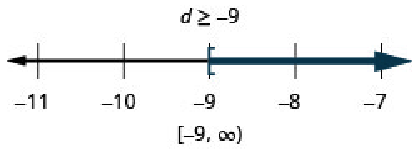
Exercise:

Problem: $9x > 54$

Exercise:

Problem: $-12d \leq 108$

Solution:



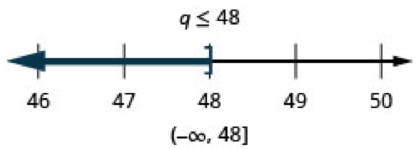
Exercise:

Problem: $\frac{5}{2}j < -60$

Exercise:

Problem: $\frac{q}{-2} \geq -24$

Solution:



Solve Inequalities That Require Simplification

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

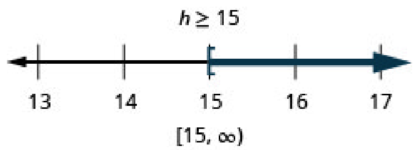
Exercise:

Problem: $6p > 15p - 30$

Exercise:

Problem: $9h - 7(h - 1) \leq 4h - 23$

Solution:



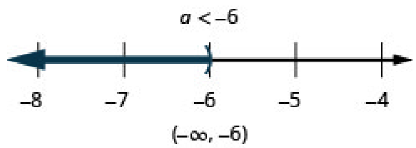
Exercise:

Problem: $5n - 15(4 - n) < 10(n - 6) + 10n$

Exercise:

Problem: $\frac{3}{8}a - \frac{1}{12}a > \frac{5}{12}a + \frac{3}{4}$

Solution:



Translate to an Inequality and Solve

In the following exercises, translate and solve. Then write the solution in interval notation and graph on the number line.

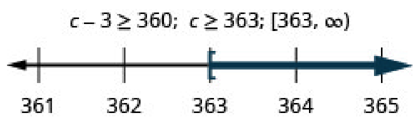
Exercise:

Problem: Five more than *z* is at most 19.

Exercise:

Problem: Three less than *c* is at least 360.

Solution:



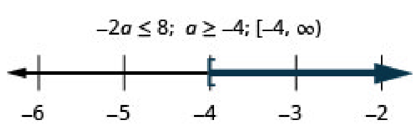
Exercise:

Problem: Nine times *n* exceeds 42.

Exercise:

Problem: Negative two times *a* is no more than 8.

Solution:



Everyday Math

Exercise:

Problem:

Describe how you have used two topics from this chapter in your life outside of your math class during the past month.

Chapter 2 Practice Test

Exercise:

Problem:

Determine whether each number is a solution to the equation $6x-3=x+20$.

ⓐ 5
ⓑ $\frac{23}{5}$

Solution:

ⓐ no ⓑ yes

In the following exercises, solve each equation.

Exercise:

Problem: $n-\frac{2}{3}=\frac{1}{4}$

Exercise:

Problem: $\frac{9}{2}c=144$

Solution:

$c=32$

Exercise:

Problem: $4y-8=16$

Exercise:

Problem: $-8x-15+9x-1=-21$

Solution:

$x = -5$

Exercise:

Problem: $-15a = 120$

Exercise:

Problem: $\frac{2}{3}x = 6$

Solution:

$x = 9$

Exercise:

Problem: $x - 3.8 = 8.2$

Exercise:

Problem: $10y = -5y - 60$

Solution:

$y = -4$

Exercise:

Problem: $8n - 2 = 6n - 12$

Exercise:

Problem: $9m - 2 - 4m - m = 42 - 8$

Solution:

$m = 9$

Exercise:

Problem: $-5(2x - 1) = 45$

Exercise:

Problem: $-(d-9)=23$

Solution:

$d=-14$

Exercise:

Problem: $\frac{1}{4}(12m-28)=6-2(3m-1)$

Exercise:

Problem: $2(6x-5)-8=-22$

Solution:

$x=-\frac{1}{3}$

Exercise:

Problem: $8(3a-5)-7(4a-3)=20-3a$

Exercise:

Problem: $\frac{1}{4}p-\frac{1}{3}=\frac{1}{2}$

Solution:

$p=\frac{10}{3}$

Exercise:

Problem: $0.1d+0.25(d+8)=4.1$

Exercise:

Problem: $14n-3(4n+5)=-9+2(n-8)$

Solution:

contradiction; no solution

Exercise:

Problem: $9(3u-2)-4[6-8(u-1)]=3(u-2)$

Exercise:

Problem: Solve the formula $x-2y=5$ for y
ⓐ when $x=-3$
ⓑ in general

Solution:

ⓐ $y=4$ ⓑ $y=\frac{5-x}{2}$

In the following exercises, graph on the number line and write in interval notation.

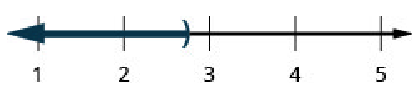
Exercise:

Problem: $x\geq -3.5$

Exercise:

Problem: $x<\frac{11}{4}$

Solution:



$\left(-\infty,\frac{11}{4}\right)$

In the following exercises,, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

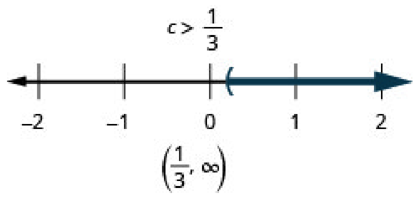
Exercise:

Problem: $8k\geq 5k-120$

Exercise:

Problem: $3c-10(c-2)<5c+16$

Solution:



In the following exercises, translate to an equation or inequality and solve.

Exercise:

Problem: 4 less than twice *x* is 16.

Exercise:

Problem: Fifteen more than *n* is at least 48.

Solution:

$n + 15 \geq 48; n \geq 33$

Exercise:

Problem:

Samuel paid $25.82 for gas this week, which was $3.47 less than he paid last week. How much had he paid last week?

Exercise:

Problem:

Jenna bought a coat on sale for $120, which was $\frac{2}{3}$ of the original price. What was the original price of the coat?

Solution:

$120 = \frac{2}{3}p$; The original price was $180.

Exercise:

Problem:

Sean took the bus from Seattle to Boise, a distance of 506 miles. If the trip took $7\frac{2}{3}$ hours, what was the speed of the bus?

Introduction
class="introduction"

This odd-looking headgear provides the user with a virtual world. (credit: fill/Pixabay)

Imagine visiting a faraway city or even outer space from the comfort of your living room. It could be possible using virtual reality. This technology creates realistic images that make you feel as if you are truly immersed in the scene and even enable you to interact with them. It is being developed for fun applications, such as video games, but also for architects to plan buildings, car companies to design prototypes, the military to train, and medical students to learn.

Developing virtual reality devices requires modeling the environment using graphs and mathematical relationships. In this chapter, you will graph different relationships and learn ways to describe and analyze graphs.

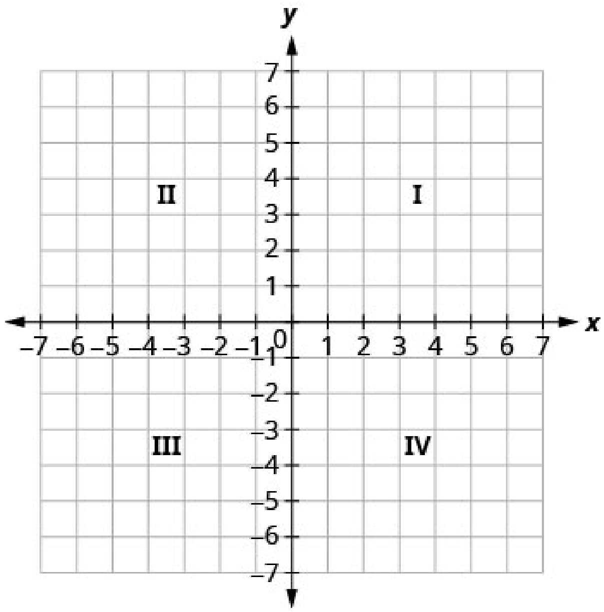
Graph Linear Equations in Two Variables: ASE
By the end of this section, you will be able to:

- Plot points in a rectangular coordinate system
- Graph a linear equation by plotting points
- Graph vertical and horizontal lines
- Find the x- and y-intercepts
- Graph a line using the intercepts

Plot Points on a Rectangular Coordinate System

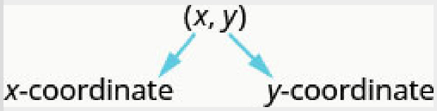
Just like maps use a grid system to identify locations, a grid system is used in algebra to show a relationship between two variables in a rectangular coordinate system. The rectangular coordinate system is also called the *xy*-plane or the “coordinate plane.”

The rectangular coordinate system is formed by two intersecting number lines, one horizontal and one vertical. The horizontal number line is called the *x*-axis. The vertical number line is called the *y*-axis. These axes divide a plane into four regions, called quadrants. The quadrants are identified by Roman numerals, beginning on the upper right and proceeding counterclockwise. See [link].



In the rectangular coordinate system, every point is represented by an **ordered pair**. The first number in the ordered pair is the *x*-coordinate of the point, and the second number is the *y*-coordinate of the point. The phrase “ordered pair” means that the order is important.

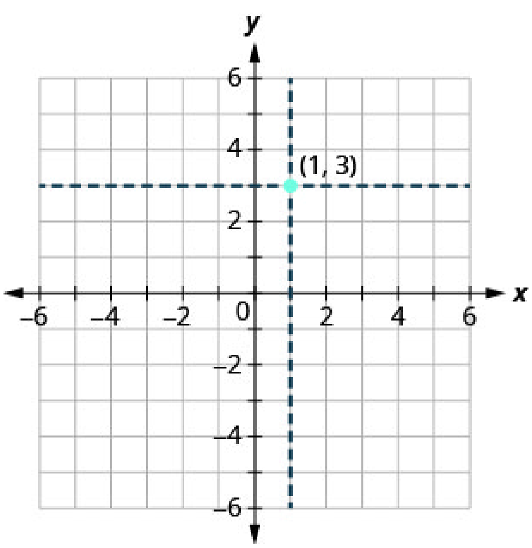
Note:
Ordered Pair
An **ordered pair**, (x, y) gives the coordinates of a point in a rectangular coordinate system. The first number is the *x*-coordinate. The second number is the *y*-coordinate.



What is the ordered pair of the point where the axes cross? At that point both coordinates are zero, so its ordered pair is $(0, 0)$. The point $(0, 0)$ has a special name. It is called the **origin**.

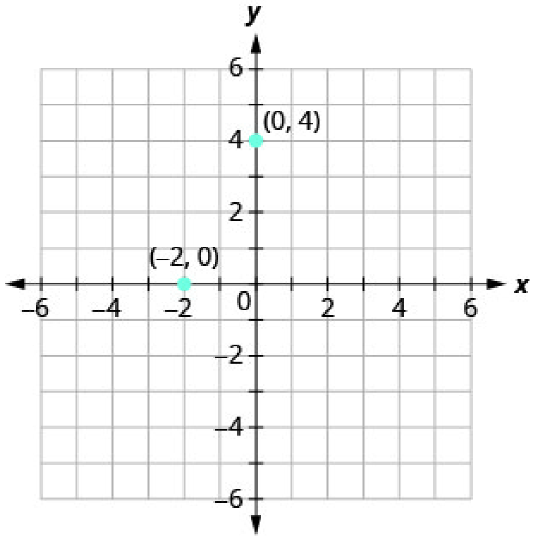
Note:
The Origin
The point $(0, 0)$ is called the **origin**. It is the point where the *x*-axis and *y*-axis intersect.

We use the coordinates to locate a point on the *xy*-plane. Let's plot the point $(1, 3)$ as an example. First, locate 1 on the *x*-axis and lightly sketch a vertical line through $x = 1$. Then, locate 3 on the *y*-axis and sketch a horizontal line through $y = 3$. Now, find the point where these two lines meet—that is the point with coordinates $(1, 3)$. See [link].



Notice that the vertical line through $x = 1$ and the horizontal line through $y = 3$ are not part of the graph. We just used them to help us locate the point $(1, 3)$.

When one of the coordinates is zero, the point lies on one of the axes. In [link] the point $(0, 4)$ is on the *y*-axis and the point $(-2, 0)$ is on the *x*-axis.



Note:
Points on the Axes
Points with a *y*-coordinate equal to 0 are on the *x*-axis, and have coordinates $(a, 0)$.
Points with an *x*-coordinate equal to 0 are on the *y*-axis, and have coordinates $(0, b)$.

Example:
Exercise:

Problem:

Plot each point in the rectangular coordinate system and identify the quadrant in which the point is located:

ⓐ $(-5, 4)$ ⓑ $(-3, -4)$ ⓒ $(2, -3)$ ⓓ $(0, -1)$ ⓔ $\left(3, \frac{5}{2}\right)$.

Solution:

The first number of the coordinate pair is the *x*-coordinate, and the second number is the *y*-coordinate. To plot each point, sketch a vertical line through the *x*-coordinate and a horizontal line through the *y*-coordinate. Their intersection is the point.

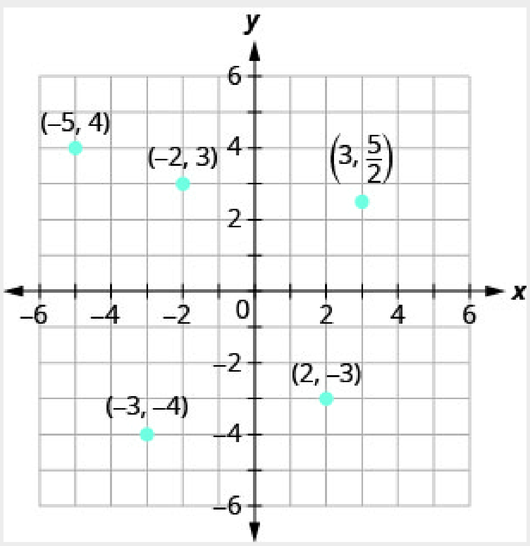
ⓐ Since $x = -5$, the point is to the left of the *y*-axis. Also, since $y = 4$, the point is above the *x*-axis. The point $(-5, 4)$ is in Quadrant II.

ⓑ Since $x = -3$, the point is to the left of the *y*-axis. Also, since $y = -4$, the point is below the *x*-axis. The point $(-3, -4)$ is in Quadrant III.

ⓒ Since $x = 2$, the point is to the right of the *y*-axis. Since $y = -3$, the point is below the *x*-axis. The point $(2, -3)$ is in Quadrant IV.

ⓓ Since $x = 0$, the point whose coordinates are $(0, -1)$ is on the *y*-axis.

ⓔ Since $x = 3$, the point is to the right of the *y*-axis. Since $y = \frac{5}{2}$, the point is above the *x*-axis. (It may be helpful to write $\frac{5}{2}$ as a mixed number or decimal.) The point $\left(3, \frac{5}{2}\right)$ is in Quadrant I.



Note:

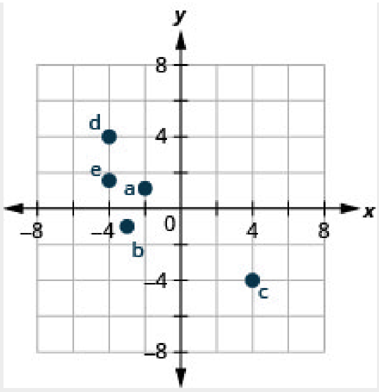
Exercise:

Problem:

Plot each point in a rectangular coordinate system and identify the quadrant in which the point is located:

ⓐ $(-2, 1)$ ⓑ $(-3, -1)$ ⓒ $(4, -4)$ ⓓ $(-4, 4)$ ⓔ $\left(-4, \frac{3}{2}\right)$

Solution:

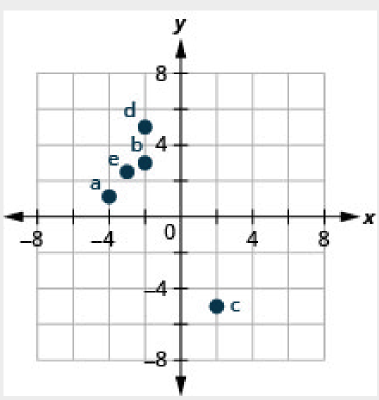


Note:
Exercise:

Problem:

Plot each point in a rectangular coordinate system and identify the quadrant in which the point is located:
ⓐ $(-4, 1)$ ⓑ $(-2, 3)$ ⓒ $(2, -5)$ ⓓ $(-2, 5)$ ⓔ $\left(-3, \frac{5}{2}\right)$

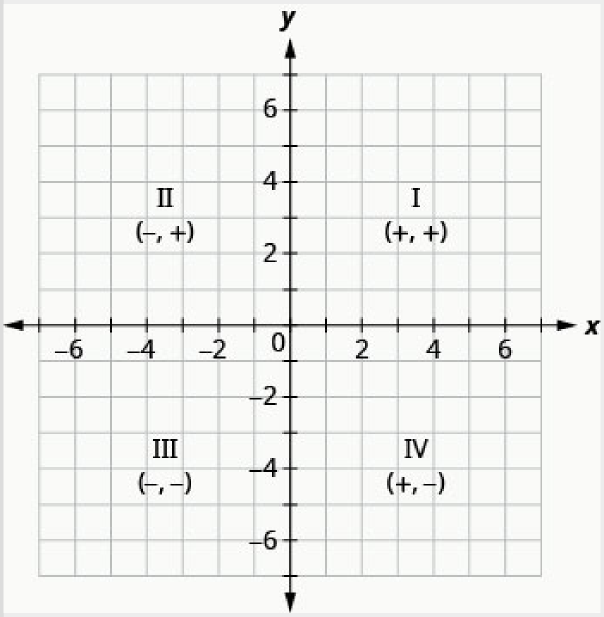
Solution:



The signs of the *x*-coordinate and *y*-coordinate affect the location of the points. You may have noticed some patterns as you graphed the points in the previous example. We can summarize sign patterns of the quadrants in this way:

Note:
Quadrants
Equation:

Quadrant I	**Quadrant II**	**Quadrant III**	**Quadrant IV**
(x, y)	(x, y)	(x, y)	(x, y)
$(+, +)$	$(-, +)$	$(-, -)$	$(+, -)$



Up to now, all the equations you have solved were equations with just one variable. In almost every case, when you solved the equation you got exactly one solution. But equations can have more than one variable. Equations with two variables may be of the form $Ax + By = C$. An equation of this form is called a **linear equation** in two variables.

Note:
Linear Equation
An equation of the form $Ax + By = C$, where *A* and *B* are not both zero, is called a **linear equation** in two variables.

Here is an example of a linear equation in two variables, *x* and *y*.

$$Ax + By = C$$

$$x + 4y = 8$$

$$A = 1,\ B = 4,\ C = 8$$

The equation $y = -3x + 5$ is also a linear equation. But it does not appear to be in the form $Ax + By = C$. We can use the Addition Property of Equality and rewrite it in $Ax + By = C$ form.

Equation:

	y	$=$	$-3x + 5$
Add to both sides.	$y + 3x$	$=$	$-3x + 5 + 3x$
Simplify.	$y + 3x$	$=$	5
Use the Commutative Property to put it in $Ax + By = C$ form.	$3x + y$	$=$	5

By rewriting $y = -3x + 5$ as $3x + y = 5$, we can easily see that it is a linear equation in two variables because it is of the form $Ax + By = C$. When an equation is in the form $Ax + By = C$, we say it is in **standard form of a linear equation**.

Note:
Standard Form of Linear Equation
A linear equation is in **standard form** when it is written $Ax + By = C$.

Most people prefer to have *A*, *B*, and *C* be integers and $A \geq 0$ when writing a linear equation in standard form, although it is not strictly necessary.

Linear equations have infinitely many solutions. For every number that is substituted for *x* there is a corresponding *y* value. This pair of values is a **solution** to the linear equation and is represented by the ordered pair (x, y). When we substitute these values of *x* and *y* into the equation, the result is a true statement, because the value on the left side is equal to the value on the right side.

Note:
Solution of a Linear Equation in Two Variables

An ordered pair (x, y) is a **solution** of the linear equation$Ax + By = C$, if the equation is a true statement when the *x*- and *y*-values of the ordered pair are substituted into the equation.

Linear equations have infinitely many solutions. We can plot these solutions in the rectangular coordinate system. The points will line up perfectly in a straight line. We connect the points with a straight line to get the graph of the equation. We put arrows on the ends of each side of the line to indicate that the line continues in both directions.

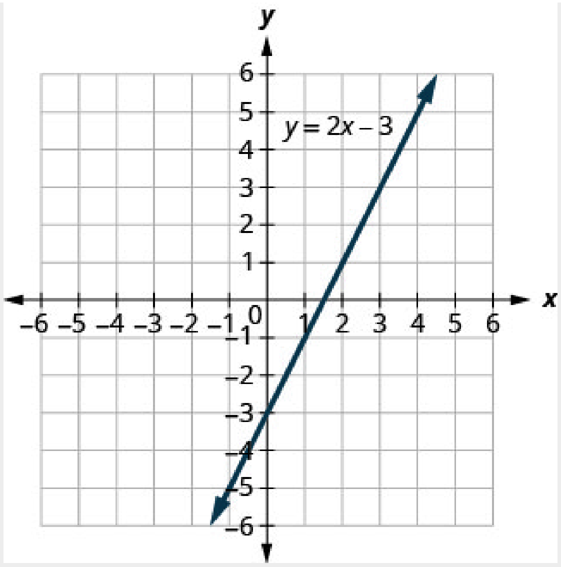
A graph is a visual representation of all the solutions of the equation. It is an example of the saying, “A picture is worth a thousand words.” The line shows you *all* the solutions to that equation. Every point on the line is a solution of the equation. And, every solution of this equation is on this line. This line is called the graph of the equation. Points *not* on the line are not solutions!

Note:
Graph of a Linear Equation
The graph of a linear equation $Ax + By = C$ is a straight line.

- Every point on the line is a solution of the equation.
- Every solution of this equation is a point on this line.

Example:
Exercise:

Problem: The graph of $y = 2x - 3$ is shown.



For each ordered pair, decide:

ⓐ Is the ordered pair a solution to the equation?

ⓑ Is the point on the line?

A: $(0, -3)$ B: $(3, 3)$ C: $(2, -3)$ D: $(-1, -5)$

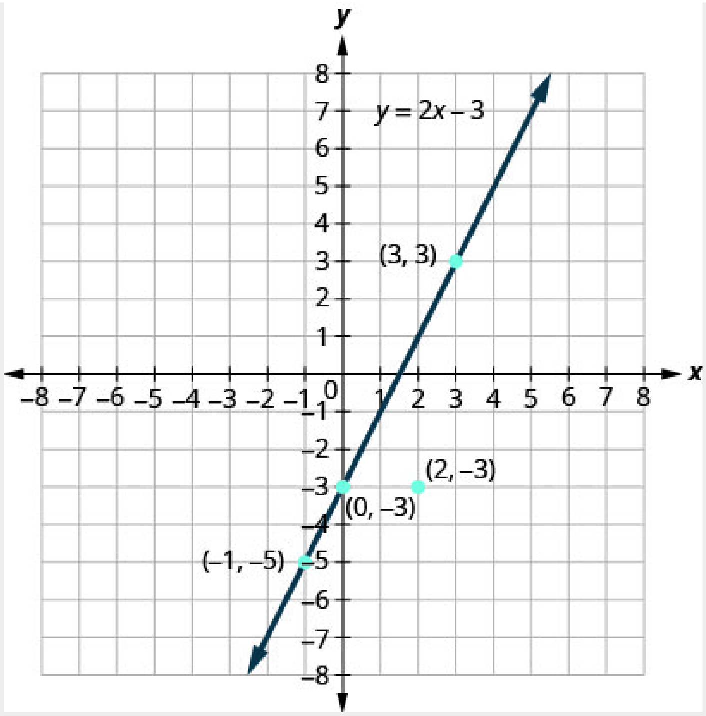
Solution:

Substitute the *x*- and *y*-values into the equation to check if the ordered pair is a solution to the equation.

ⓐ

A: $(0, -3)$	B: $(3, 3)$	C: $(2, -3)$	D: $(-1, -5)$
$y = 2x - 3$	$y = 2x - 3$	$y = 2x - 3$	$y = 2x - 3$
$-3 \stackrel{?}{=} 2(0) - 3$	$3 \stackrel{?}{=} 2(3) - 3$	$-3 \stackrel{?}{=} 2(2) - 3$	$-5 \stackrel{?}{=} 2(-1) - 3$
$-3 = -3$ ✓	$3 = 3$ ✓	$-3 \neq 1$	$-5 = -5$ ✓
$(0, -3)$ is a solution.	$(3, 3)$ is a solution.	$(2, -3)$ is not a solution.	$(-1, -5)$ is a solution.

ⓑ Plot the points $(0, -3)$,$(3, 3)$,$(2, -3)$, and $(-1, -5)$.



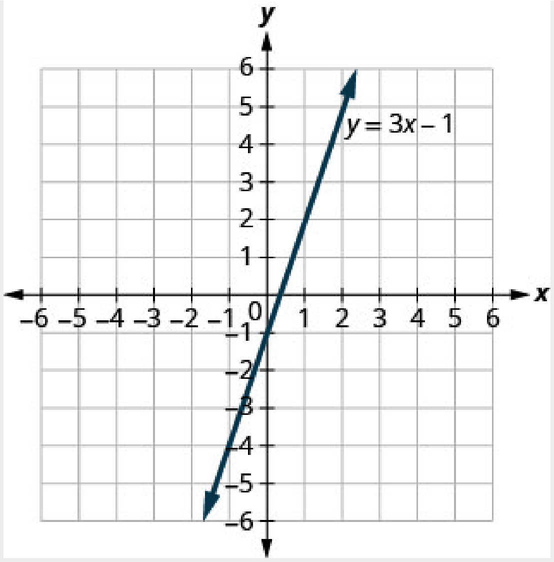
The points $(0, 3)$,$(3, -3)$, and $(-1, -5)$ are on the line $y = 2x - 3$, and the point $(2, -3)$ is not on the line.
The points that are solutions to $y = 2x - 3$ are on the line, but the point that is not a solution is not on the line.

Note:
Exercise:

Problem: Use graph of $y = 3x - 1$. For each ordered pair, decide:

ⓐ Is the ordered pair a solution to the equation?
ⓑ Is the point on the line?

A $(0, -1)$ B $(2, 5)$



Solution:

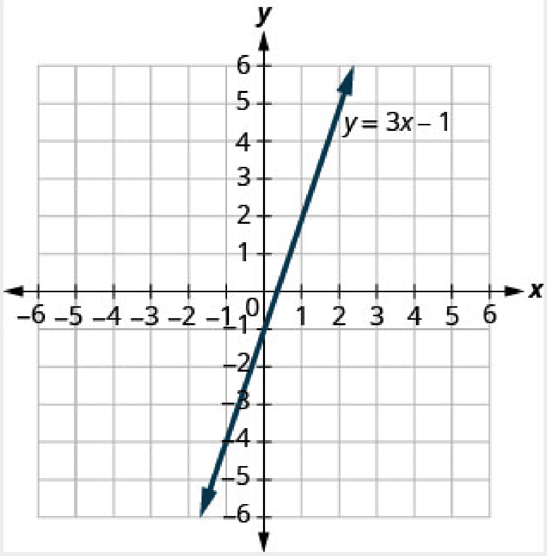
ⓐ yes, yes ⓑ yes, yes

Note:
Exercise:

Problem: Use graph of $y = 3x - 1$. For each ordered pair, decide:

ⓐ Is the ordered pair a solution to the equation?
ⓑ Is the point on the line?

A$(3, -1)$ B$(-1, -4)$



Solution:

ⓐ no, no ⓑ yes, yes

Graph a Linear Equation by Plotting Points

There are several methods that can be used to graph a linear equation. The first method we will use is called plotting points, or the Point-Plotting Method. We find three points whose coordinates are solutions to the equation and then plot them in a rectangular coordinate system. By connecting these points in a line, we have the graph of the linear equation.

Example:
How to Graph a Linear Equation by Plotting Points
Exercise:

Problem: Graph the equation $y = 2x + 1$ by plotting points.

Solution:

Step 1. Find three points whose coordinates are solutions to the equation.

You can choose any values for x or y.

In this case, since y is isolated on the left side of the equation, it is easier to choose values for x.

$$y = 2x + 1$$

$$x = 0$$
$$y = 2x + 1$$
$$y = 2 \cdot 0 + 1$$
$$y = 0 + 1$$
$$y = 1$$

$$x = 1$$
$$y = 2x + 1$$
$$y = 2 \cdot 1 + 1$$
$$y = 2 + 1$$
$$y = 3$$

$$x = -2$$
$$y = 2x + 1$$
$$y = 2(-2) + 1$$
$$y = -4 + 1$$
$$y = -3$$

Organize the solutions in a table.

Put the three solutions in a table.

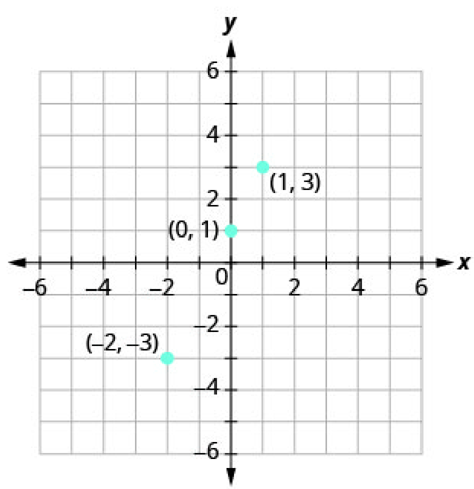
$y = 2x + 1$		
x	y	(x, y)
0	1	(0, 1)
1	3	(1, 3)
–2	–3	(–2, –3)

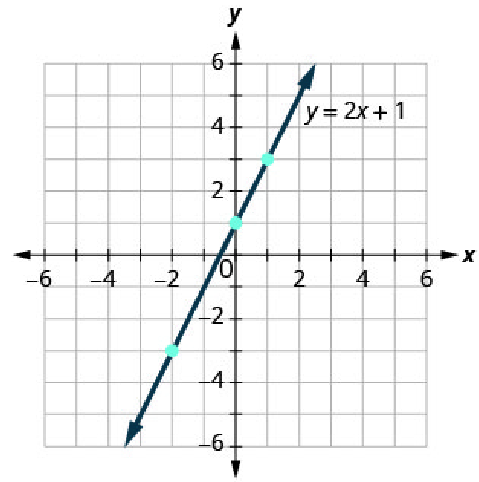
Step 2. Plot the points in a rectangular coordinate system.

Check that the points line up. If they do not, carefully check your work!

Plot:
(0, 1), (1, 3), (–2, –3).

Do the points line up?
Yes, the points line up.

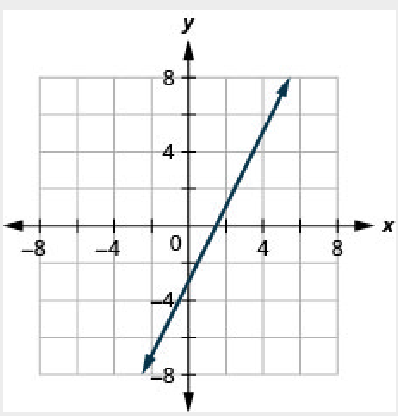


Step 3. Draw the line through the three points. Extend the line to fill the grid and put arrows on both ends of the line.	This line is the graph of $y = 2x + 1$.	

Note:
Exercise:

Problem: Graph the equation by plotting points: $y = 2x - 3$.

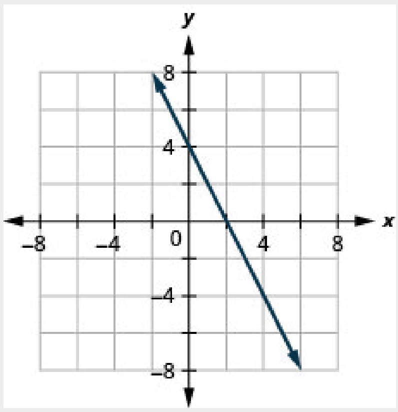
Solution:



Note:
Exercise:

Problem: Graph the equation by plotting points: $y = -2x + 4$.

Solution:



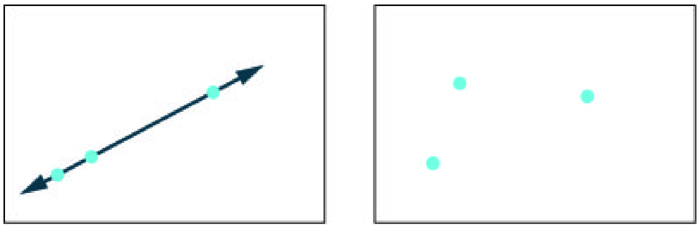
The steps to take when graphing a linear equation by plotting points are summarized here.

Note:
Graph a linear equation by plotting points.

Find three points whose coordinates are solutions to the equation. Organize them in a table.
Plot the points in a rectangular coordinate system. Check that the points line up. If they do not, carefully check your work.
Draw the line through the three points. Extend the line to fill the grid and put arrows on both ends of the line.

It is true that it only takes two points to determine a line, but it is a good habit to use three points. If you only plot two points and one of them is incorrect, you can still draw a line but it will not represent the solutions to the equation. It will be the wrong line.

If you use three points, and one is incorrect, the points will not line up. This tells you something is wrong and you need to check your work. Look at the difference between these illustrations.



(a) (b)

When an equation includes a fraction as the coefficient of x, we can still substitute any numbers for *x*. But the arithmetic is easier if we make "good" choices for the values of *x*. This way we will avoid fractional answers, which are hard to graph precisely.

Example:
Exercise:

Problem: Graph the equation: $y = \frac{1}{2}x + 3$.

Solution:

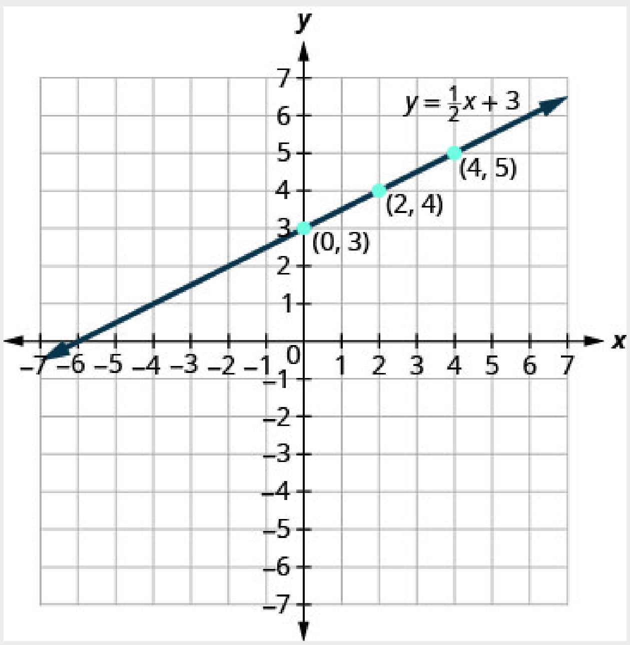
Find three points that are solutions to the equation. Since this equation has the fraction $\frac{1}{2}$ as a coefficient of *x*, we will choose values of *x* carefully. We will use zero as one choice and multiples of 2 for the other choices. Why are multiples of two a good choice for values of *x*? By choosing multiples of 2 the multiplication by $\frac{1}{2}$ simplifies to a whole number

$x = 0$	$x = 2$	$x = 4$
$y = \frac{1}{2}x + 3$	$y = \frac{1}{2}x + 3$	$y = \frac{1}{2}x + 3$
$y = \frac{1}{2}(0) + 3$	$y = \frac{1}{2}(2) + 3$	$y = \frac{1}{2}(4) + 3$
$y = 0 + 3$	$y = 1 + 3$	$y = 2 + 3$
$y = 3$	$y = 4$	$y = 5$

The points are shown in [link].

$y = \frac{1}{2}x + 3$		
x	***y***	(x, y)
0	3	$(0, 3)$
2	4	$(2, 4)$
4	5	$(4, 5)$

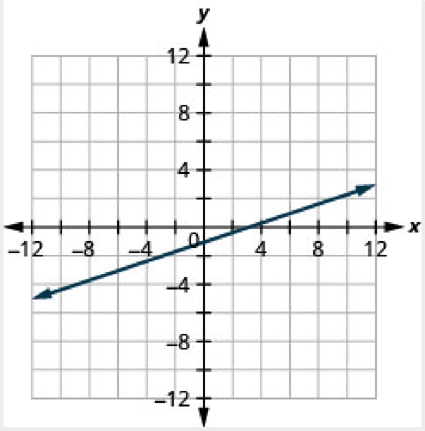
Plot the points, check that they line up, and draw the line.



Note:
Exercise:

Problem: Graph the equation: $y = \frac{1}{3}x - 1$.

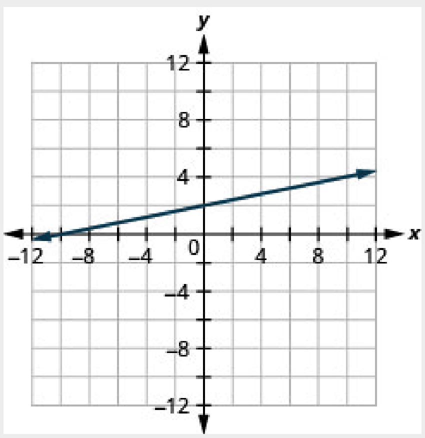
Solution:



Note:
Exercise:

Problem: Graph the equation: $y = \frac{1}{4}x + 2.$

Solution:



Graph Vertical and Horizontal Lines

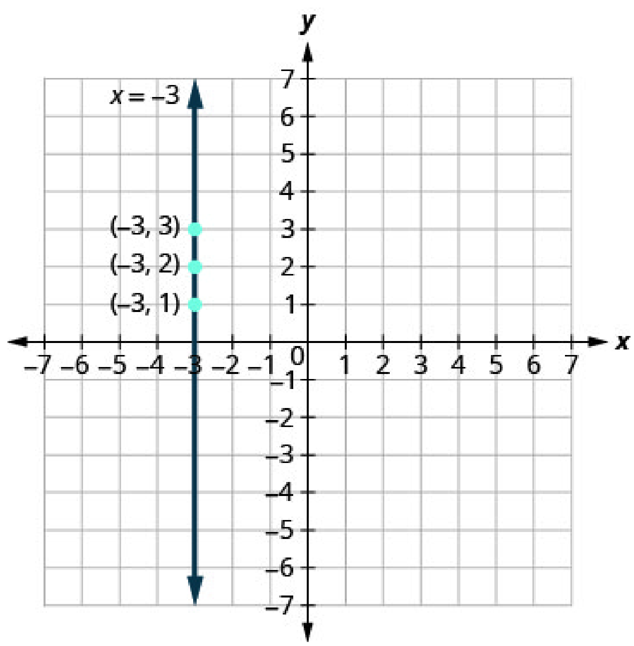
Some linear equations have only one variable. They may have just *x* and no *y*, or just *y* without an *x*. This changes how we make a table of values to get the points to plot.

Let’s consider the equation $x=-3$. This equation has only one variable, *x*. The equation says that *x* is *always* equal to -3, so its value does not depend on *y*. No matter what is the value of *y*, the value of *x* is always -3.

So to make a table of values, write -3 in for all the *x*-values. Then choose any values for *y*. Since *x* does not depend on *y*, you can choose any numbers you like. But to fit the points on our coordinate graph, we’ll use 1, 2, and 3 for the *y*-coordinates. See [link].

$x=-3$		
x	***y***	(x,y)
-3	1	$(-3,1)$
-3	2	$(-3,2)$
-3	3	$(-3,3)$

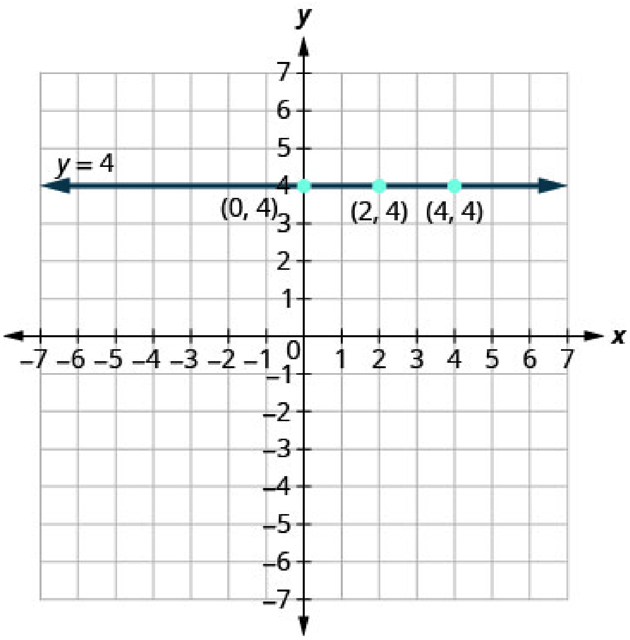
Plot the points from the table and connect them with a straight line. Notice that we have graphed a **vertical line**.



What if the equation has *y* but no *x*? Let's graph the equation $y = 4$. This time the *y*-value is a constant, so in this equation, *y* does not depend on *x*. Fill in 4 for all the *y*'s in [link] and then choose any values for *x*. We'll use 0, 2, and 4 for the *x*-coordinates.

$y = 4$		
x	***y***	(x, y)
0	4	$(0, 4)$
2	4	$(2, 4)$
4	4	$(4, 4)$

In this figure, we have graphed a **horizontal line** passing through the *y*-axis at 4.



Note:
Vertical and Horizontal Lines
A **vertical line** is the graph of an equation of the form $x = a$.

The line passes through the *x*-axis at $(a, 0)$.

A **horizontal line** is the graph of an equation of the form $y = b$.

The line passes through the *y*-axis at $(0, b)$.

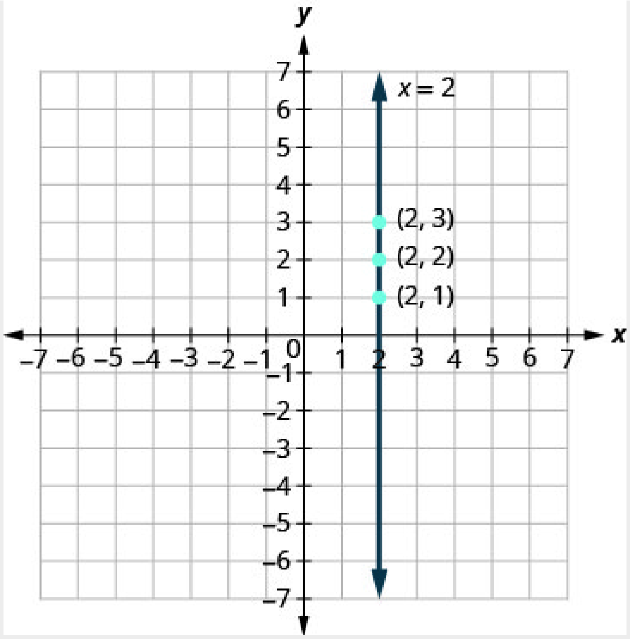
Example:
Exercise:

Problem: Graph: ⓐ $x = 2$ ⓑ $y = -1$.

Solution:

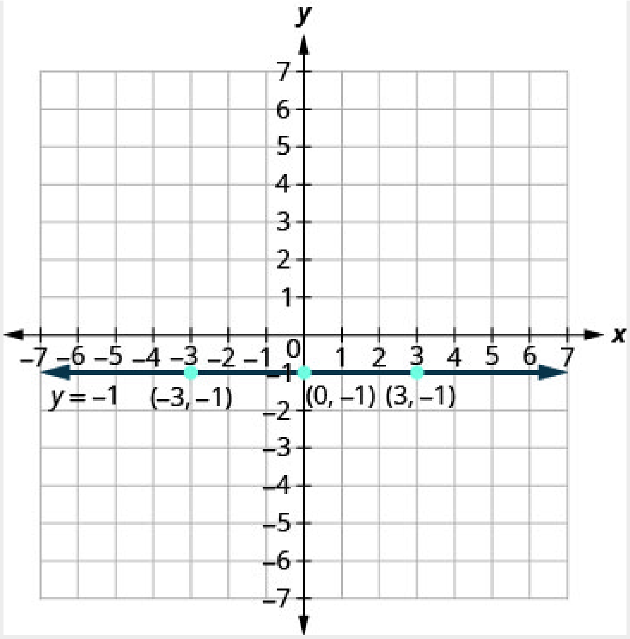
ⓐ The equation has only one variable, *x*, and *x* is always equal to 2. We create a table where *x* is always 2 and then put in any values for *y*. The graph is a vertical line passing through the *x*-axis at 2.

$x = 2$		
x	***y***	(x, y)
2	1	$(2, 1)$
2	2	$(2, 2)$
2	3	$(2, 3)$



ⓑ Similarly, the equation $y=-1$ has only one variable, *y*. The value of *y* is constant. All the ordered pairs in the next table have the same *y*-coordinate. The graph is a horizontal line passing through the *y*-axis at -1.

$y=-1$		
x	***y***	(x,y)
0	-1	$(0,-1)$
3	-1	$(3,-1)$
-3	-1	$(-3,-1)$

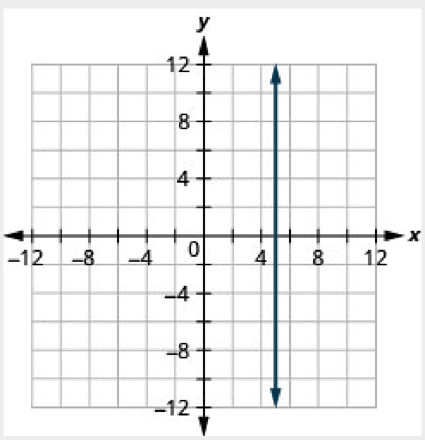


Note:
Exercise:

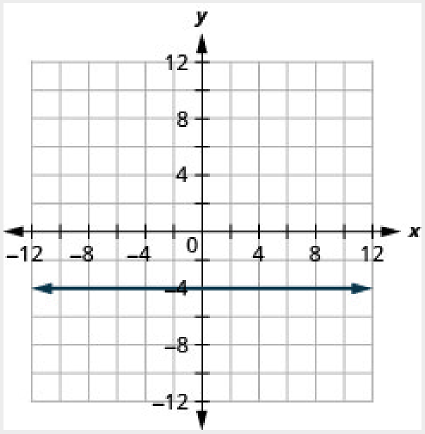
Problem: Graph the equations: ⓐ $x = 5$ ⓑ $y = -4$.

Solution:

ⓐ



ⓑ

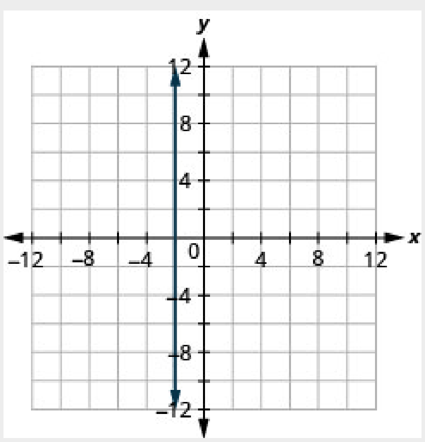


Note:
Exercise:

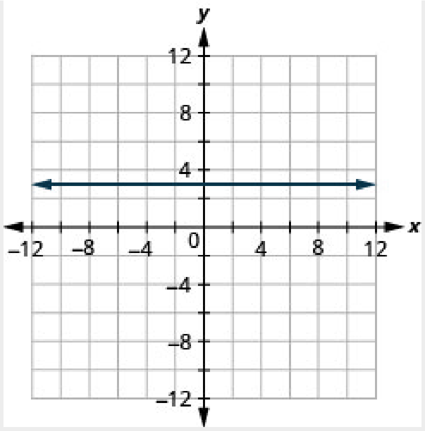
Problem: Graph the equations: ⓐ $x = -2$ ⓑ $y = 3$.

Solution:

ⓐ



ⓑ

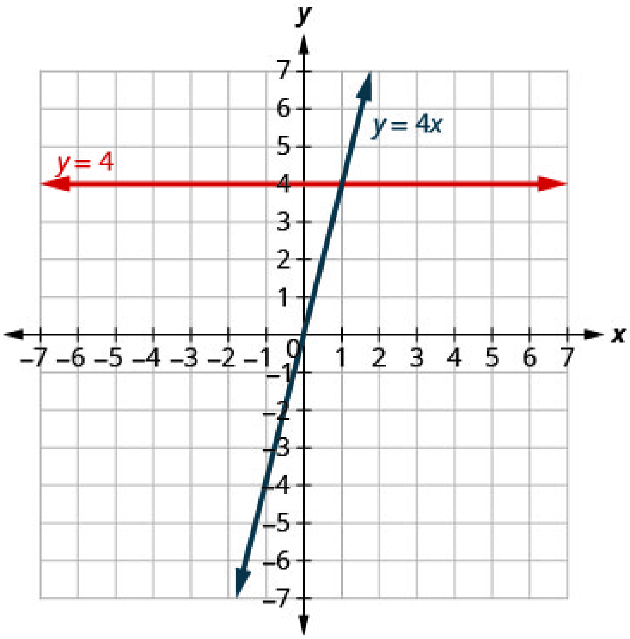


What is the difference between the equations $y = 4x$ and $y = 4$?

The equation $y = 4x$ has both *x* and *y*. The value of *y* depends on the value of *x*, so the *y* -coordinate changes according to the value of *x*. The equation $y = 4$ has only one variable. The value of *y* is constant, it does not depend on the value of *x*, so the *y*-coordinate is always 4.

$y = 4x$		
x	*y*	(*x*, *y*)
0	0	(0, 0)
1	4	(1, 4)
2	8	(2, 8)

$y = 4$		
x	*y*	(*x*, *y*)
0	4	(0, 4)
1	4	(1, 4)
2	4	(2, 4)



Notice, in the graph, the equation $y = 4x$ gives a slanted line, while $y = 4$ gives a horizontal line.

Example:
Exercise:

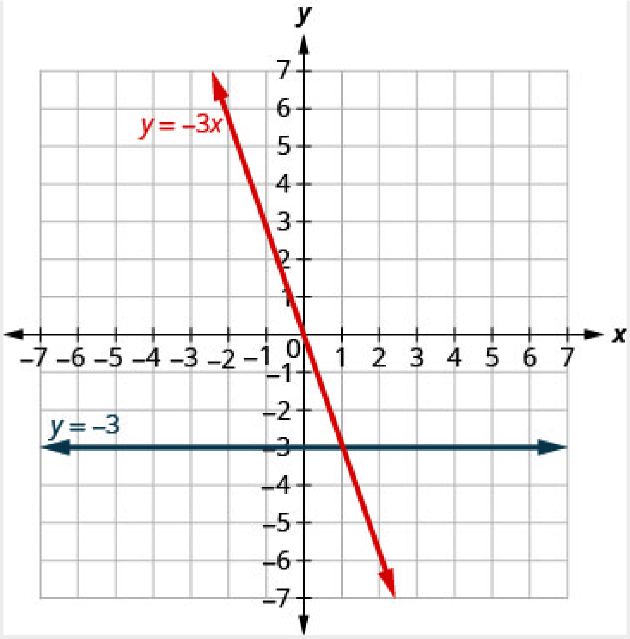
Problem: Graph $y = -3x$ and $y = -3$ in the same rectangular coordinate system.

Solution:

We notice that the first equation has the variable *x*, while the second does not. We make a table of points for each equation and then graph the lines. The two graphs are shown.

$y = -3x$		
x	y	(x, y)
0	0	(0, 0)
1	–3	(1, –3)
2	–6	(2, –6)

$y = -3$		
x	y	(x, y)
0	–3	(0, –3)
1	–3	(1, –3)
2	–3	(2, –3)

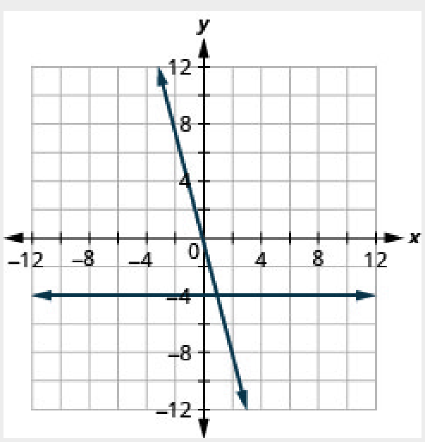


Note:
Exercise:

Problem:

Graph the equations in the same rectangular coordinate system: $y = -4x$ and $y = -4$.

Solution:

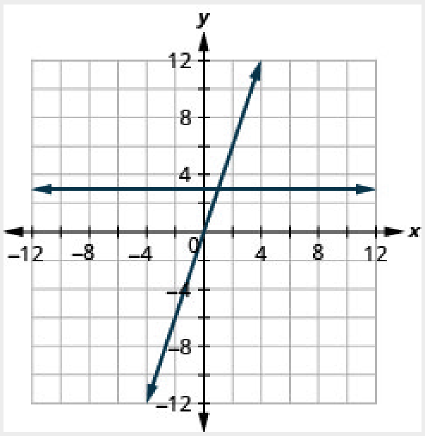


Note:
Exercise:

Problem:

Graph the equations in the same rectangular coordinate system: $y = 3$ and $y = 3x$.

Solution:



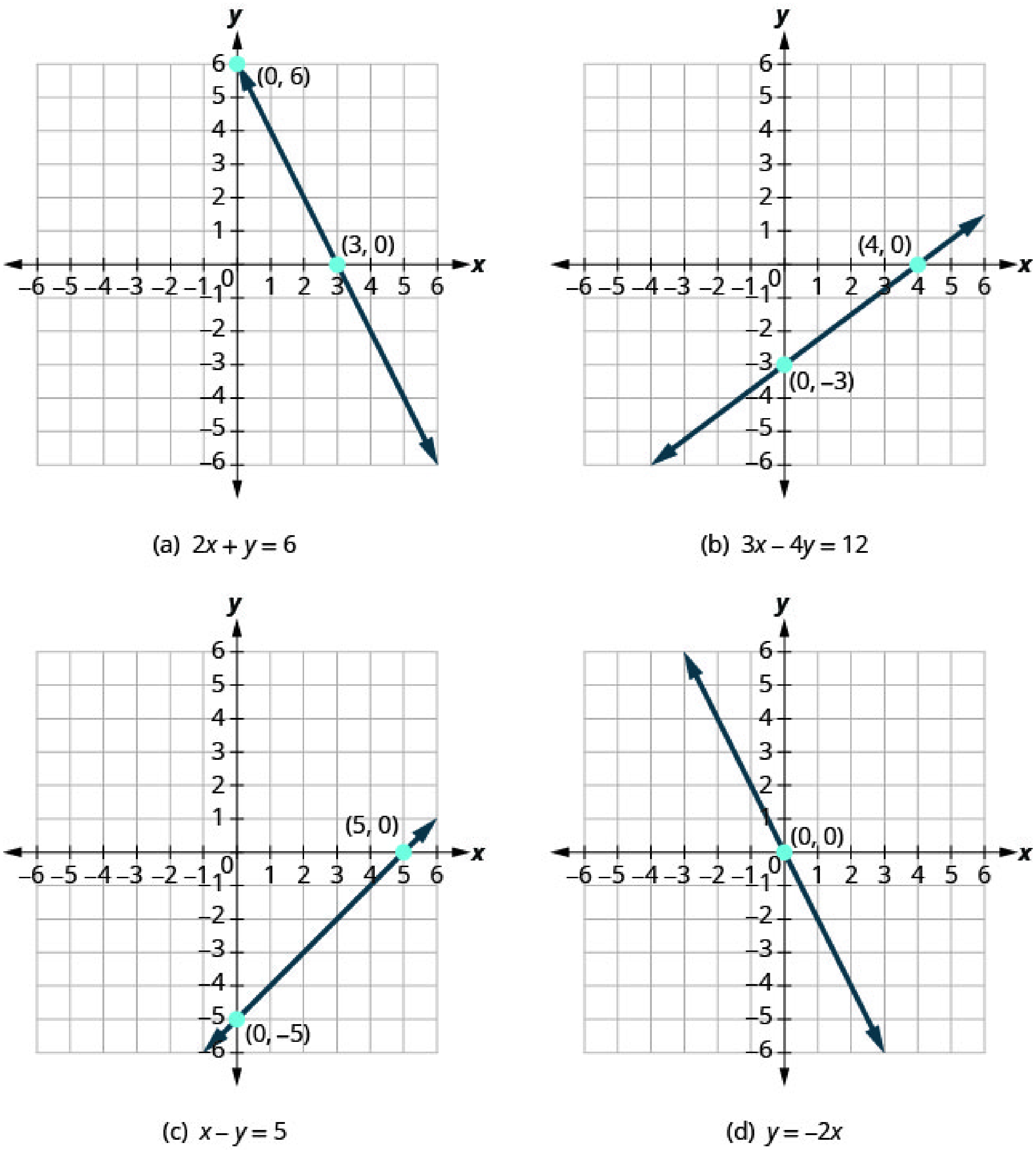
Find *x*- and *y*-intercepts

Every linear equation can be represented by a unique line that shows all the solutions of the equation. We have seen that when graphing a line by plotting points, you can use any three solutions to graph. This means that two people graphing the line might use different sets of three points.

At first glance, their two lines might not appear to be the same, since they would have different points labeled. But if all the work was done correctly, the lines should be exactly the same. One way to recognize that they are indeed the same line is to look at where the line crosses the *x*-axis and the *y*-axis. These points are called the **intercepts of a line**.

Note:
Intercepts of a Line
The points where a line crosses the *x*-axis and the *y*-axis are called the **intercepts of the line**.

Let’s look at the graphs of the lines.



(a) $2x + y = 6$

(b) $3x - 4y = 12$

(c) $x - y = 5$

(d) $y = -2x$

First, notice where each of these lines crosses the *x*-axis. See [link].

Now, let’s look at the points where these lines cross the *y*-axis.

Figure	**The line crosses the *x*-axis at:**	**Ordered pair for this point**	**The line crosses the *y*-axis at:**	**Ordered pair for this point**
Figure (a)	3	$(3, 0)$	6	$(0, 6)$
Figure (b)	4	$(4, 0)$	-3	$(0, -3)$
Figure (c)	5	$(5, 0)$	-5	$(0, 5)$
Figure (d)	0	$(0, 0)$	0	$(0, 0)$
General Figure	a	$(a, 0)$	b	$(0, b)$

Do you see a pattern?

For each line, the *y*-coordinate of the point where the line crosses the *x*-axis is zero. The point where the line crosses the *x*-axis has the form $(a, 0)$ and is called the *x-intercept* of the line. The *x*-intercept occurs when *y* is zero.

In each line, the *x*-coordinate of the point where the line crosses the *y*-axis is zero. The point where the line crosses the *y*-axis has the form $(0, b)$ and is called the *y-intercept* of the line. The *y*-intercept occurs when *x* is zero.

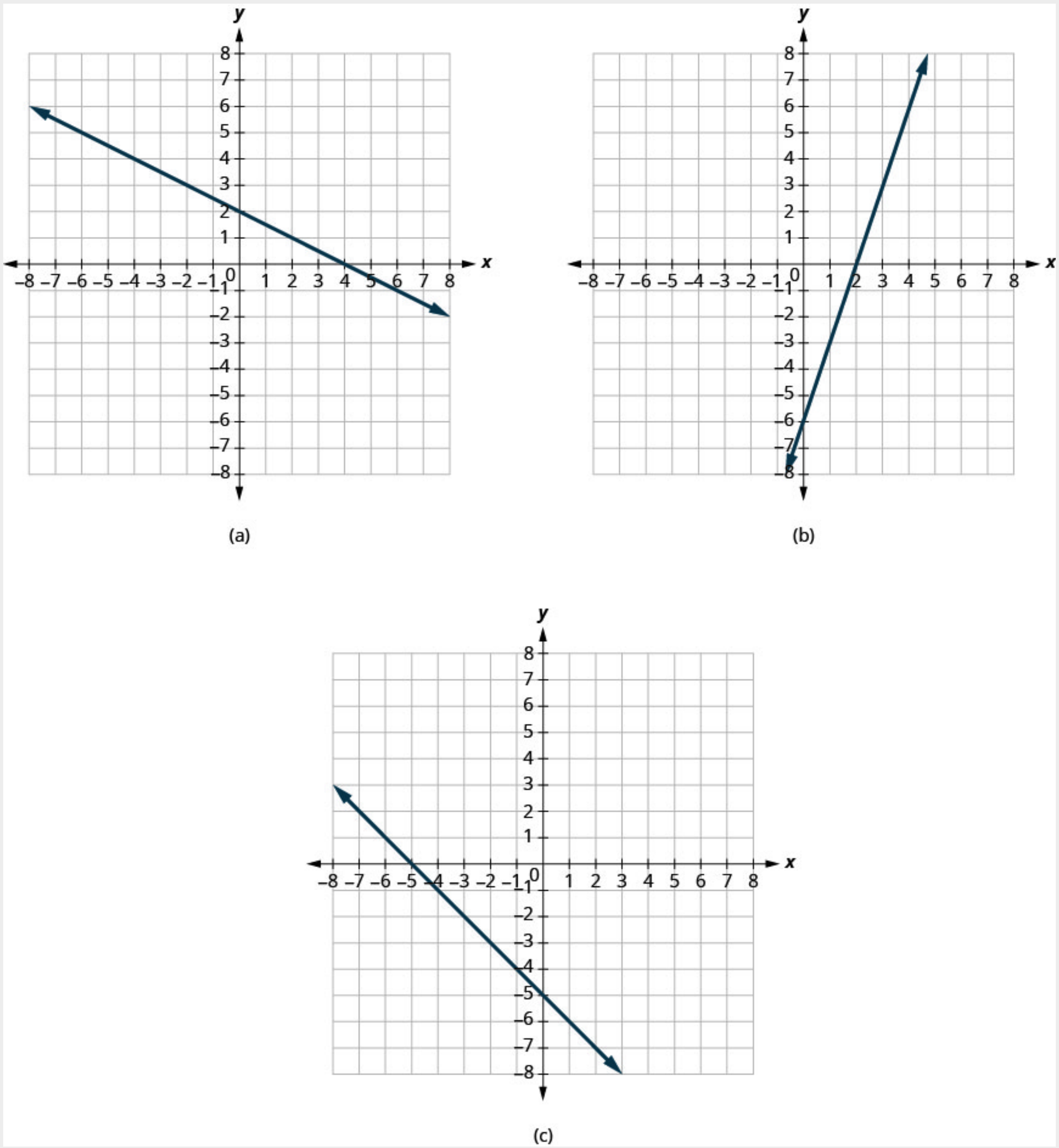
Note:
x-intercept and *y*-intercept of a Line
The *x*-intercept is the point $(a, 0)$ where the line crosses the *x*-axis.
The *y*-intercept is the point $(0, b)$ where the line crosses the *y*-axis.

- The *x*-intercept occurs when *y* is zero.
- The *y*-intercept occurs when *x* is zero.

x	*y*
a	0
0	*b*

Example:
Exercise:

Problem: Find the *x*- and *y*-intercepts on each graph shown.



Solution:

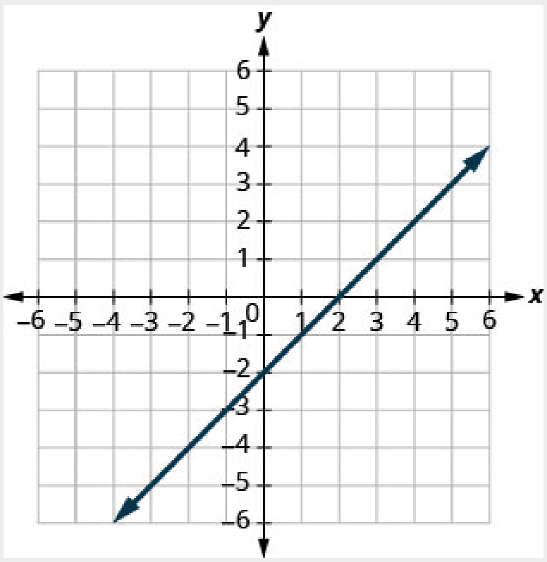
ⓐ The graph crosses the *x*-axis at the point $(4, 0)$. The *x*-intercept is $(4, 0)$.
The graph crosses the *y*-axis at the point $(0, 2)$. The *y*-intercept is $(0, 2)$.

ⓑ The graph crosses the *x*-axis at the point $(2, 0)$. The *x*-intercept is $(2, 0)$.
The graph crosses the *y*-axis at the point $(0, -6)$. The *y*-intercept is $(0, -6)$.

ⓒ The graph crosses the *x*-axis at the point $(-5, 0)$. The *x*-intercept is $(-5, 0)$. The graph crosses the *y*-axis at the point $(0, -5)$. The *y*-intercept is $(0, -5)$.

Note:
Exercise:

Problem: Find the *x*- and *y*-intercepts on the graph.

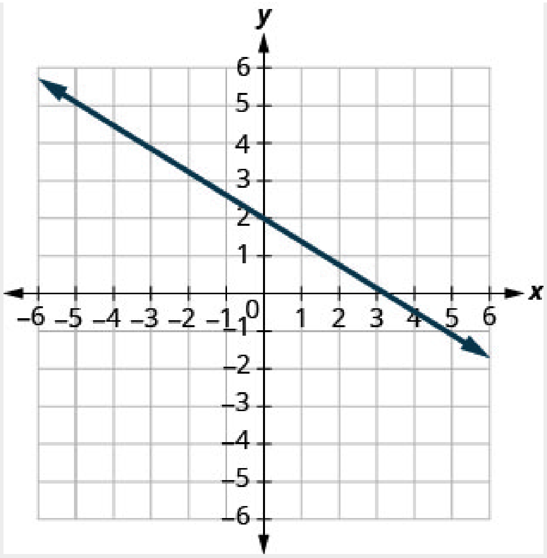


Solution:

x-intercept: $(2, 0)$,
y-intercept: $(0, -2)$

Note:
Exercise:

Problem: Find the *x*- and *y*-intercepts on the graph.



Solution:

x-intercept: $(3, 0)$,
y-intercept: $(0, 2)$

Recognizing that the *x*-intercept occurs when *y* is zero and that the *y*-intercept occurs when *x* is zero, gives us a method to find the intercepts of a line from its equation. To find the *x*-intercept, let $y = 0$ and solve for *x*. To find the *y*-intercept, let $x = 0$ and solve for *y*.

Note:
Find the *x*- and *y*-intercepts from the Equation of a Line
Use the equation of the line. To find:

- the *x*-intercept of the line, let $y = 0$ and solve for *x*.
- the *y*-intercept of the line, let $x = 0$ and solve for *y*.

Example:
Exercise:

Problem: Find the intercepts of $2x + y = 8$.

Solution:

We will let $y=0$ to find the x-intercept, and let $x=0$ to find the y-intercept. We will fill in a table, which reminds us of what we need to find.

$2x+y=8$		
x	y	
	0	x-intercept
0		y-intercept

To find the x-intercept, let $y=0$.	
	$2x+y=8$
Let $y=0$.	$2x+0=8$
Simplify.	$2x=8$
	$x=4$
The x-intercept is:	$(4,0)$
To find the y-intercept, let $x=0$.	
	$2x+y=8$
Let $x=0$.	$2\cdot 0+y=8$

Simplify.	$0 + y = 8$
	$y = 8$
The *y*-intercept is:	$(0, 8)$

The intercepts are the points $(4, 0)$ and $(0, 8)$ as shown in the table.

$2x + y = 8$	
x	***y***
4	0
0	8

Note:
Exercise:

Problem: Find the intercepts: $3x + y = 12$.

Solution:

x-intercept: $(4, 0)$,
y-intercept: $(0, 12)$

Note:

Exercise:

Problem: Find the intercepts: $x + 4y = 8$.

Solution:

x-intercept: $(8, 0)$,
y-intercept: $(0, 2)$

Graph a Line Using the Intercepts

To graph a linear equation by plotting points, you need to find three points whose coordinates are solutions to the equation. You can use the *x*- and *y*- intercepts as two of your three points. Find the intercepts, and then find a third point to ensure accuracy. Make sure the points line up—then draw the line. This method is often the quickest way to graph a line.

Example:
How to Graph a Line Using the Intercepts
Exercise:

Problem: Graph $-x + 2y = 6$ using the intercepts.

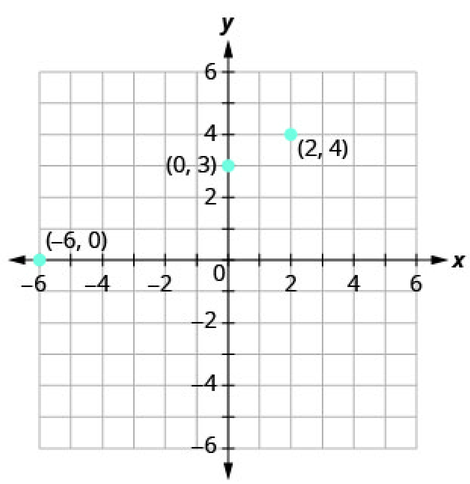
Solution:

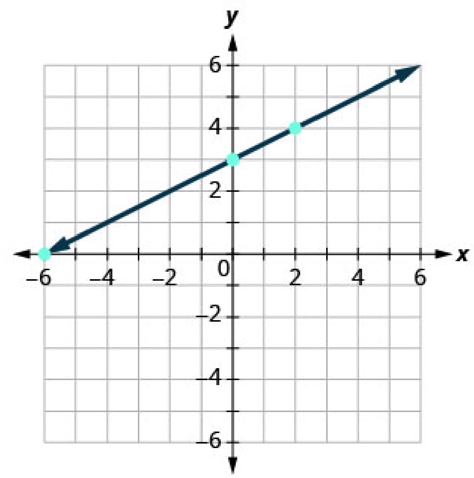
Step 1. Find the x- and y-intercepts of the line. Let $y = 0$ and solve for x.	Find the x-intercept.	Let $y = 0$ $-x + 2y = 6$ $-x + 2(0) = 6$ $-x = 6$ $x = -6$ The x-intercept is $(-6, 0)$.
Let $x = 0$ and solve for y.	Find the y-intercept.	Let $x = 0$. $-x + 2y = 6$ $-0 + 2y = 6$ $2y = 6$ $y = 3$ The y-intercept is $(0, 3)$.

Step 2. Find another solution to the equation.	We'll use $x = 2$.	Let $x = 2$. $-x + 2y = 6$ $-2 + 2y = 6$ $2y = 8$ $y = 4$ A third point is $(2, 4)$.

Step 3. Plot the three points. Check that the points line up.

x	y	(x, y)
−6	0	(−6, 0)
0	3	(0, 3)
2	4	(2, 4)

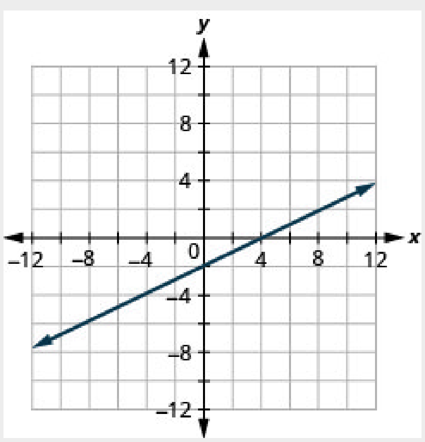


Step 4. Draw the line.	See the graph.	

Note:
Exercise:

Problem: Graph using the intercepts: $x-2y=4$.

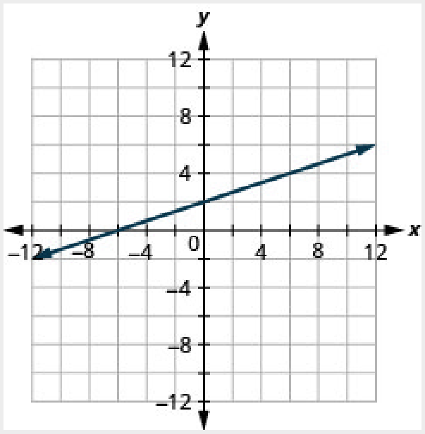
Solution:



Note:
Exercise:

Problem: Graph using the intercepts: $-x+3y=6$.

Solution:



The steps to graph a linear equation using the intercepts are summarized here.

Note:
Graph a linear equation using the intercepts.

Find the*x*- and*y*-intercepts of the line.

- Let $y = 0$ and solve for *x*.
- Let $x = 0$ and solve for *y*.

Find a third solution to the equation.
Plot the three points and check that they line up.
Draw the line.

Example:
Exercise:

Problem: Graph $4x - 3y = 12$ using the intercepts.

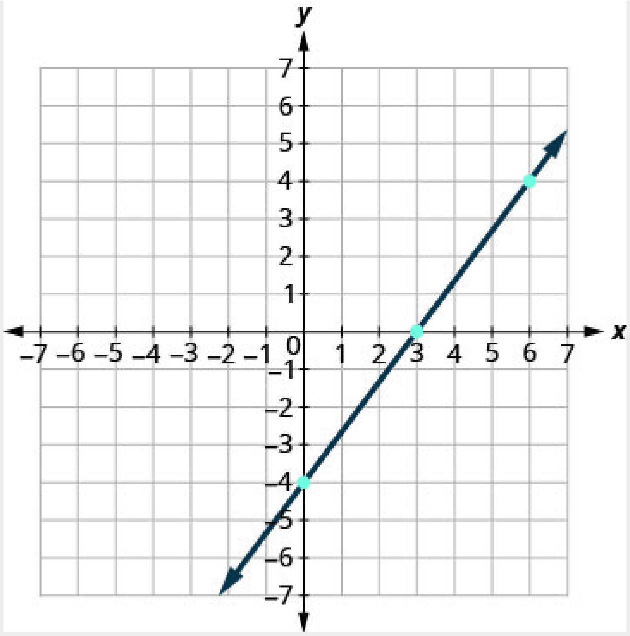
Solution:

Find the intercepts and a third point.

x-intercept, let $y = 0$	y-intercept, let $x = 0$	third point, let $y = 4$
$4x - 3y = 12$	$4x - 3y = 12$	$4x - 3y = 12$
$4x - 3(0) = 12$	$4(0) - 3y = 12$	$4x - 3(4) = 12$
$4x = 12$	$-3y = 12$	$4x - 12 = 12$
$x = 3$	$y = -4$	$4x = 24$
		$x = 6$

We list the points in the table and show the graph.

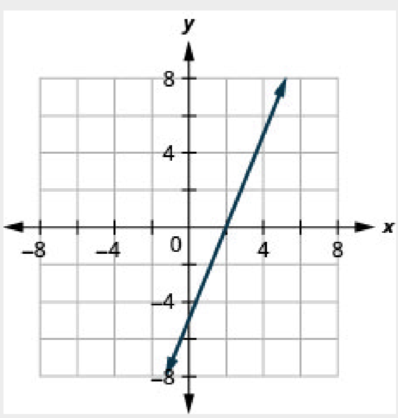
$4x - 3y = 12$		
x	***y***	(x, y)
3	0	$(3, 0)$
0	-4	$(0, -4)$
6	4	$(6, 4)$



Note:
Exercise:

Problem: Graph using the intercepts: $5x - 2y = 10$.

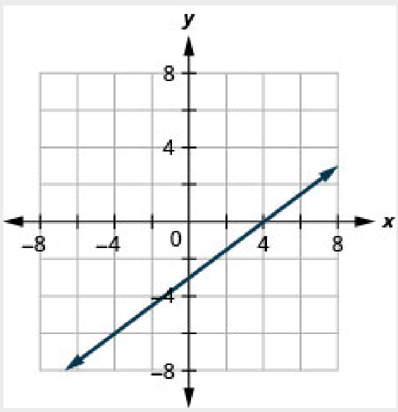
Solution:



Note:
Exercise:

Problem: Graph using the intercepts: $3x - 4y = 12$.

Solution:



When the line passes through the origin, the *x*-intercept and the *y*-intercept are the same point.

Example:
Exercise:

Problem: Graph $y = 5x$ using the intercepts.

Solution:

x-intercept	*y*-intercept
Let $y = 0$.	Let $x = 0$.
$y = 5x$	$y = 5x$
$0 = 5x$	$y = 5 \cdot 0$
$0 = x$	$y = 0$
$(0, 0)$	$(0, 0)$

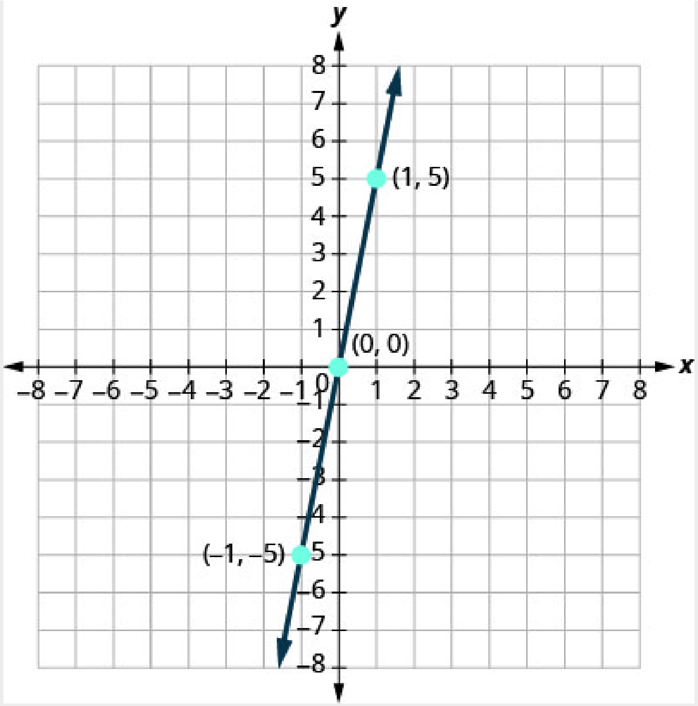
This line has only one intercept. It is the point $(0, 0)$.
To ensure accuracy, we need to plot three points. Since the *x*- and *y*-intercepts are the same point, we need *two* more points to graph the line.

Let $x = 1$.	Let $x = -1$.
$y = 5x$	$y = 5x$
$y = 5 \cdot 1$	$y = 5(-1)$
$y = 5$	$y = -5$

The resulting three points are summarized in the table.

$y = 5x$		
x	***y***	(x, y)
0	0	$(0, 0)$
1	5	$(1, 5)$
-1	-5	$(-1, -5)$

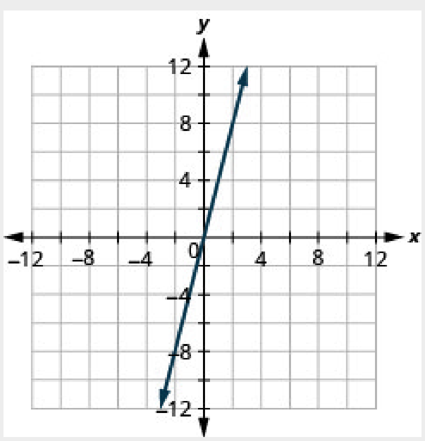
Plot the three points, check that they line up, and draw the line.



Note:
Exercise:

Problem: Graph using the intercepts: $y = 4x$.

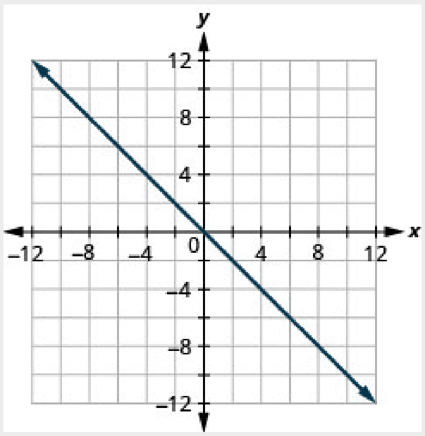
Solution:



Note:
Exercise:

Problem: Graph the intercepts: $y = -x$.

Solution:

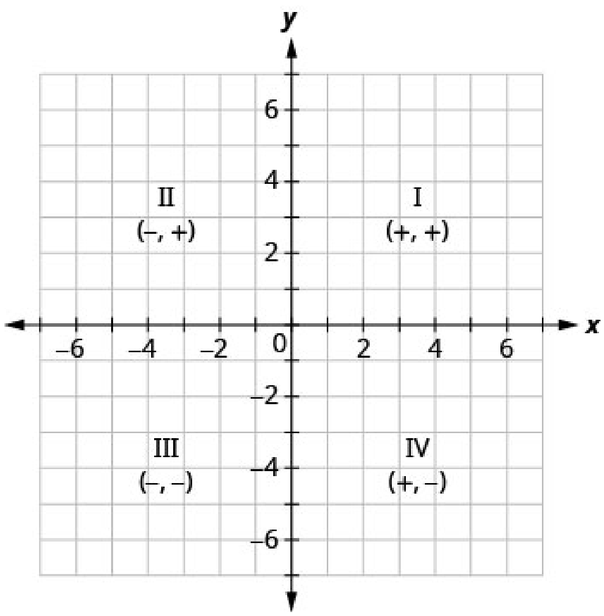


Key Concepts

- **Points on the Axes**
  - Points with a *y*-coordinate equal to 0 are on the *x*-axis, and have coordinates $(a, 0)$.
  - Points with an *x*-coordinate equal to 0 are on the *y*-axis, and have coordinates $(0, b)$.

- **Quadrant**
  Equation:

Quadrant I	Quadrant II	Quadrant III	Quadrant IV
(x, y)	(x, y)	(x, y)	(x, y)
$(+, +)$	$(-, +)$	$(-, -)$	$(+, -)$



- **Graph of a Linear Equation:** The graph of a linear equation $Ax + By = C$ is a straight line.
  Every point on the line is a solution of the equation.
  Every solution of this equation is a point on this line.
- **How to graph a linear equation by plotting points.**

  Find three points whose coordinates are solutions to the equation. Organize them in a table.
  Plot the points in a rectangular coordinate system. Check that the points line up. If they do not, carefully check your work.
  Draw the line through the three points. Extend the line to fill the grid and put arrows on both ends of the line.

- ***x*-intercept and *y*-intercept of a Line**

  - The *x*-intercept is the point $(a, 0)$ where the line crosses the *x*-axis.
  - The *y*-intercept is the point $(0, b)$ where the line crosses the *y*-axis.

  - The *x*-intercept occurs when *y* is zero.
  - The *y*-intercept occurs when *x* is zero.

x	*y*
a	0
0	*b*

- **Find the *x*- and *y*-intercepts from the Equation of a Line**

  - Use the equation of the line. To find:
    the *x*-intercept of the line, let $y = 0$ and solve for *x*.
    the *y*-intercept of the line, let $x = 0$ and solve for *y*.

- **How to graph a linear equation using the intercepts.**

  Find the x- and y-intercepts of the line. Let $y = 0$ and solve for x. Let $x = 0$ and solve for y.
  Find a third solution to the equation.
  Plot the three points and check that they line up.
  Draw the line

Practice Makes Perfect

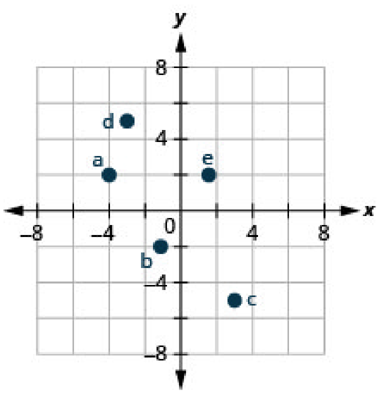
Plot Points in a Rectangular Coordinate System

In the following exercises, plot each point in a rectangular coordinate system and identify the quadrant in which the point is located.

Exercise:

Problem: ⓐ $(-4, 2)$ ⓑ $(-1, -2)$ ⓒ $(3, -5)$ ⓓ $(-3, 0)$ ⓔ $\left(\frac{5}{3}, 2\right)$

Solution:



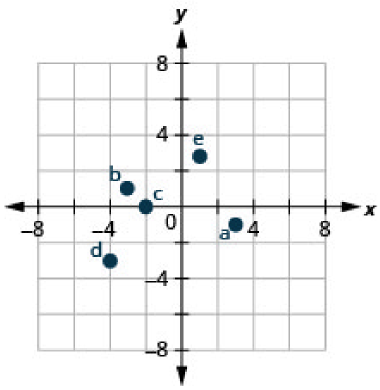
Exercise:

Problem: ⓐ $(-2, -3)$ ⓑ $(3, -3)$ ⓒ $(-4, 1)$ ⓓ $(4, -1)$ ⓔ $\left(\frac{3}{2}, 1\right)$

Exercise:

Problem: ⓐ $(3, -1)$ ⓑ $(-3, 1)$ ⓒ $(-2,\ 0)$ ⓓ $(-4, -3)$ ⓔ $\left(1,\ \frac{14}{5}\right)$

Solution:



Exercise:

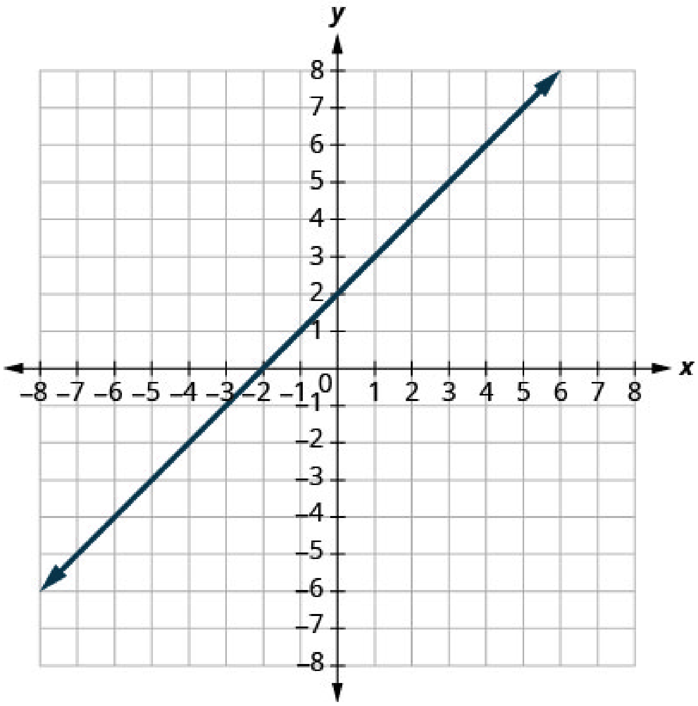
Problem: ⓐ $(-1,1)$ ⓑ $(-2,-1)$ ⓒ $(2,0)$ ⓓ $(1,-4)$ ⓔ $\left(3, \frac{7}{2}\right)$

In the following exercises, for each ordered pair, decide

ⓐ is the ordered pair a solution to the equation? ⓑ is the point on the line?

Exercise:

Problem: $y = x + 2$; A: $(0,2)$; B: $(1,2)$; C: $(-1,1)$; D: $(-3,-1)$.

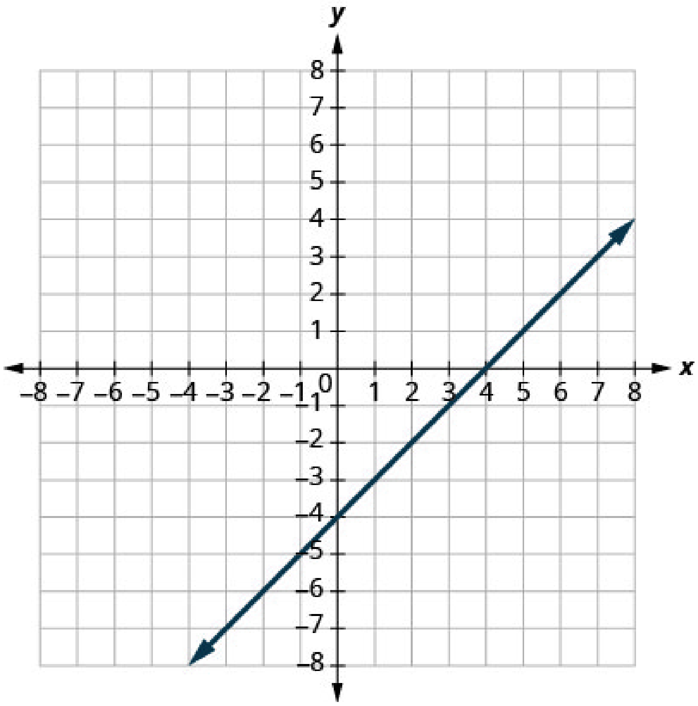


Solution:

ⓐ A: yes, B: no, C: yes, D: yes ⓑ A: yes, B: no, C: yes, D: yes

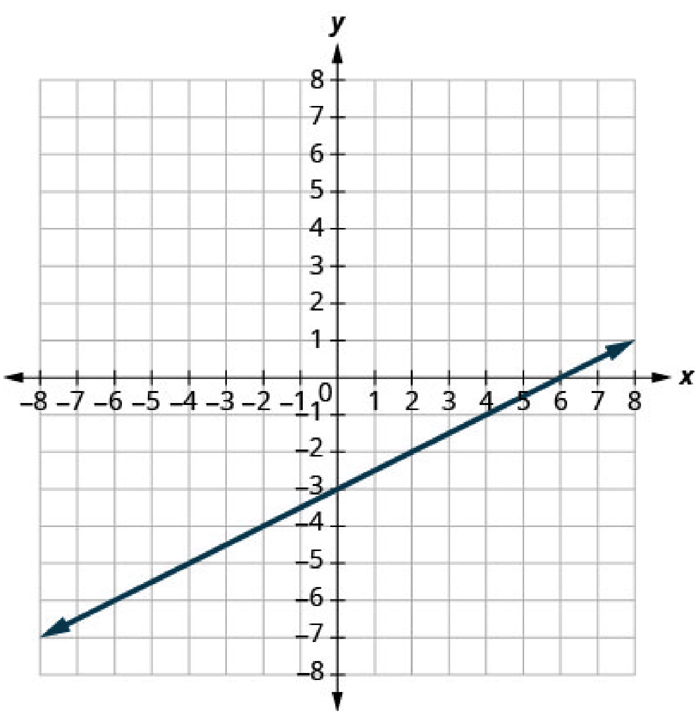
Exercise:

Problem: $y = x - 4;$
A: $(0, -4)$; B: $(3, -1)$; C: $(2, 2)$; D: $(1, -5)$.



Exercise:

Problem: $y = \frac{1}{2}x - 3$; A: $(0, -3)$; B: $(2, -2)$; C: $(-2, -4)$; D: $(4, 1)$



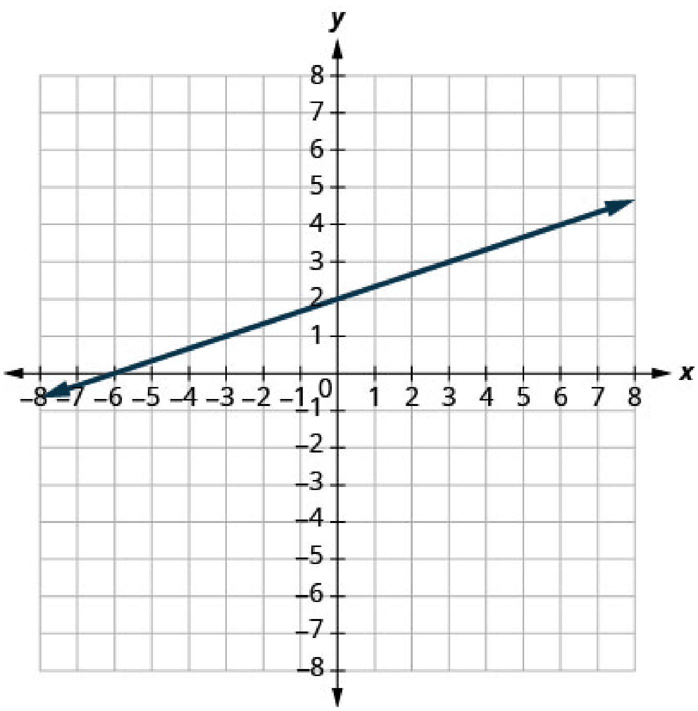
Solution:

ⓐ A: yes, B: yes, C: yes, D: no ⓑ A: yes, B: yes, C: yes, D: no

Exercise:

$$y = \frac{1}{3}x + 2;$$

Problem: A: $(0, 2)$; B: $(3, 3)$; C: $(-3, 2)$; D: $(-6, 0)$.



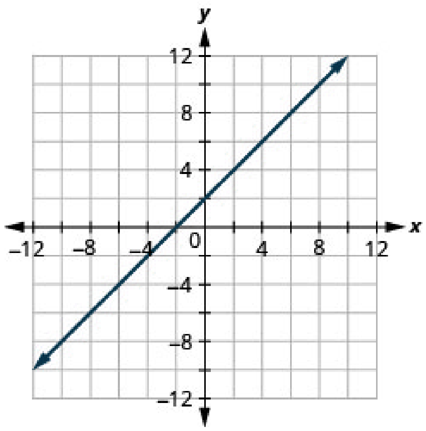
Graph a Linear Equation by Plotting Points

In the following exercises, graph by plotting points.

Exercise:

Problem: $y = x + 2$

Solution:



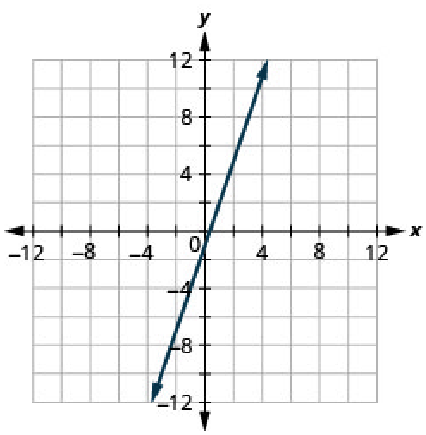
Exercise:

Problem: $y = x - 3$

Exercise:

Problem: $y = 3x - 1$

Solution:



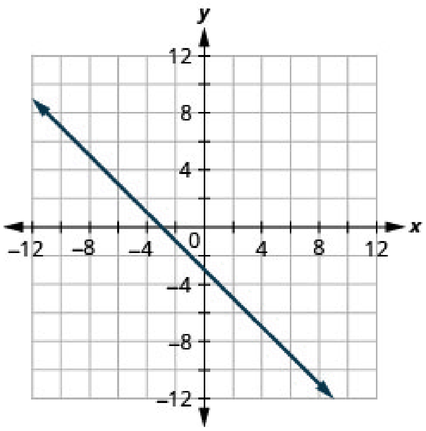
Exercise:

Problem: $y = -2x + 2$

Exercise:

Problem: $y = -x - 3$

Solution:



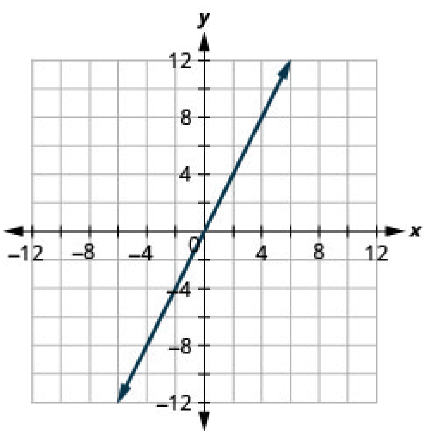
Exercise:

Problem: $y = -x - 2$

Exercise:

Problem: $y = 2x$

Solution:



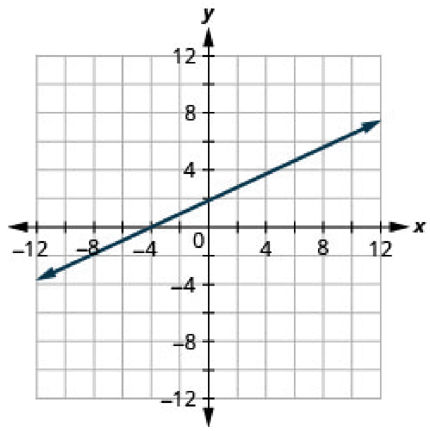
Exercise:

Problem: $y = -2x$

Exercise:

Problem: $y = \frac{1}{2}x + 2$

Solution:



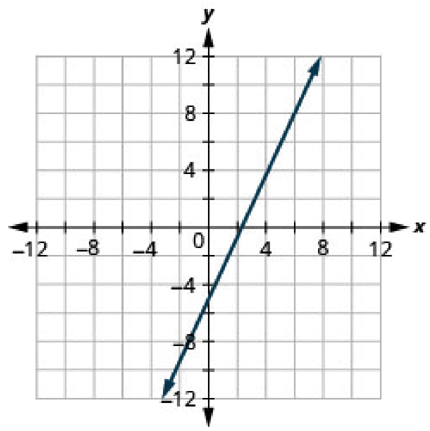
Exercise:

Problem: $y = \frac{1}{3}x - 1$

Exercise:

Problem: $y = \frac{4}{3}x - 5$

Solution:



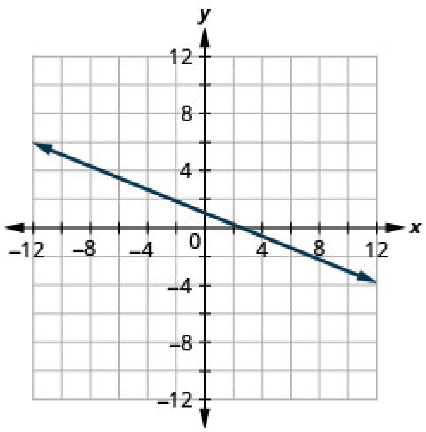
Exercise:

Problem: $y = \frac{3}{2}x - 3$

Exercise:

Problem: $y = -\frac{2}{5}x + 1$

Solution:



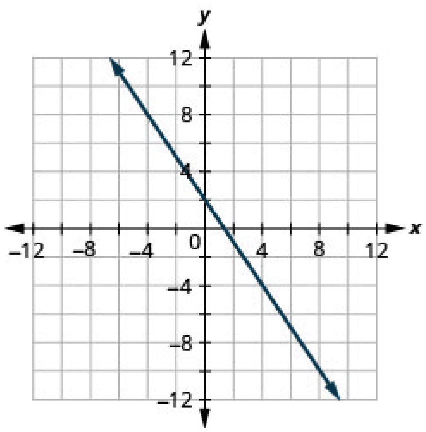
Exercise:

Problem: $y = -\frac{4}{5}x - 1$

Exercise:

Problem: $y = -\frac{3}{2}x + 2$

Solution:



Exercise:

Problem: $y = -\frac{5}{3}x + 4$

Graph Vertical and Horizontal lines

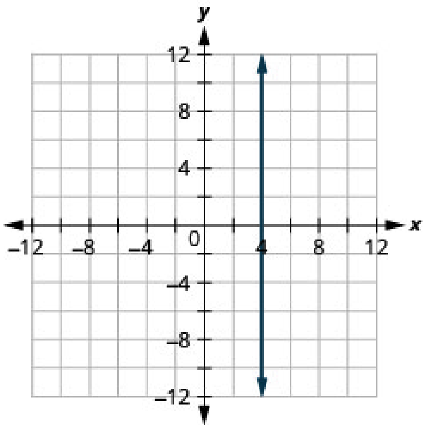
In the following exercises, graph each equation.

Exercise:

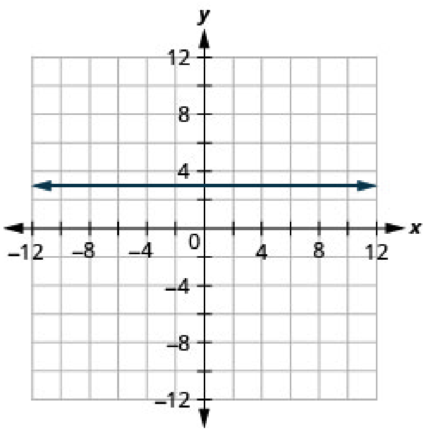
Problem: ⓐ $x=4$ ⓑ $y=3$

Solution:

ⓐ



ⓑ



Exercise:

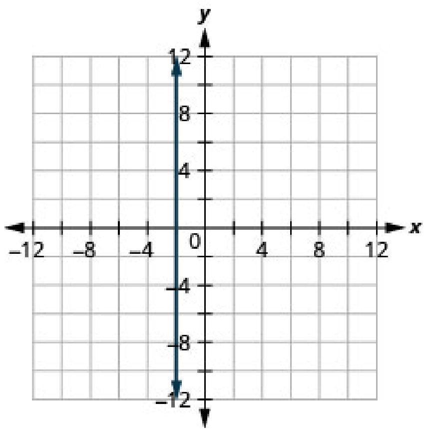
Problem: ⓐ $x=3$ ⓑ $y=1$

Exercise:

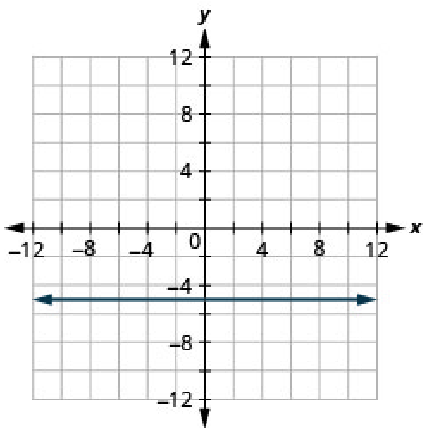
Problem: ⓐ $x=-2$ ⓑ $y=-5$

Solution:

ⓐ



ⓑ



Exercise:

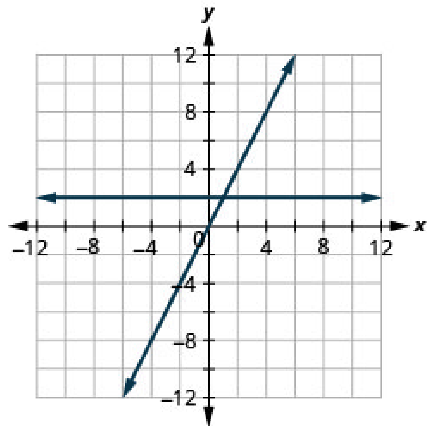
Problem: ⓐ $x=-5$ ⓑ $y=-2$

In the following exercises, graph each pair of equations in the same rectangular coordinate system.

Exercise:

Problem: $y = 2x$ and $y = 2$

Solution:



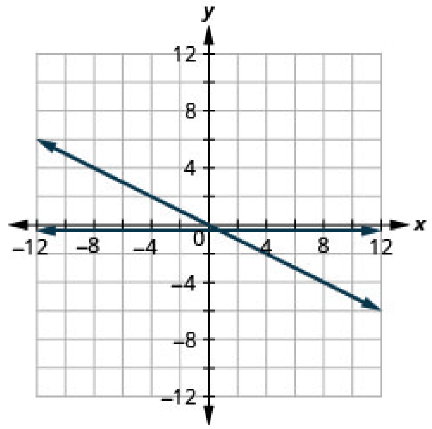
Exercise:

Problem: $y = 5x$ and $y = 5$

Exercise:

Problem: $y = -\frac{1}{2}x$ and $y = -\frac{1}{2}$

Solution:



Exercise:

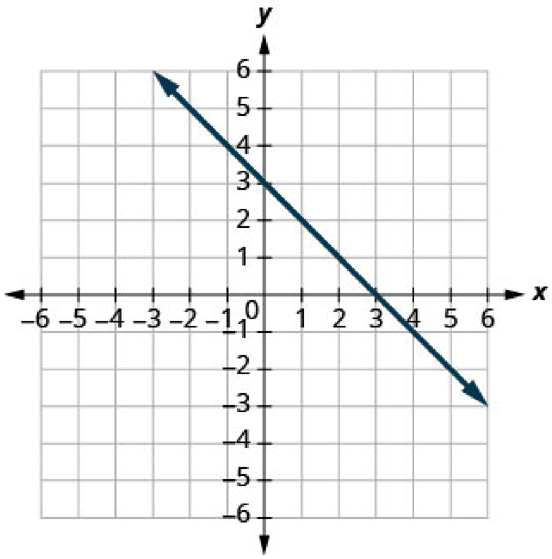
Problem: $y = -\frac{1}{3}x$ and $y = -\frac{1}{3}$

Find *x*- and *y*-Intercepts

In the following exercises, find the *x*- and *y*-intercepts on each graph.

Exercise:

Problem:

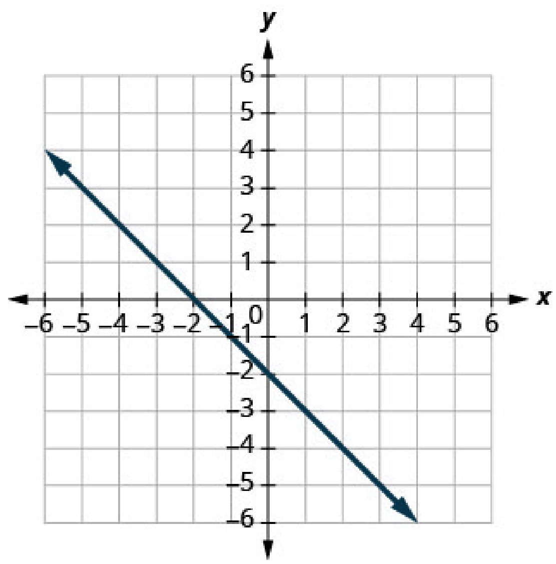


Solution:

$(3,0),(0,3)$

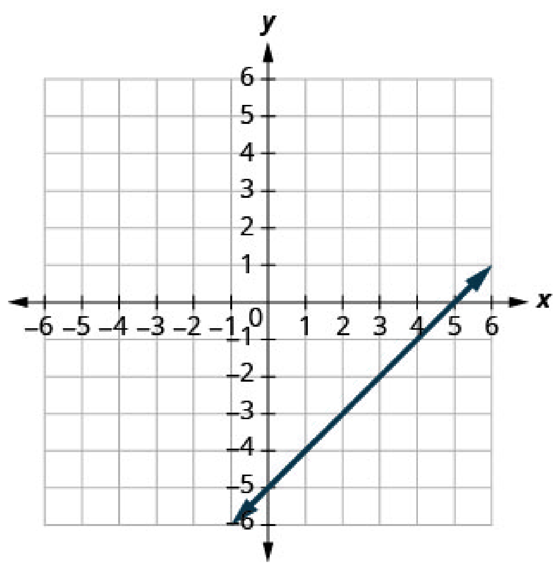
Exercise:

Problem:



Exercise:

Problem:

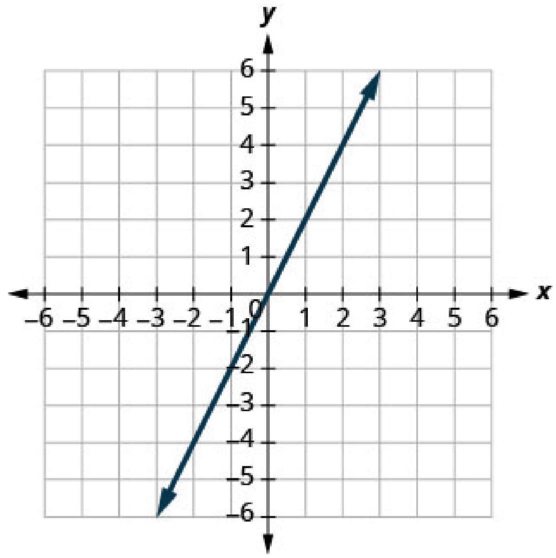


Solution:

$(5,0),(0,-5)$

Exercise:

Problem:



In the following exercises, find the intercepts for each equation.

Exercise:

Problem: $x - y = 5$

Solution:

$(5, 0), (0, -5)$

Exercise:

Problem: $x - y = -4$

Exercise:

Problem: $3x + y = 6$

Solution:

$(2, 0), (0, 6)$

Exercise:

Problem: $x - 2y = 8$

Exercise:

Problem: $4x - y = 8$

Solution:

$(2, 0), (0, -8)$

Exercise:

Problem: $5x - y = 5$

Exercise:

Problem: $2x + 5y = 10$

Solution:

$(5, 0), (0, 2)$

Exercise:

Problem: $3x - 2y = 12$

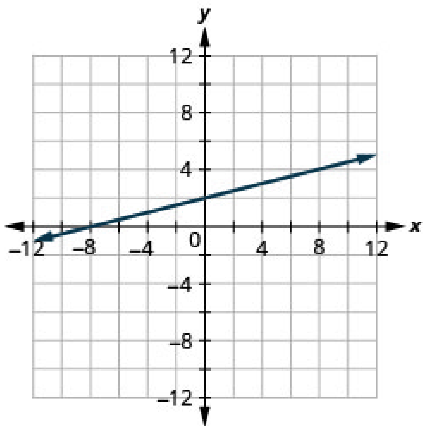
Graph a Line Using the Intercepts

In the following exercises, graph using the intercepts.

Exercise:

Problem: $-x + 4y = 8$

Solution:



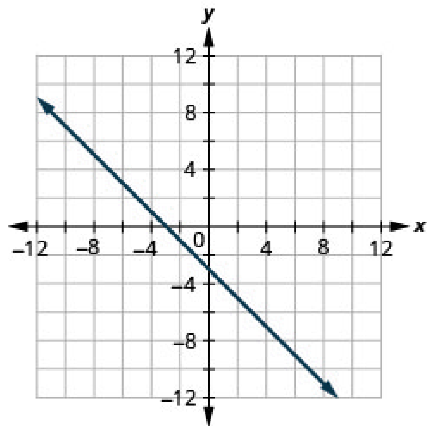
Exercise:

Problem: $x + 2y = 4$

Exercise:

Problem: $x + y = -3$

Solution:



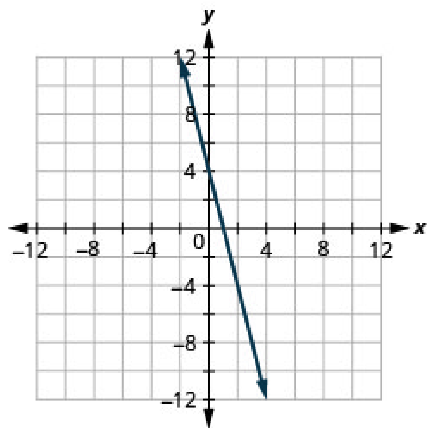
Exercise:

Problem: $x - y = -4$

Exercise:

Problem: $4x + y = 4$

Solution:



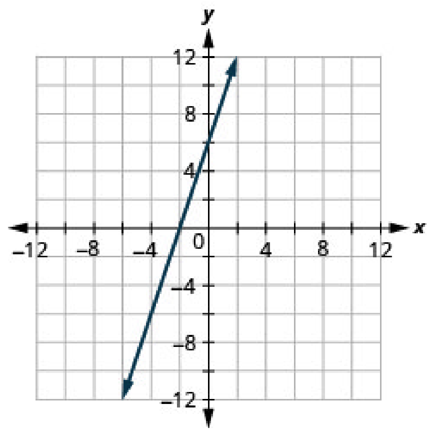
Exercise:

Problem: $3x + y = 3$

Exercise:

Problem: $3x - y = -6$

Solution:



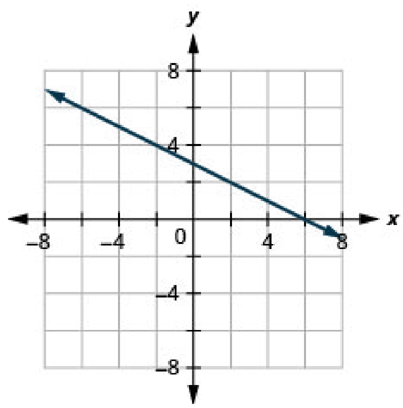
Exercise:

Problem: $2x - y = -8$

Exercise:

Problem: $2x + 4y = 12$

Solution:



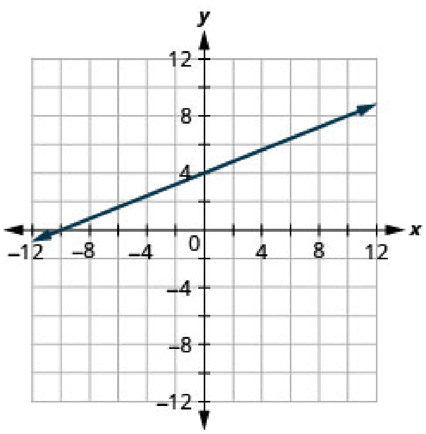
Exercise:

Problem: $3x - 2y = 6$

Exercise:

Problem: $2x - 5y = -20$

Solution:



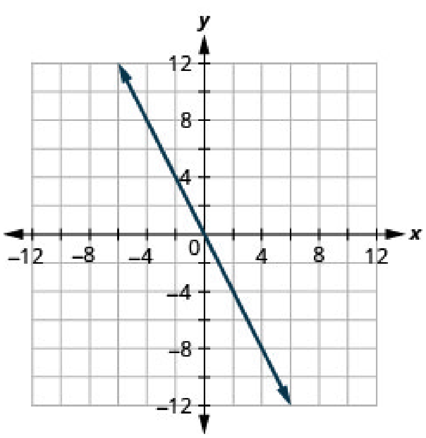
Exercise:

Problem: $3x - 4y = -12$

Exercise:

Problem: $y = -2x$

Solution:



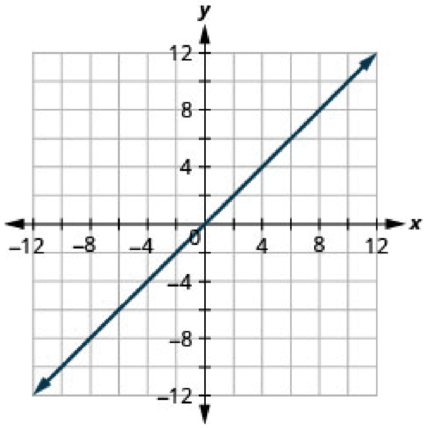
Exercise:

Problem: $y = 5x$

Exercise:

Problem: $y = x$

Solution:



Exercise:

Problem: $y = -x$

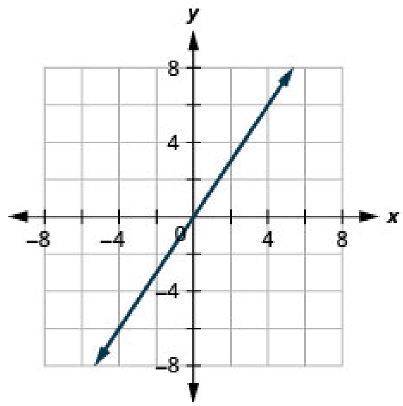
Mixed Practice

In the following exercises, graph each equation.

Exercise:

Problem: $y = \frac{3}{2}x$

Solution:



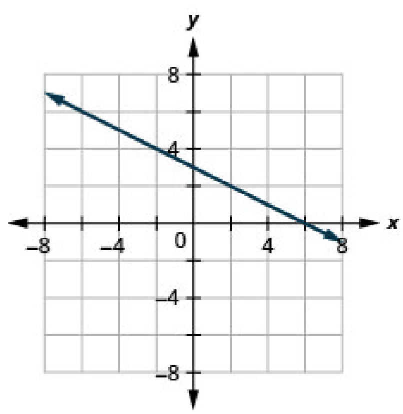
Exercise:

Problem: $y = -\frac{2}{3}x$

Exercise:

Problem: $y = -\frac{1}{2}x + 3$

Solution:



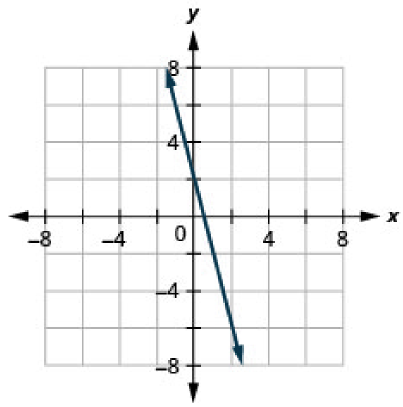
Exercise:

Problem: $y = \frac{1}{4}x - 2$

Exercise:

Problem: $4x + y = 2$

Solution:



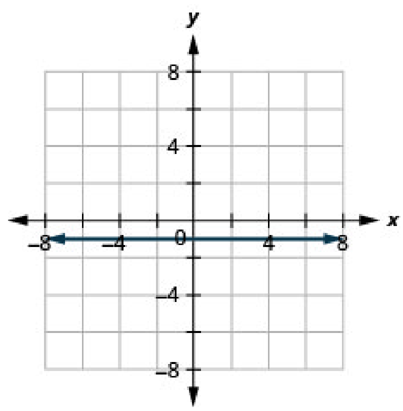
Exercise:

Problem: $5x + 2y = 10$

Exercise:

Problem: $y = -1$

Solution:



Exercise:

Problem: $x = 3$

Writing Exercises

Exercise:

Problem:

Explain how you would choose three *x*-values to make a table to graph the line $y=\frac{1}{5}x-2$.

Solution:

Answers will vary.

Exercise:

Problem:

What is the difference between the equations of a vertical and a horizontal line?

Exercise:

Problem:

Do you prefer to use the method of plotting points or the method using the intercepts to graph the equation $4x+y=-4$? Why?

Solution:

Answers will vary.

Exercise:

Problem:

Do you prefer to use the method of plotting points or the method using the intercepts to graph the equation $y=\frac{2}{3}x-2$? Why?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
plot points on a rectangular coordinate system.			
graph a linear equation by plotting points.			
graph vertical and horizontal lines.			
find *x*- and *y*-intercepts.			
graph a line using the intercepts.			

ⓑ If most of your checks were:

Confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

With some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

No, I don't get it. This is a warning sign and you must address it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

Glossary

horizontal line
: A horizontal line is the graph of an equation of the form $y=b$. The line passes through the *y*-axis at $(0,b)$.

intercepts of a line
: The points where a line crosses the *x*-axis and the *y*-axis are called the intercepts of the line.

linear equation
: An equation of the form $Ax+By=C$, where *A* and *B* are not both zero, is called a linear equation in two variables.

ordered pair

An ordered pair, (x, y) gives the coordinates of a point in a rectangular coordinate system. The first number is the *x*-coordinate. The second number is the *y*-coordinate.

origin
: The point $(0, 0)$ is called the origin. It is the point where the *x*-axis and *y*-axis intersect.

solution of a linear equation in two variables
: An ordered pair (x, y) is a solution of the linear equation$Ax + By = C$, if the equation is a true statement when the *x*- and *y*-values of the ordered pair are substituted into the equation.

standard form of a linear equation
: A linear equation is in standard form when it is written $Ax + By = C$.

vertical line
: A vertical line is the graph of an equation of the form $x = a$. The line passes through the *x*-axis at $(a, 0)$.

Slope of a Line: ASE
By the end of this section, you will be able to:

- Find the slope of a line
- Graph a line given a point and the slope
- Graph a line using its slope and intercept
- Choose the most convenient method to graph a line
- Graph and interpret applications of slope–intercept
- Use slopes to identify parallel and perpendicular lines

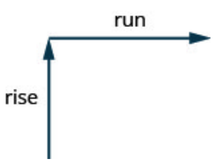
Find the Slope of a Line

When you graph linear equations, you may notice that some lines tilt up as they go from left to right and some lines tilt down. Some lines are very steep and some lines are flatter.

In mathematics, the measure of the steepness of a line is called the *slope* of the line.

The concept of slope has many applications in the real world. In construction the pitch of a roof, the slant of the plumbing pipes, and the steepness of the stairs are all applications of slope. and as you ski or jog down a hill, you definitely experience slope.

We can assign a numerical value to the slope of a line by finding the ratio of the rise and run. The *rise* is the amount the vertical distance changes while the *run* measures the horizontal change, as shown in this illustration. Slope is a rate of change. See [link].



Note:
Slope of a Line
The slope of a line is $m=\frac{\text{rise}}{\text{run}}$.
The rise measures the vertical change and the run measures the horizontal change.

To find the slope of a line, we locate two points on the line whose coordinates are integers. Then we sketch a right triangle where the two points are vertices and one side is horizontal and one side is vertical.

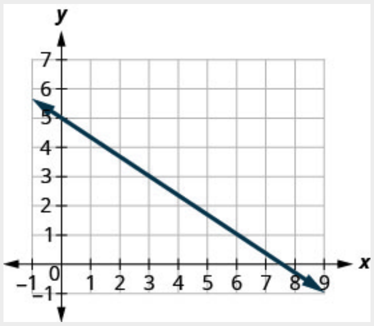
To find the slope of the line, we measure the distance along the vertical and horizontal sides of the triangle. The vertical distance is called the *rise* and the horizontal distance is called the *run*,

Note:
Find the slope of a line from its graph using $m=\frac{\text{rise}}{\text{run}}$.

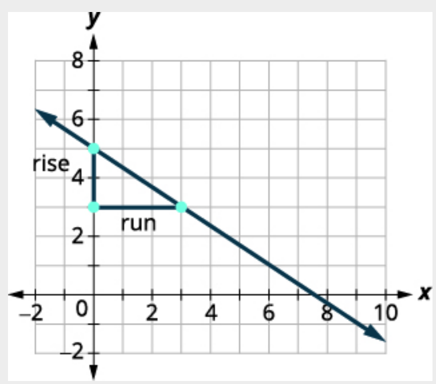
Locate two points on the line whose coordinates are integers.
Starting with one point, sketch a right triangle, going from the first point to the second point.
Count the rise and the run on the legs of the triangle.
Take the ratio of rise to run to find the slope:$m=\frac{\text{rise}}{\text{run}}$.

Example:
Exercise:

Problem: Find the slope of the line shown.



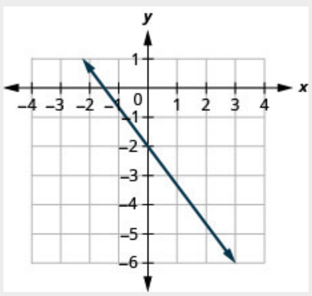
Solution:

Locate two points on the graph whose coordinates are integers.	$(0,5)$ and $(3,3)$
Starting at $(0,5)$, sketch a right triangle to $(3,3)$ as shown in this graph.	
Count the rise— since it goes down, it is negative.	The rise is -2.
Count the run.	The run is 3.
Use the slope formula.	$m=\frac{\text{rise}}{\text{run}}$
Substitute the values of the rise and run.	$m=\frac{-2}{3}$
Simplify.	$m=-\frac{2}{3}$

	The slope of the line is $-\frac{2}{3}$.
	So *y* decreases by 2 units as *x* increases by 3 units.

Note:
Exercise:

Problem: Find the slope of the line shown.

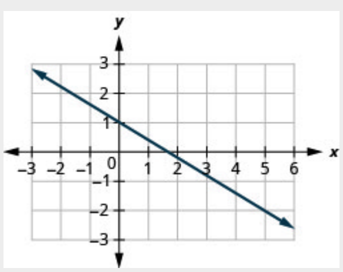


Solution:

$-\frac{4}{3}$

Note:
Exercise:

Problem: Find the slope of the line shown.

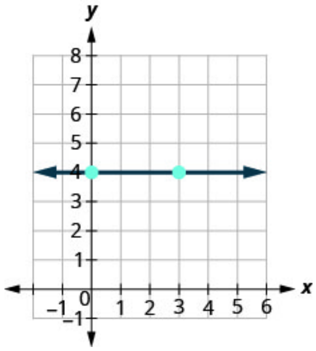


Solution:

$-\frac{3}{5}$

How do we find the slope of horizontal and vertical lines? To find the slope of the horizontal line, $y = 4$, we could graph the line, find two points on it, and count the rise and the run. Let’s see what happens when we do this, as

shown in the graph below.



What is the rise? The rise is 0.

What is the run? The run is 3.

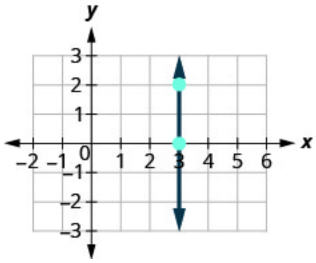
What is the slope? $m = \frac{\text{rise}}{\text{run}}$

$m = \frac{0}{3}$

$m = 0$

The slope of the horizontal line $y = 4$ is 0.

Let's also consider a vertical line, the line $x = 3$, as shown in the graph.



What is the rise? The rise is 2.

What is the run? The run is 0.

What is the slope? $m = \frac{\text{rise}}{\text{run}}$

$m = \frac{2}{0}$

The slope is undefined since division by zero is undefined. So we say that the slope of the vertical line $x = 3$ is undefined.

All horizontal lines have slope 0. When the *y*-coordinates are the same, the rise is 0.

The slope of any vertical line is undefined. When the *x*-coordinates of a line are all the same, the run is 0.

Note:
Slope of a Horizontal and Vertical Line
The slope of a horizontal line, $y = b$, is 0.
The slope of a vertical line, $x = a$, is undefined.

Example:
Exercise:

Problem: Find the slope of each line: ⓐ $x=8$ ⓑ $y=-5$.

Solution:

ⓐ $x=8$
This is a vertical line. Its slope is undefined.
ⓑ $y=-5$
This is a horizontal line. It has slope 0.

Note:
Exercise:

Problem: Find the slope of the line: $x=-4$.

Solution:

undefined

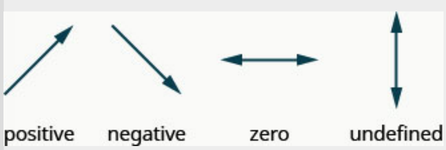
Note:
Exercise:

Problem: Find the slope of the line: $y=7$.

Solution:

0

Note:
Quick Guide to the Slopes of Lines



Sometimes we'll need to find the slope of a line between two points when we don't have a graph to count out the rise and the run. We could plot the points on grid paper, then count out the rise and the run, but as we'll see, there is a way to find the slope without graphing. Before we get to it, we need to introduce some algebraic notation.

We have seen that an ordered pair (x,y) gives the coordinates of a point. But when we work with slopes, we use two points. How can the same symbol (x,y) be used to represent two different points? Mathematicians use subscripts to distinguish the points.

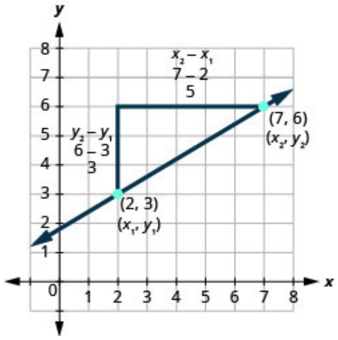
Equation:

$$\begin{array}{ll}(x_1, y_1) & \text{read "}x\text{ sub 1, }y\text{ sub 1"} \\ (x_2, y_2) & \text{read "}x\text{ sub 2, }y\text{ sub 2"}\end{array}$$

We will use (x_1, y_1) to identify the first point and (x_2, y_2) to identify the second point.

If we had more than two points, we could use (x_3, y_3),(x_4, y_4), and so on.

Let's see how the rise and run relate to the coordinates of the two points by taking another look at the slope of the line between the points $(2, 3)$ and $(7, 6)$, as shown in this graph.



Since we have two points, we will use subscript notation.	$\begin{pmatrix} x_1,y_1 \\ 2,3 \end{pmatrix} \begin{pmatrix} x_2,y_2 \\ 7,6 \end{pmatrix}$
	$m = \frac{\text{rise}}{\text{run}}$
On the graph, we counted the rise of 3 and the run of 5.	$m = \frac{3}{5}$
Notice that the rise of 3 can be found by subtracting the y-coordinates, 6 and 3, and the run of 5 can be found by subtracting the x-coordinates 7 and 2.	
We rewrite the rise and run by putting in the coordinates.	$m = \frac{6-3}{7-2}$
But 6 is y_2, the y-coordinate of the second point and 3 is y_1, the y-coordinate of the first point. So we can rewrite the slope using subscript notation.	$m = \frac{y_2-y_1}{7-2}$
Also 7 is the x-coordinate of the second point and 2 is the x-coordinate of the first point. So again we rewrite the slope using subscript notation.	$m = \frac{y_2-y_1}{x_2-x_1}$

We've shown that $m = \frac{y_2-y_1}{x_2-x_1}$ is really another version of $m = \frac{\text{rise}}{\text{run}}$. We can use this formula to find the slope of a line when we have two points on the line.

Note:
Slope of a line between two points
The slope of the line between two points (x_1, y_1) and (x_2, y_2) is:
Equation:

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

The slope is:
Equation:

$$y \text{ of the second point minus } y \text{ of the first point}$$
$$\text{over}$$
$$x \text{ of the second point minus } x \text{ of the first point.}$$

Example:
Exercise:

Problem: Use the slope formula to find the slope of the line through the points $(-2,-3)$ and $(-7,4)$.

Solution:

We'll call $(-2,-3)$ point #1 and $(-7,4)$ point #2. $\begin{pmatrix} x_1, & y_1 \\ -2, & -3 \end{pmatrix} \begin{pmatrix} x_2, & y_2 \\ -7, & 4 \end{pmatrix}$

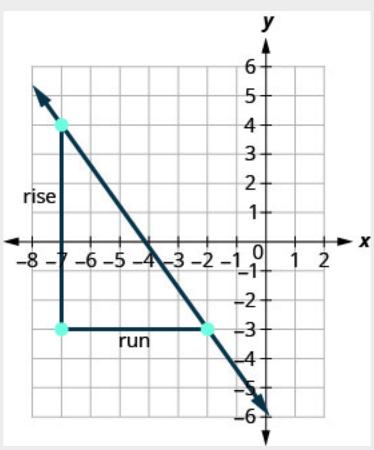
Use the slope formula. $m = \frac{y_2 - y_1}{x_2 - x_1}$

Substitute the values.
y of the second point minus y of the first point
x of the second point minus x of the first point $m = \frac{4-(-3)}{-7-(-2)}$

$m = \frac{7}{-5}$

Simplify. $m = -\frac{7}{5}$

Let's verify this slope on the graph shown.



Equation:

$$m = \frac{\text{rise}}{\text{run}}$$
$$m = \frac{7}{-5}$$
$$m = -\frac{7}{5}$$

Note:
Exercise:

Problem: Use the slope formula to find the slope of the line through the pair of points: $(-3,4)$ and $(2,-1)$.

Solution:

-1

Note:
Exercise:

Problem:

Use the slope formula to find the slope of the line through the pair of points: $(-2,6)$ and $(-3,-4)$.

Solution:

10

Graph a Line Given a Point and the Slope

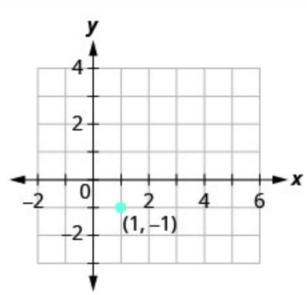
Up to now, in this chapter, we have graphed lines by plotting points, by using intercepts, and by recognizing horizontal and vertical lines.

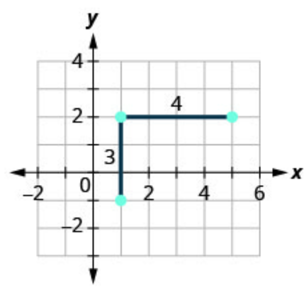
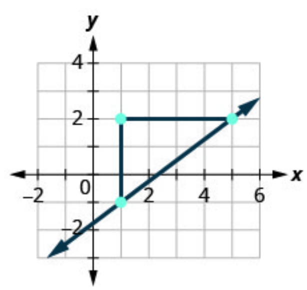
We can also graph a line when we know one point and the slope of the line. We will start by plotting the point and then use the definition of slope to draw the graph of the line.

Example:
How to graph a Line Given a Point and the Slope
Exercise:

Problem: Graph the line passing through the point $(1,-1)$ whose slope is $m=\frac{3}{4}$.

Solution:

Step 1. Plot the given point.	Plot (1, –1).	

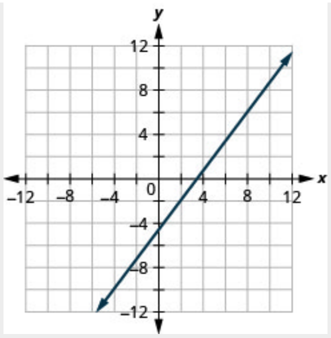
Step 2. Use the slope formula $m = \frac{\text{rise}}{\text{run}}$ to identify the rise and the run.	Identify the rise and the run.	$m = \frac{3}{4}$ $\frac{\text{rise}}{\text{run}} = \frac{3}{4}$ $\text{rise} = 3$ $\text{run} = 4$
Step 3. Starting at the given point, count out the rise and run to mark the second point.	Start at (1, –1) and count the rise and the run. Up 3 units, right 4 units.	
Step 4. Connect the points with a line.	Connect the two points with a line.	

You can check your work by finding a third point. Since the slope is $m = \frac{3}{4}$, it can also be written as $m = \frac{-3}{-4}$ (negative divided by negative is positive!). Go back to $(1, -1)$ and count out the rise, -3, and the run, -4.

Note:
Exercise:

Problem: Graph the line passing through the point $(2, -2)$ with the slope$m = \frac{4}{3}$.

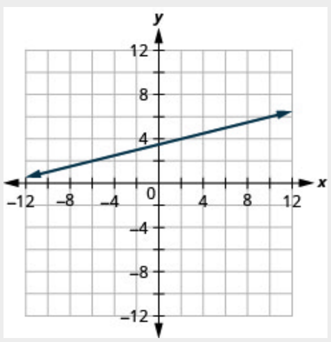
Solution:



Note:
Exercise:

Problem: Graph the line passing through the point $(-2, 3)$ with the slope $m = \frac{1}{4}$.

Solution:



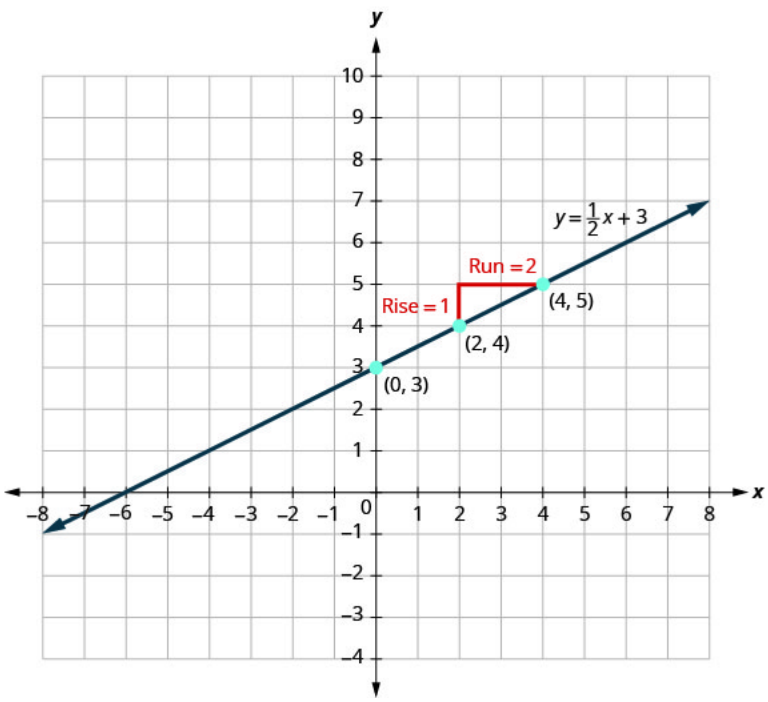
Note:
Graph a line given a point and the slope.

Plot the given point.
Use the slope formula$m = \frac{\text{rise}}{\text{run}}$to identify the rise and the run.
Starting at the given point, count out the rise and run to mark the second point.
Connect the points with a line.

Graph a Line Using its Slope and Intercept

We have graphed linear equations by plotting points, using intercepts, recognizing horizontal and vertical lines, and using one point and the slope of the line. Once we see how an equation in slope–intercept form and its graph are related, we'll have one more method we can use to graph lines.

See [link]. Let’s look at the graph of the equation $y=\frac{1}{2}x+3$ and find its slope and *y*-intercept.



The red lines in the graph show us the rise is 1 and the run is 2. Substituting into the slope formula:
Equation:

$$m=\frac{\text{rise}}{\text{run}}$$
$$m=\frac{1}{2}$$

The *y*-intercept is $(0,3)$.

Look at the equation of this line.

$y=\frac{1}{2}x+3$

Look at the slope and *y*-intercept.

slope $m=\frac{1}{2}$ and *y*-intercept (0, 3).

When a linear equation is solved for *y*, the coefficient of the *x* term is the slope and the constant term is the *y*-coordinate of the *y*-intercept. We say that the equation $y=\frac{1}{2}x+3$ is in slope–intercept form. Sometimes the slope–intercept form is called the “*y*-form.”

$m = \frac{1}{2}$; y-intercept is $(0, 3)$

$y = \frac{1}{2}x + 3$

$y = mx + b$

Note:
Slope Intercept Form of an Equation of a Line
The slope–intercept form of an equation of a line with slope m and y-intercept, $(0, b)$ is $y = mx + b$.

Let's practice finding the values of the slope and y-intercept from the equation of a line.

Example:
Exercise:

Problem: Identify the slope and y-intercept of the line from the equation:

ⓐ $y = -\frac{4}{7}x - 2$ ⓑ $x + 3y = 9$

Solution:

ⓐ We compare our equation to the slope–intercept form of the equation.

Write the slope–intercept form of the equation of the line.	$y = mx + b$
Write the equation of the line.	$y = -\frac{4}{7}x - 2$
Identify the slope.	$m = -\frac{4}{7}$
Identify the y-intercept.	y-intercept is $(0, -2)$

ⓑ When an equation of a line is not given in slope–intercept form, our first step will be to solve the equation for y.

Solve for *y*.	$x + 3y = 9$
Subtract *x* from each side.	$3y = -x + 9$
Divide both sides by 3.	$\frac{3y}{3} = \frac{-x + 9}{3}$
Simplify.	$y = -\frac{1}{3}x + 3$
Write the slope–intercept form of the equation of the line.	$y = mx + b$
Write the equation of the line.	$y = -\frac{1}{3}x + 3$
Identify the slope.	$m = -\frac{1}{3}$
Identify the *y*-intercept.	*y*-intercept is (0, 3)

Note:
Exercise:

Problem: Identify the slope and *y*-intercept from the equation of the line.

ⓐ $y = \frac{2}{5}x - 1$ ⓑ $x + 4y = 8$

Solution:

ⓐ $m = \frac{2}{5}; (0, -1)$
ⓑ $m = -\frac{1}{4}; (0, 2)$

Note:
Exercise:

Problem: Identify the slope and *y*-intercept from the equation of the line.

ⓐ $y = -\frac{4}{3}x + 1$ ⓑ $3x + 2y = 12$

Solution:

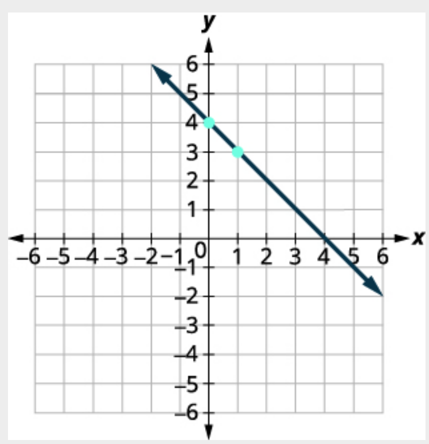
ⓐ $m = -\frac{4}{3}$; $(0, 1)$
ⓑ $m = -\frac{3}{2}$; $(0, 6)$

We have graphed a line using the slope and a point. Now that we know how to find the slope and *y*-intercept of a line from its equation, we can use the *y*-intercept as the point, and then count out the slope from there.

Example:
Exercise:

Problem: Graph the line of the equation $y = -x + 4$ using its slope and *y*-intercept.

Solution:

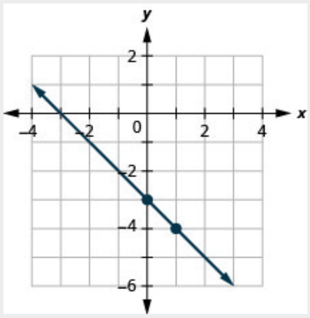
	$y = mx + b$
The equation is in slope–intercept form.	$y = -x + 4$
Identify the slope and *y*-intercept.	$m = -1$ *y*-intercept is $(0, 4)$
Plot the *y*-intercept.	See the graph.
Identify the rise over the run.	$m = \frac{-1}{1}$
Count out the rise and run to mark the second point.	rise -1, run 1 

Draw the line as shown in the graph.

Note:
Exercise:

Problem: Graph the line of the equation $y = -x - 3$ using its slope and *y*-intercept.

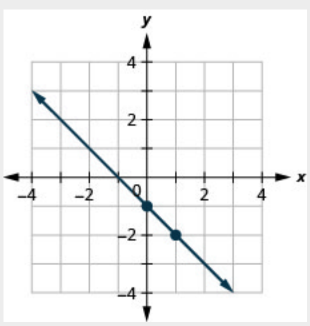
Solution:



Note:
Exercise:

Problem: Graph the line of the equation$y = -x - 1$ using its slope and *y*-intercept.

Solution:



Now that we have graphed lines by using the slope and *y*-intercept, let's summarize all the methods we have used to graph lines.

Methods to Graph Lines			
Point Plotting	**Slope–Intercept**	**Intercepts**	**Recognize Vertical and Horizontal Lines**
x | y	$y = mx + b$	x | y; 0 |; | 0	
Find three points. Plot the points, make sure they line up, then draw the line.	Find the slope and y-intercept. Start at the y-intercept, then count the slope to get a second point.	Find the intercepts and a third point. Plot the points, make sure they line up, then draw the line.	The equation has only one variable. $x = a$ vertical $y = b$ horizontal

Choose the Most Convenient Method to Graph a Line

Now that we have seen several methods we can use to graph lines, how do we know which method to use for a given equation?

While we could plot points, use the slope–intercept form, or find the intercepts for *any* equation, if we recognize the most convenient way to graph a certain type of equation, our work will be easier.

Generally, plotting points is not the most efficient way to graph a line. Let’s look for some patterns to help determine the most convenient method to graph a line.

Here are five equations we graphed in this chapter, and the method we used to graph each of them.
Equation:

	Equation	**Method**
#1	$x = 2$	Vertical line
#2	$y = -1$	Horizontal line
#3	$-x + 2y = 6$	Intercepts
#4	$4x - 3y = 12$	Intercepts
#5	$y = -x + 4$	Slope–intercept

Equations #1 and #2 each have just one variable. Remember, in equations of this form the value of that one variable is constant; it does not depend on the value of the other variable. Equations of this form have graphs that are vertical or horizontal lines.

In equations #3 and #4, both *x* and *y* are on the same side of the equation. These two equations are of the form $Ax + By = C$. We substituted $y = 0$ to find the *x*- intercept and $x = 0$ to find the *y*-intercept, and then found a third point by choosing another value for *x* or *y*.

Equation #5 is written in slope–intercept form. After identifying the slope and *y*-intercept from the equation we used them to graph the line.

This leads to the following strategy.

Note:
Strategy for Choosing the Most Convenient Method to Graph a Line
Consider the form of the equation.

- If it only has one variable, it is a vertical or horizontal line.
    - $x = a$ is a vertical line passing through the *x*-axis at *a*.
    - $y = b$ is a horizontal line passing through the *y*-axis at *b*.
- If *y* is isolated on one side of the equation, in the form $y = mx + b$, graph by using the slope and *y*-intercept.
    - Identify the slope and *y*-intercept and then graph.
- If the equation is of the form $Ax + By = C$, find the intercepts.
    - Find the *x*- and *y*-intercepts, a third point, and then graph.

Example:
Exercise:

Problem: Determine the most convenient method to graph each line:

ⓐ $y = 5$ ⓑ $4x - 5y = 20$ ⓒ $x = -3$ ⓓ $y = -\frac{5}{9}x + 8$

Solution:

ⓐ $y = 5$
This equation has only one variable, *y*. Its graph is a horizontal line crossing the *y*-axis at 5.
ⓑ $4x - 5y = 20$
This equation is of the form $Ax + By = C$. The easiest way to graph it will be to find the intercepts and one more point.
ⓒ $x = -3$
There is only one variable, *x*. The graph is a vertical line crossing the *x*-axis at -3.
ⓓ $y = -\frac{5}{9}x + 8$
Since this equation is in $y = mx + b$ form, it will be easiest to graph this line by using the slope and *y*-intercepts.

Note:
Exercise:

Problem: Determine the most convenient method to graph each line:

ⓐ $3x + 2y = 12$ ⓑ $y = 4$ ⓒ $y = \frac{1}{5}x - 4$ ⓓ $x = -7$.

Solution:

ⓐ intercepts ⓑ horizontal line ⓒ slope-intercept ⓓ vertical line

Note:
Exercise:

Problem: Determine the most convenient method to graph each line:

ⓐ $x=6$ ⓑ $y=-\frac{3}{4}x+1$ ⓒ $y=-8$ ⓓ $4x-3y=-1$.

Solution:

ⓐ vertical line ⓑ slope-intercept ⓒ horizontal line
ⓓ intercepts

Graph and Interpret Applications of Slope–Intercept

Many real-world applications are modeled by linear equations. We will take a look at a few applications here so you can see how equations written in slope–intercept form relate to real world situations.

Usually, when a linear equation models uses real-world data, different letters are used for the variables, instead of using only *x* and *y*. The variable names remind us of what quantities are being measured.

Also, we often will need to extend the axes in our rectangular coordinate system to bigger positive and negative numbers to accommodate the data in the application.

Example:
Exercise:

Problem:

The equation $F=\frac{9}{5}C+32$ is used to convert temperatures, *C*, on the Celsius scale to temperatures, *F*, on the Fahrenheit scale.

ⓐ Find the Fahrenheit temperature for a Celsius temperature of 0.

ⓑ Find the Fahrenheit temperature for a Celsius temperature of 20.

ⓒ Interpret the slope and *F*-intercept of the equation.

ⓓ Graph the equation.

Solution:

ⓐ

Find the Fahrenheit temperature for a Celsius temperature of 0.	$F=\frac{9}{5}C+32$
Find F when $C=0$.	$F=\frac{9}{5}(0)+32$
Simplify.	$F=32$

ⓑ

Find the Fahrenheit temperature for a Celsius temperature of 20.	$F=\frac{9}{5}C+32$
Find F when $C=20$.	$F=\frac{9}{5}(20)+32$
Simplify.	$F=36+32$
Simplify.	$F=68$

ⓒ

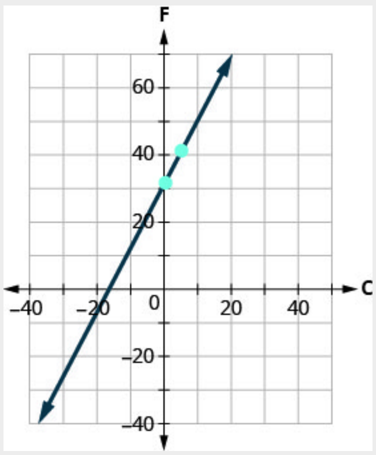
Interpret the slope and *F*-intercept of the equation.
Even though this equation uses *F* and *C*, it is still in slope–intercept form.

$y = mx + b$

$F = mC + b$

$F = \frac{9}{5}C + 32$

The slope, $\frac{9}{5}$, means that the temperature Fahrenheit (F) increases 9 degrees when the temperature Celsius (C) increases 5 degrees.
The F-intercept means that when the temperature is 0° on the Celsius scale, it is 32° on the Fahrenheit scale.
ⓓ Graph the equation.
We'll need to use a larger scale than our usual. Start at the F-intercept $(0, 32)$, and then count out the rise of 9 and the run of 5 to get a second point as shown in the graph.



Note:
Exercise:

Problem:

The equation $h = 2s + 50$ is used to estimate a woman's height in inches, h, based on her shoe size, s.

ⓐ Estimate the height of a child who wears women's shoe size 0.

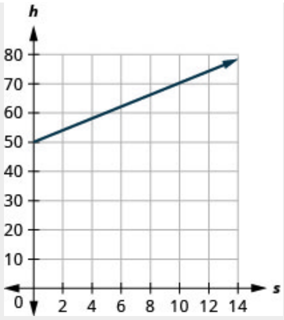
ⓑ Estimate the height of a woman with shoe size 8.

ⓒ Interpret the slope and h-intercept of the equation.

ⓓ Graph the equation.

Solution:

ⓐ 50 inches
ⓑ 66 inches
ⓒ The slope, 2, means that the height, h, increases by 2 inches when the shoe size, s, increases by 1. The h-intercept means that when the shoe size is 0, the height is 50 inches.
ⓓ



Note:
Exercise:

Problem:

The equation $T = \frac{1}{4}n + 40$ is used to estimate the temperature in degrees Fahrenheit, *T*, based on the number of cricket chirps, *n*, in one minute.

ⓐ Estimate the temperature when there are no chirps.

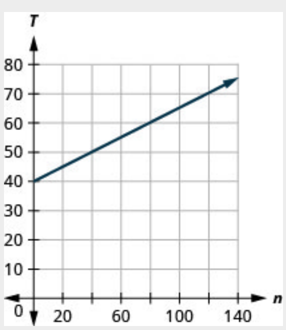
ⓑ Estimate the temperature when the number of chirps in one minute is 100.

ⓒ Interpret the slope and *T*-intercept of the equation.

ⓓ Graph the equation.

Solution:

ⓐ 40 degrees
ⓑ 65 degrees
ⓒ The slope, $\frac{1}{4}$, means that the temperature Fahrenheit (*F*) increases 1 degree when the number of chirps, *n*, increases by 4. The *T*-intercept means that when the number of chirps is 0, the temperature is 40°.
ⓓ



The cost of running some types business have two components—a *fixed cost* and a *variable cost*. The fixed cost is always the same regardless of how many units are produced. This is the cost of rent, insurance, equipment, advertising, and other items that must be paid regularly. The variable cost depends on the number of units produced. It is for the material and labor needed to produce each item.

Example:
Exercise:

Problem:

Sam drives a delivery van. The equation $C = 0.5m + 60$ models the relation between his weekly cost, *C*, in dollars and the number of miles, *m*, that he drives.

ⓐ Find Sam's cost for a week when he drives 0 miles.

ⓑ Find the cost for a week when he drives 250 miles.

ⓒ Interpret the slope and *C*-intercept of the equation.

ⓓ Graph the equation.

Solution:

ⓐ

Find Sam's cost for a week when he drives 0 miles.	$C = 0.5m + 60$
Find C when $m = 0$.	$C = 0.5\,(0) + 60$
Simplify.	$C = 60$
	Sam's costs are \$60 when he drives 0 miles.

ⓑ

Find the cost for a week when he drives 250 miles.	$C = 0.5m + 60$
Find C when $m = 250$.	$C = 0.5\,(250) + 60$
Simplify.	$C = 185$
	Sam's costs are \$185 when he drives 250 miles.

ⓒ Interpret the slope and *C*-intercept of the equation.

$$y = mx + b$$

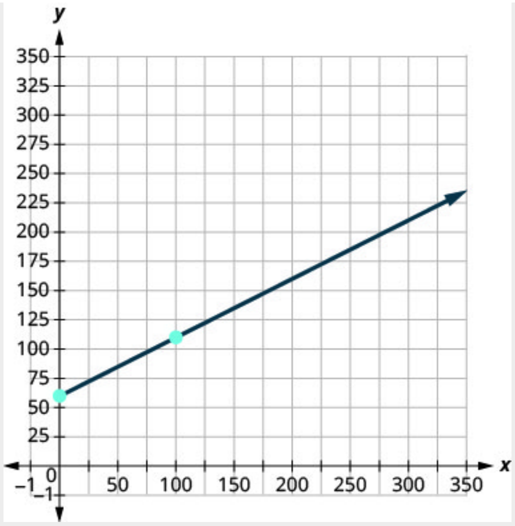
$$C = 0.5p + 60$$

The slope, 0.5, means that the weekly cost, *C*, increases by \$0.50 when the number of miles driven, *n*, increases by 1.
The *C*-intercept means that when the number of miles driven is 0, the weekly cost is \$60.
ⓓ Graph the equation.
We'll need to use a larger scale than our usual. Start at the *C*-intercept $(0, 60)$.

To count out the slope *m* = 0.5, we rewrite it as an equivalent fraction that will make our graphing easier.

	$m = 0.5$
Rewrite as a fraction.	$m = \frac{0.5}{1}$
Multiply numerator and denominator by 100.	$m = \frac{0.5(100)}{1(100)}$
Simplify.	$m = \frac{50}{100}$

So to graph the next point go up 50 from the intercept of 60 and then to the right 100. The second point will be (100, 110).



Note:
Exercise:

Problem:

Stella has a home business selling gourmet pizzas. The equation $C = 4p + 25$ models the relation between her weekly cost, *C*, in dollars and the number of pizzas, *p*, that she sells.

ⓐ Find Stella's cost for a week when she sells no pizzas.

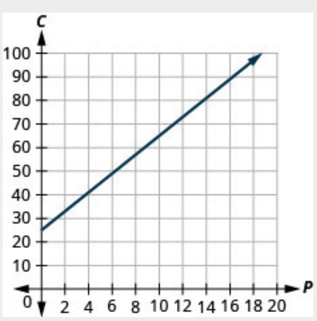
ⓑ Find the cost for a week when she sells 15 pizzas.

ⓒ Interpret the slope and *C*-intercept of the equation.

ⓓ Graph the equation.

Solution:

ⓐ $25
ⓑ $85
ⓒ The slope, 4, means that the weekly cost, *C*, increases by $4 when the number of pizzas sold, *p*, increases by 1. The *C*-intercept means that when the number of pizzas sold is 0, the weekly cost is $25.
ⓓ



Note:
Exercise:

Problem:

Loreen has a calligraphy business. The equation $C = 1.8n + 35$ models the relation between her weekly cost, *C*, in dollars and the number of wedding invitations, *n*, that she writes.

ⓐ Find Loreen’s cost for a week when she writes no invitations.

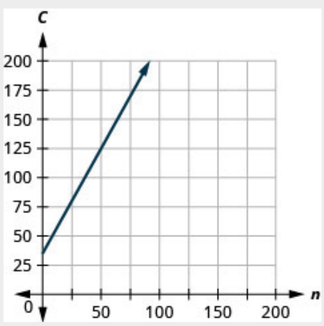
ⓑ Find the cost for a week when she writes 75 invitations.

ⓒ Interpret the slope and *C*-intercept of the equation.

ⓓ Graph the equation.

Solution:

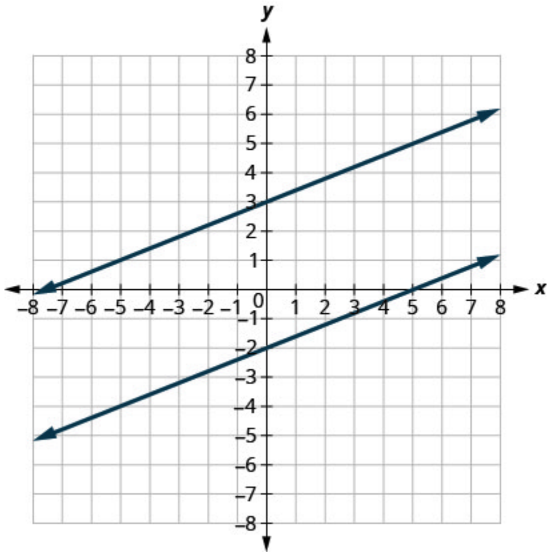
ⓐ $35
ⓑ $170
ⓒ The slope, 1.8, means that the weekly cost, *C*, increases by $1.80 when the number of invitations, *n*, increases by 1.
The *C*-intercept means that when the number of invitations is 0, the weekly cost is $35.
ⓓ



Use Slopes to Identify Parallel and Perpendicular Lines

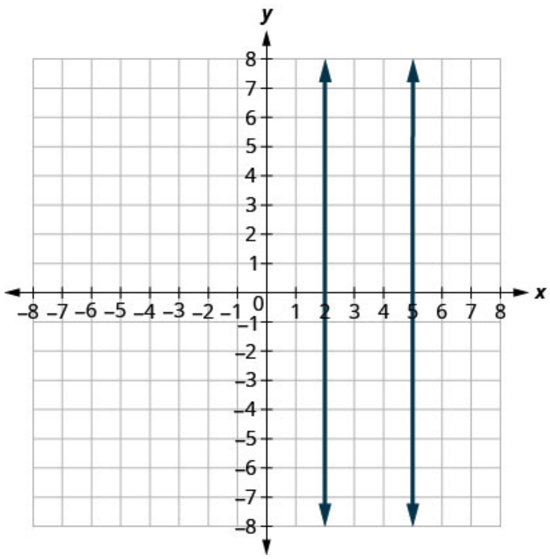
Two lines that have the same slope are called **parallel lines**. Parallel lines have the same steepness and never intersect.

We say this more formally in terms of the rectangular coordinate system. Two lines that have the same slope and different *y*-intercepts are called parallel lines. See [link].



Verify that both lines have the same slope, $m = \frac{2}{5}$, and different *y*-intercepts.

What about vertical lines? The slope of a vertical line is undefined, so vertical lines don't fit in the definition above. We say that vertical lines that have different *x*-intercepts are parallel, like the lines shown in this graph.



Note:
Parallel Lines
Parallel lines are lines in the same plane that do not intersect.

- Parallel lines have the same slope and different *y*-intercepts.
- If m_1 and m_2 are the slopes of two parallel lines then $m_1 = m_2$.
- Parallel vertical lines have different *x*-intercepts.

Since parallel lines have the same slope and different y-intercepts, we can now just look at the slope–intercept form of the equations of lines and decide if the lines are parallel.

Example:
Exercise:

Problem: Use slopes and y-intercepts to determine if the lines are parallel:

ⓐ $3x - 2y = 6$ and $y = \frac{3}{2}x + 1$ ⓑ $y = 2x - 3$ and $-6x + 3y = -9$.

Solution:

ⓐ

$3x - 2y = 6$ and $y = \frac{3}{2}x +$

Solve the first equation for y.

$$-2y = -3x + 6$$
$$\frac{-2y}{-2} = \frac{-3x+6}{-2}$$

The equation is now in slope–intercept form. $y = \frac{3}{2}x - 3$

The equation of the second line is already in slope–intercept form. $y = \frac{3}{2}x +$

Identify the slope and y-intercept of both lines.

$$y = \frac{3}{2}x - 3 \qquad y = \frac{3}{2}x +$$
$$y = mx + b \qquad y = mx +$$
$$m = \frac{3}{2} \qquad y = \frac{3}{2}$$

y-intercept is $(0, -3)$ y-intercept is

The lines have the same slope and different y-intercepts and so they are parallel.
You may want to graph the lines to confirm whether they are parallel.

ⓑ

$y = 2x - 3$ and $-6x + 3y = -$

The first equation is already in slope–intercept form. $y = 2x - 3$

Solve the second equation for y.

$$-6x + 3y = -$$
$$3y = 6$$
$$\frac{3y}{3} = \frac{6}{}$$
$$y = 2$$

The second equation is now in slope–intercept form. $y = 2x$

Identify the slope and y-intercept of both lines.

$$y = 2x - 3 \qquad y = 2x$$
$$y = mx + b \qquad y = m$$
$$m = 2 \qquad m = 2$$

y-intercept is $(0, -3)$ y-intercept

The lines have the same slope, but they also have the same y-intercepts. Their equations represent the same line and we say the lines are coincident. They are not parallel; they are the same line.

Note:
Exercise:

Problem: Use slopes and *y*-intercepts to determine if the lines are parallel:

ⓐ $2x+5y=5$ and $y=-\frac{2}{5}x-4$ ⓑ $y=-\frac{1}{2}x-1$ and $x+2y=-2$.

Solution:

ⓐ parallel ⓑ not parallel; same line

Note:
Exercise:

Problem: Use slopes and *y*-intercepts to determine if the lines are parallel:

ⓐ $4x-3y=6$ and $y=\frac{4}{3}x-1$ ⓑ $y=\frac{3}{4}x-3$ and $3x-4y=12$.

Solution:

ⓐ parallel ⓑ not parallel; same line

Example:
Exercise:

Problem: Use slopes and *y*-intercepts to determine if the lines are parallel:

ⓐ $y=-4$ and $y=3$ ⓑ $x=-2$ and $x=-5$.

Solution:

ⓐ $y=-4$ and $y=3$
We recognize right away from the equations that these are horizontal lines, and so we know their slopes are both 0.
Since the horizontal lines cross the *y*-axis at $y=-4$ and at $y=3$, we know the *y*-intercepts are $(0,-4)$ and $(0,3)$.
The lines have the same slope and different *y*-intercepts and so they are parallel.
ⓑ $x=-2$ and $x=-5$
We recognize right away from the equations that these are vertical lines, and so we know their slopes are undefined.
Since the vertical lines cross the *x*-axis at $x=-2$ and $x=-5$, we know the *y*-intercepts are $(-2,0)$ and $(-5,0)$.
The lines are vertical and have different *x*-intercepts and so they are parallel.

Note:
Exercise:

Problem: Use slopes and *y*-intercepts to determine if the lines are parallel:

ⓐ $y=8$ and $y=-6$ ⓑ $x=1$ and $x=-5$.

Solution:

ⓐ parallel ⓑ parallel

Note:
Exercise:

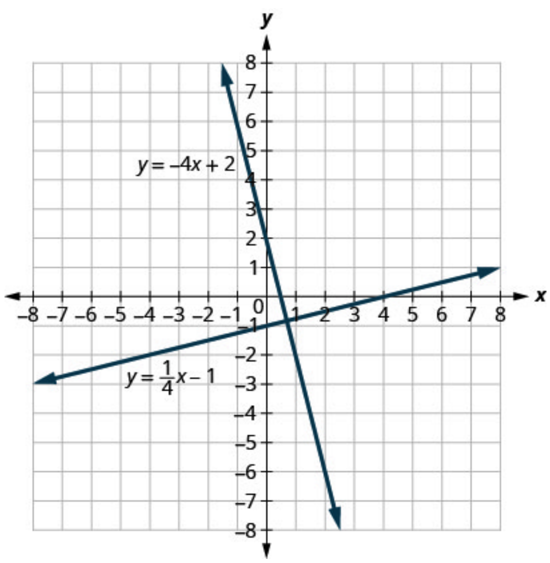
Problem: Use slopes and *y*-intercepts to determine if the lines are parallel:

ⓐ $y = 1$ and $y = -5$ ⓑ $x = 8$ and $x = -6$.

Solution:

ⓐ parallel ⓑ parallel

Let's look at the lines whose equations are $y = \frac{1}{4}x - 1$ and $y = -4x + 2$, shown in [link].



These lines lie in the same plane and intersect in right angles. We call these lines perpendicular.

If we look at the slope of the first line, $m_1 = \frac{1}{4}$, and the slope of the second line, $m_2 = -4$, we can see that they are *negative reciprocals* of each other. If we multiply them, their product is -1.
Equation:

$$\begin{array}{l} m_1 \cdot m_2 \\ \frac{1}{4}(-4) \\ -1 \end{array}$$

This is always true for **perpendicular lines** and leads us to this definition.

Note:
Perpendicular Lines
Perpendicular lines are lines in the same plane that form a right angle.

- If m_1 and m_2 are the slopes of two perpendicular lines, then:
  - their slopes are negative reciprocals of each other, $m_1 = -\frac{1}{m_2}$.
  - the product of their slopes is -1, $m_1 \cdot m_2 = -1$.
- A vertical line and a horizontal line are always perpendicular to each other.

We were able to look at the slope–intercept form of linear equations and determine whether or not the lines were parallel. We can do the same thing for perpendicular lines.

We find the slope–intercept form of the equation, and then see if the slopes are opposite reciprocals. If the product of the slopes is -1, the lines are perpendicular.

Example:
Exercise:

Problem: Use slopes to determine if the lines are perpendicular:

ⓐ $y = -5x - 4$ and $x - 5y = 5$ ⓑ $7x + 2y = 3$ and $2x + 7y = 5$

Solution:

ⓐ
The first equation is in slope–intercept form. $y = -5x - 4$

Solve the second equation for y.

$$\begin{aligned} x - 5y &= 5 \\ -5y &= -x + 5 \\ \frac{-5y}{-5} &= \frac{-x+5}{-5} \\ y &= \frac{1}{5}x - 1 \end{aligned}$$

Identify the slope of each line.

$$\begin{aligned} y &= -5x - 4 \\ y &= mx + b \\ m_1 &= -5 \end{aligned} \qquad \begin{aligned} y &= \frac{1}{5}x - 1 \\ y &= mx + b \\ m_2 &= \frac{1}{5} \end{aligned}$$

The slopes are negative reciprocals of each other, so the lines are perpendicular. We check by multiplying the slopes, Since $-5\left(\frac{1}{5}\right) = -1$, it checks.

ⓑ

Solve the equations for y.

$$\begin{aligned} 7x + 2y &= 3 \\ 2y &= -7x + 3 \\ \frac{2y}{2} &= \frac{-7x+3}{2} \\ y &= -\frac{7}{2}x + \frac{3}{2} \end{aligned} \qquad \begin{aligned} 2x + 7y &= 5 \\ 7y &= -2x + 5 \\ \frac{7y}{7} &= \frac{-2x+5}{7} \\ y &= -\frac{2}{7}x + \frac{5}{7} \end{aligned}$$

Identify the slope of each line.

$$\begin{aligned} y &= mx + b \\ m_1 &= -\frac{7}{2} \end{aligned} \qquad \begin{aligned} y &= mx + b \\ m_1 &= -\frac{2}{7} \end{aligned}$$

The slopes are reciprocals of each other, but they have the same sign. Since they are not negative reciprocals, the lines are not perpendicular.

Note:
Exercise:

Problem: Use slopes to determine if the lines are perpendicular:

ⓐ $y = -3x + 2$ and $x - 3y = 4$ ⓑ $5x + 4y = 1$ and $4x + 5y = 3$.

Solution:

ⓐ perpendicular ⓑ not perpendicular

Note:
Exercise:

Problem: Use slopes to determine if the lines are perpendicular:

ⓐ $y = 2x - 5$ and $x + 2y = -6$ ⓑ $2x - 9y = 3$ and $9x - 2y = 1$.

Solution:

ⓐ perpendicular ⓑ not perpendicular

Key Concepts

- **Slope of a Line**
  - The slope of a line is $m = \frac{\text{rise}}{\text{run}}$.
  - The rise measures the vertical change and the run measures the horizontal change.

- **How to find the slope of a line from its graph using $m = \frac{\text{rise}}{\text{run}}$.**

  Locate two points on the line whose coordinates are integers.
  Starting with one point, sketch a right triangle, going from the first point to the second point.
  Count the rise and the run on the legs of the triangle.
  Take the ratio of rise to run to find the slope:$m = \frac{\text{rise}}{\text{run}}$.

- **Slope of a line between two points.**
  - The slope of the line between two points (x_1, y_1) and (x_2, y_2) is:
    Equation:

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

- **How to graph a line given a point and the slope.**

  Plot the given point.
  Use the slope formula $m = \frac{\text{rise}}{\text{run}}$ to identify the rise and the run.
  Starting at the given point, count out the rise and run to mark the second point.
  Connect the points with a line.

- **Slope Intercept Form of an Equation of a Line**
  - The slope–intercept form of an equation of a line with slope *m* and *y*-intercept, $(0, b)$ is $y = mx + b$

Methods to Graph Lines			
Point Plotting	**Slope–Intercept**	**Intercepts**	**Recognize Vertical and Horizontal Lines**
x | *y* (blank table)	$y = mx + b$	*x* | *y*: 0 | ; | 0; |	
Find three points. Plot the points, make sure they line up, then draw the line.	Find the slope and *y*-intercept. Start at the *y*-intercept, then count the slope to get a second point.	Find the intercepts and a third point. Plot the points, make sure they line up, then draw the line.	The equation has only one variable. $x = a$ vertical $y = b$ horizontal

- **Parallel Lines**
  - Parallel lines are lines in the same plane that do not intersect.
    Parallel lines have the same slope and different *y*-intercepts.
    If m_1 and m_2 are the slopes of two parallel lines then $m_1 = m_2$.
    Parallel vertical lines have different *x*-intercepts.

- **Perpendicular Lines**
  - Perpendicular lines are lines in the same plane that form a right angle.
  - If m_1 and m_2 are the slopes of two perpendicular lines, then:
    their slopes are negative reciprocals of each other, $m_1 = -\frac{1}{m_2}$.
    the product of their slopes is -1, $m_1 \cdot m_2 = -1$.
  - A vertical line and a horizontal line are always perpendicular to each other.

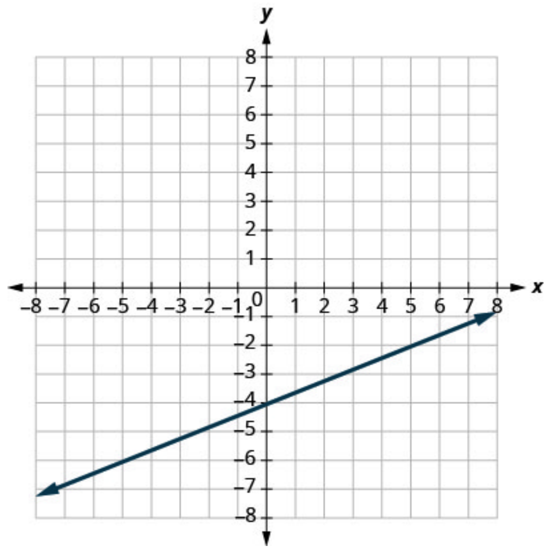
Practice Makes Perfect

Find the Slope of a Line

In the following exercises, find the slope of each line shown.

Exercise:

Problem:

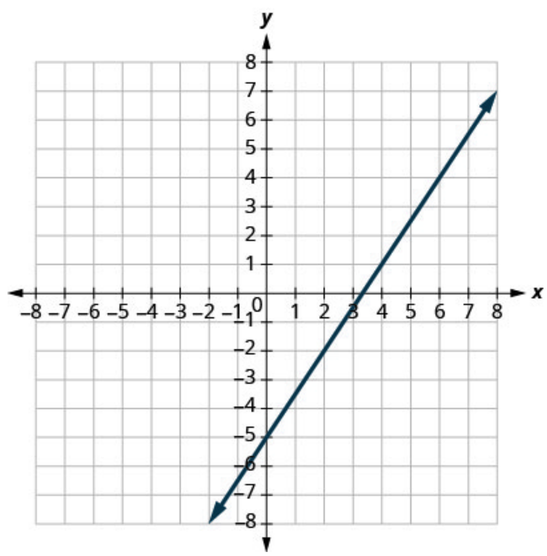


Solution:

$\frac{2}{5}$

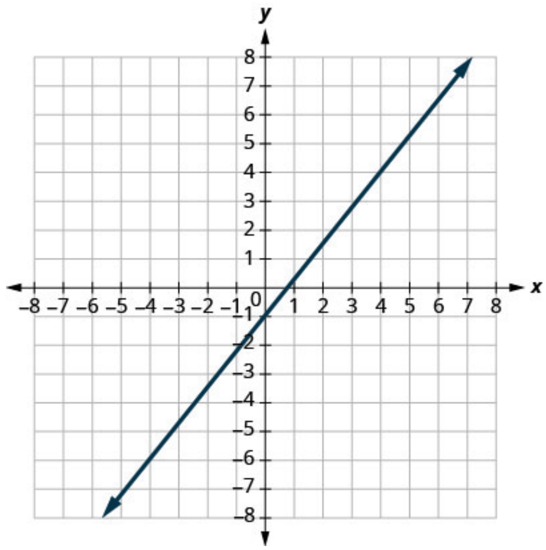
Exercise:

Problem:



Exercise:

Problem:

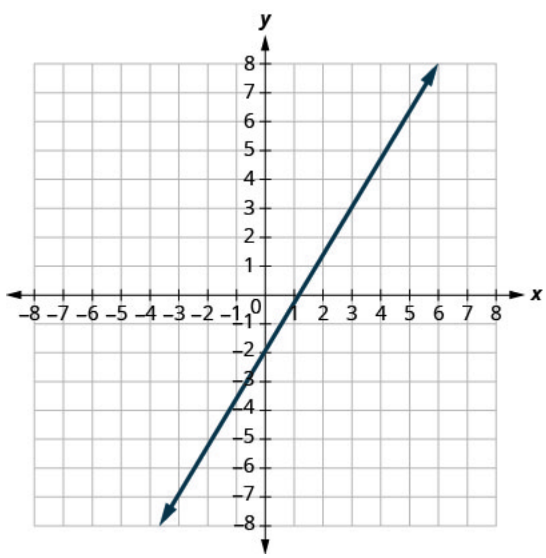


Solution:

$\frac{5}{4}$

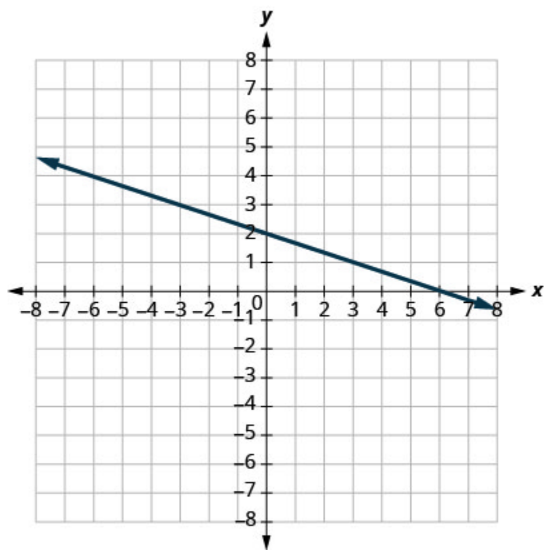
Exercise:

Problem:



Exercise:

Problem:

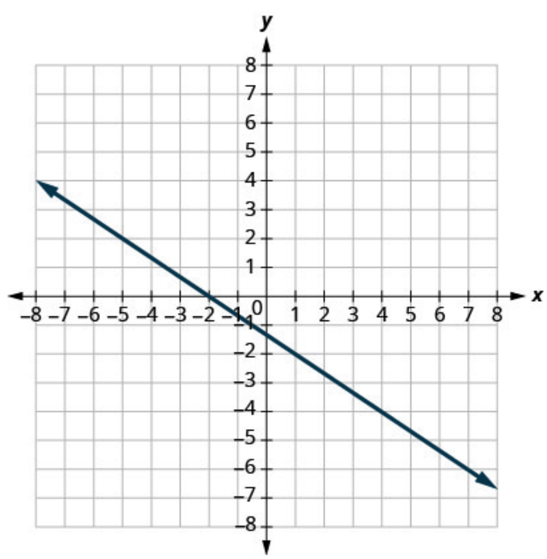


Solution:

$-\frac{1}{3}$

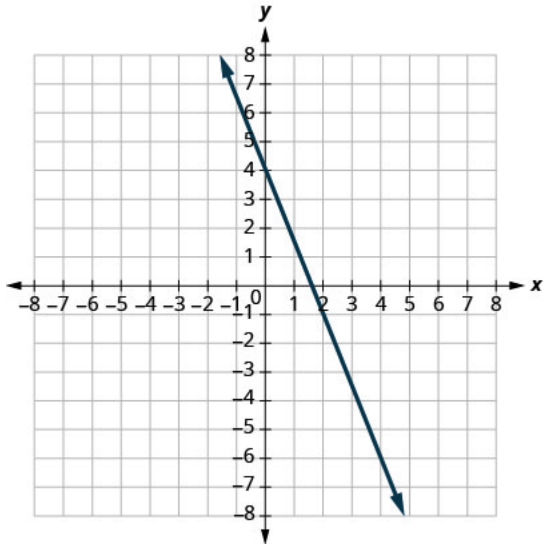
Exercise:

Problem:



Exercise:

Problem:

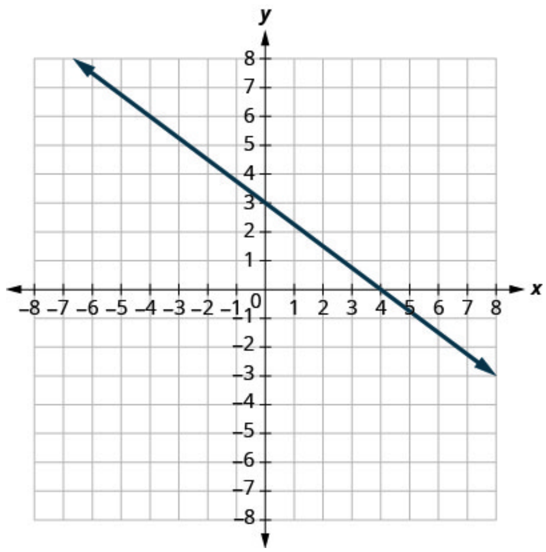


Solution:

$-\frac{5}{2}$

Exercise:

Problem:



In the following exercises, find the slope of each line.

Exercise:

Problem: $y = 3$

Solution:

0

Exercise:

Problem: $y=-2$

Exercise:

Problem: $x=-5$

Solution:

undefined

Exercise:

Problem: $x=4$

In the following exercises, use the slope formula to find the slope of the line between each pair of points.

Exercise:

Problem: $(2,5),(4,0)$

Solution:

$-\frac{5}{2}$

Exercise:

Problem: $(3,6),(8,0)$

Exercise:

Problem: $(-3,3),(4,-5)$

Solution:

$-\frac{8}{7}$

Exercise:

Problem: $(-2,4),(3,-1)$

Exercise:

Problem: $(-1,-2),(2,5)$

Solution:

$\frac{7}{3}$

Exercise:

Problem: $(-2,-1),(6,5)$

Exercise:

Problem: $(4,-5),(1,-2)$

Solution:

-1

Exercise:

Problem: $(3, -6), (2, -2)$

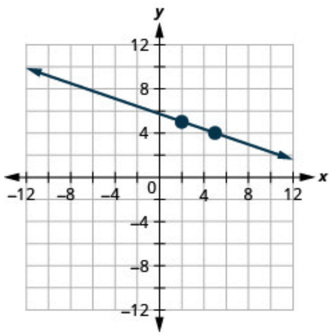
Graph a Line Given a Point and the Slope

In the following exercises, graph each line with the given point and slope.

Exercise:

Problem: $(2, 5)$;$m = -\frac{1}{3}$

Solution:



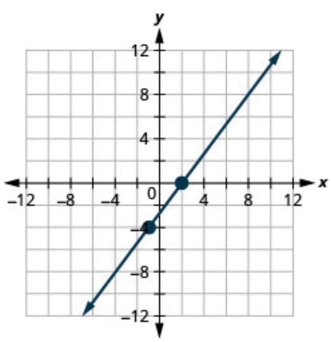
Exercise:

Problem: $(1, 4)$; $m = -\frac{1}{2}$

Exercise:

Problem: $(-1, -4)$; $m = \frac{4}{3}$

Solution:



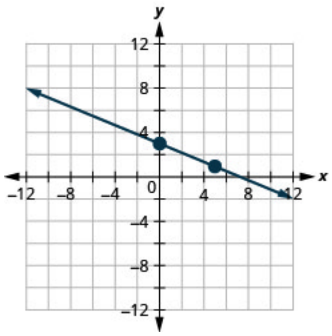
Exercise:

Problem: $(-3, -5)$; $m = \frac{3}{2}$

Exercise:

Problem: y-intercept 3; $m = -\frac{2}{5}$

Solution:



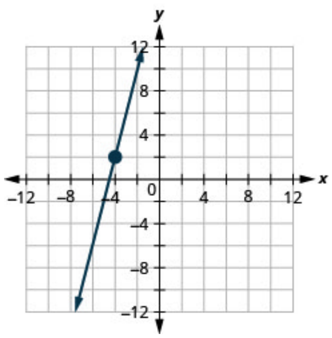
Exercise:

Problem: x-intercept -2; $m = \frac{3}{4}$

Exercise:

Problem: $(-4, 2)$; $m = 4$

Solution:



Exercise:

Problem: $(1, 5)$; $m = -3$

Graph a Line Using Its Slope and Intercept

In the following exercises, identify the slope and y-intercept of each line.

Exercise:

Problem: $y = -7x + 3$

Solution:

$m = -7; (0, 3)$

Exercise:

Problem: $y = 4x - 10$

Exercise:

Problem: $3x + y = 5$

Solution:

$m = -3; (0, 5)$

Exercise:

Problem: $4x + y = 8$

Exercise:

Problem: $6x + 4y = 12$

Solution:

$m = -\frac{3}{2}; (0, 3)$

Exercise:

Problem: $8x + 3y = 12$

Exercise:

Problem: $5x - 2y = 6$

Solution:

$m = \frac{5}{2}; (0, -3)$

Exercise:

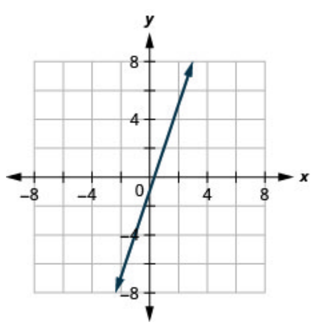
Problem: $7x - 3y = 9$

In the following exercises, graph the line of each equation using its slope and *y*-intercept.

Exercise:

Problem: $y = 3x - 1$

Solution:



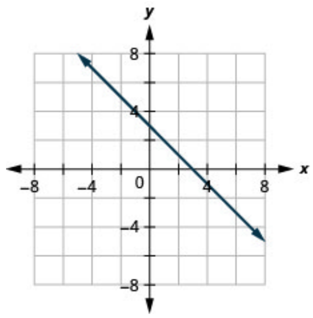
Exercise:

Problem: $y = 2x - 3$

Exercise:

Problem: $y = -x + 3$

Solution:



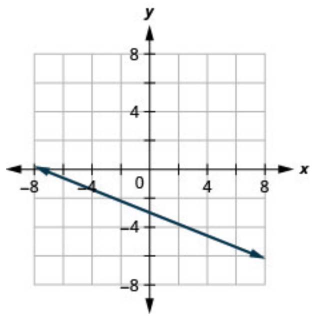
Exercise:

Problem: $y = -x - 4$

Exercise:

Problem: $y = -\frac{2}{5}x - 3$

Solution:



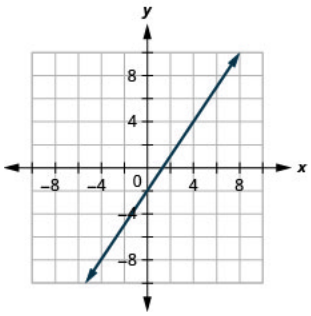
Exercise:

Problem: $y = -\frac{3}{5}x + 2$

Exercise:

Problem: $3x - 2y = 4$

Solution:



Exercise:

Problem: $3x - 4y = 8$

Choose the Most Convenient Method to Graph a Line

In the following exercises, determine the most convenient method to graph each line.

Exercise:

Problem: $x = 2$

Solution:

vertical line

Exercise:

Problem: $y = 5$

Exercise:

Problem: $y = -3x + 4$

Solution:

slope-intercept

Exercise:

Problem: $x - y = 5$

Exercise:

Problem: $x - y = 1$

Solution:

intercepts

Exercise:

Problem: $y = \frac{2}{3}x - 1$

Exercise:

Problem: $3x - 2y = -12$

Solution:

intercepts

Exercise:

Problem: $2x - 5y = -10$

Graph and Interpret Applications of Slope–Intercept

Exercise:

Problem:

The equation $P = 31 + 1.75w$ models the relation between the amount of Tuyet's monthly water bill payment, *P*, in dollars, and the number of units of water, *w*, used.

ⓐ Find Tuyet's payment for a month when 0 units of water are used.

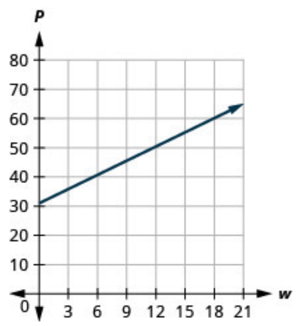
ⓑ Find Tuyet's payment for a month when 12 units of water are used.

ⓒ Interpret the slope and *P*-intercept of the equation.

ⓓ Graph the equation.

Solution:

ⓐ \$31
ⓑ \$52
ⓒ The slope, 1.75, means that the payment, *P*, increases by $\$1.75$ when the number of units of water used, *w*, increases by 1. The *P*-intercept means that when the number units of water Tuyet used is 0, the payment is \$31.
ⓓ



Exercise:

Problem:

The equation $P = 28 + 2.54w$ models the relation between the amount of R and y's monthly water bill payment, *P*, in dollars, and the number of units of water, *w*, used.

ⓐ Find the payment for a month when R and y used 0 units of water.

ⓑ Find the payment for a month when R and y used 15 units of water.

ⓒ Interpret the slope and P-intercept of the equation.

ⓓ Graph the equation.

Exercise:

Problem:

Bruce drives his car for his job. The equation $R = 0.575m + 42$ models the relation between the amount in dollars, *R*, that he is reimbursed and the number of miles, *m*, he drives in one day.

ⓐ Find the amount Bruce is reimbursed on a day when he drives 0 miles.

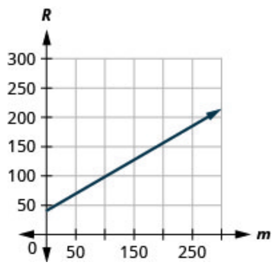
ⓑ Find the amount Bruce is reimbursed on a day when he drives 220 miles.

ⓒ Interpret the slope and R-intercept of the equation.

ⓓ Graph the equation.

Solution:

ⓐ $42
ⓑ $168.50
ⓒ The slope, 0.575 means that the amount he is reimbursed, *R*, increases by $0.575 when the number of miles driven, *m*, increases by 1. The *R*-intercept means that when the number miles driven is 0, the amount reimbursed is $42.
ⓓ



Exercise:

Problem:

Janelle is planning to rent a car while on vacation. The equation $C = 0.32m + 15$ models the relation between the cost in dollars, *C*, per day and the number of miles, *m*, she drives in one day.

ⓐ Find the cost if Janelle drives the car 0 miles one day.

ⓑ Find the cost on a day when Janelle drives the car 400 miles.

ⓒ Interpret the slope and C-intercept of the equation.

ⓓ Graph the equation.

Exercise:

Problem:

Cherie works in retail and her weekly salary includes commission for the amount she sells. The equation $S = 400 + 0.15c$ models the relation between her weekly salary, *S*, in dollars and the amount of her sales, *c*, in dollars.

ⓐ Find Cherie’s salary for a week when her sales were $0.

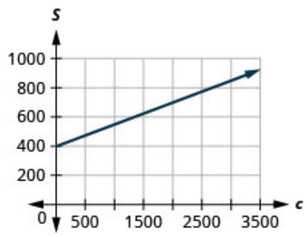
ⓑ Find Cherie’s salary for a week when her sales were $3,600.

ⓒ Interpret the slope and *S*-intercept of the equation.

ⓓ Graph the equation.

Solution:

ⓐ $400
ⓑ $940
ⓒ The slope, 0.15, means that Cherie’s salary, *S*, increases by $0.15 for every $1 increase in her sales. The *S*-intercept means that when her sales are $0, her salary is $400.
ⓓ



Exercise:

Problem:

Patel’s weekly salary includes a base pay plus commission on his sales. The equation $S = 750 + 0.09c$ models the relation between his weekly salary, *S*, in dollars and the amount of his sales, *c*, in dollars.

ⓐ Find Patel’s salary for a week when his sales were 0.

ⓑ Find Patel’s salary for a week when his sales were 18,540.

ⓒ Interpret the slope and *S*-intercept of the equation.

ⓓ Graph the equation.

Exercise:

Problem:

Costa is planning a lunch banquet. The equation $C = 450 + 28g$ models the relation between the cost in dollars, *C*, of the banquet and the number of guests, *g*.

ⓐ Find the cost if the number of guests is 40.

ⓑ Find the cost if the number of guests is 80.

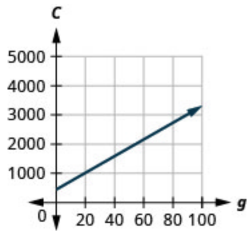
ⓒ Interpret the slope and *C*-intercept of the equation.

ⓓ Graph the equation.

Solution:

ⓐ $1570
ⓑ $5690
ⓒ The slope gives the cost per guest. The slope, 28, means that the cost, *C*, increases by $28 when the

number of guests increases by 1. The C-intercept means that if the number of guests was 0, the cost would be \$450.
ⓓ



Exercise:

Problem:

Margie is planning a dinner banquet. The equation $C = 750 + 42g$ models the relation between the cost in dollars, C of the banquet and the number of guests, g.

ⓐ Find the cost if the number of guests is 50.

ⓑ Find the cost if the number of guests is 100.

ⓒ Interpret the slope and C-intercept of the equation.

ⓓ Graph the equation.

Use Slopes to Identify Parallel and Perpendicular Lines

In the following exercises, use slopes and y-intercepts to determine if the lines are parallel, perpendicular, or neither.

Exercise:

Problem: $y = \frac{3}{4}x - 3; \quad 3x - 4y = -2$

Solution:

parallel

Exercise:

Problem: $3x - 4y = -2; \quad y = \frac{3}{4}x - 3$

Exercise:

Problem: $2x - 4y = 6; \quad x - 2y = 3$

Solution:

neither

Exercise:

Problem: $8x + 6y = 6; \quad 12x + 9y = 12$

Exercise:

Problem: $x = 5; \quad x = -6$

Solution:

parallel

Exercise:

Problem: $x = -3; \quad x = -2$

Exercise:

Problem: $4x - 2y = 5; \quad 3x + 6y = 8$

Solution:

perpendicular

Exercise:

Problem: $8x - 2y = 7; \quad 3x + 12y = 9$

Exercise:

Problem: $3x - 6y = 12; \quad 6x - 3y = 3$

Solution:

neither

Exercise:

Problem: $9x - 5y = 4; \quad 5x + 9y = -1$

Exercise:

Problem: $7x - 4y = 8; \quad 4x + 7y = 14$

Solution:

perpendicular

Exercise:

Problem: $5x - 2y = 11; \quad 5x - y = 7$

Exercise:

Problem: $3x - 2y = 8; \quad 2x + 3y = 6$

Solution:

perpendicular

Exercise:

Problem: $2x + 3y = 5; \quad 3x - 2y = 7$

Exercise:

Problem: $3x - 2y = 1; \quad 2x - 3y = 2$

Solution:

neither

Exercise:

Problem: $2x + 4y = 3; \quad 6x + 3y = 2$

Exercise:

Problem: $y = 2; \quad y = 6$

Solution:

parallel

Exercise:

Problem: $y = -1; \quad y = 2$

Writing Exercises

Exercise:

Problem: How does the graph of a line with slope $m = \frac{1}{2}$ differ from the graph of a line with slope $m = 2$?

Solution:

Answers will vary.

Exercise:

Problem: Why is the slope of a vertical line "undefined"?

Exercise:

Problem: Explain how you can graph a line given a point and its slope.

Solution:

Answers will vary.

Exercise:

Problem: Explain in your own words how to decide which method to use to graph a line.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
find the slope of a line.			
graph a line given a point and the slope.			
graph a line using its slope and intercept.			
choose the most convenient method to graph a line.			
graph and interpret applications of slope-intercept.			
use slopes to identify parallel and perpendicular lines.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

Glossary

parallel lines
: Parallel lines are lines in the same plane that do not intersect.

perpendicular lines
: Perpendicular lines are lines in the same plane that form a right angle.

Find the Equation of a Line: ASE
By the end of this section, you will be able to:

- Find an equation of the line given the slope and y-intercept
- Find an equation of the line given the slope and a point
- Find an equation of the line given two points
- Find an equation of a line parallel to a given line
- Find an equation of a line perpendicular to a given line

How do online companies know that “you may also like” a particular item based on something you just ordered? How can economists know how a rise in the minimum wage will affect the unemployment rate? How do medical researchers create drugs to target cancer cells? How can traffic engineers predict the effect on your commuting time of an increase or decrease in gas prices? It’s all mathematics.

The physical sciences, social sciences, and the business world are full of situations that can be modeled with linear equations relating two variables. To create a mathematical model of a linear relation between two variables, we must be able to find the equation of the line. In this section, we will look at several ways to write the equation of a line. The specific method we use will be determined by what information we are given.

Find an Equation of the Line Given the Slope and *y*-Intercept

We can easily determine the slope and intercept of a line if the equation is written in slope-intercept form, $y=mx+b$. Now we will do the reverse—we will start with the slope and *y*-intercept and use them to find the equation of the line.

Example:
Exercise:

Problem: Find the equation of a line with slope -9 and *y*-intercept $(0,-4)$.

Solution:

Since we are given the slope and *y*-intercept of the line, we can substitute the needed values into the slope-intercept form, $y=mx+b$.

Name the slope.	$m=-9$

Name the *y*-intercept.	*y*-intercept (0, −4)
Substitute the values into $y = mx + b$.	$y = mx + b$
	$y = -9x + (-4)$
	$y = -9x - 4$

Note:
Exercise:

Problem: Find the equation of a line with slope $\frac{2}{5}$ and *y*-intercept $(0, 4)$.

Solution:

$y = \frac{2}{5}x + 4$

Note:
Exercise:

Problem: Find the equation of a line with slope -1 and *y*-intercept $(0, -3)$.

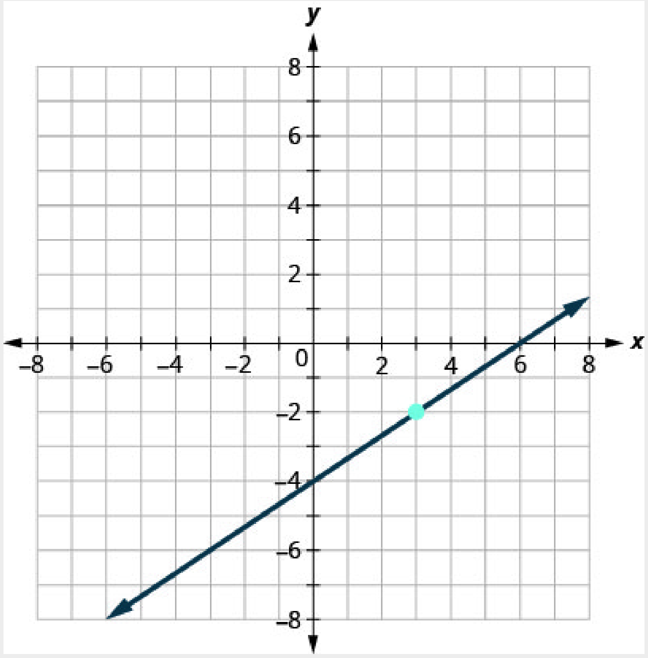
Solution:

$y = -x - 3$

Sometimes, the slope and intercept need to be determined from the graph.

Example:
Exercise:

Problem: Find the equation of the line shown in the graph.



Solution:

We need to find the slope and *y*-intercept of the line from the graph so we can substitute the needed values into the slope-intercept form, $y = mx + b$.

To find the slope, we choose two points on the graph.

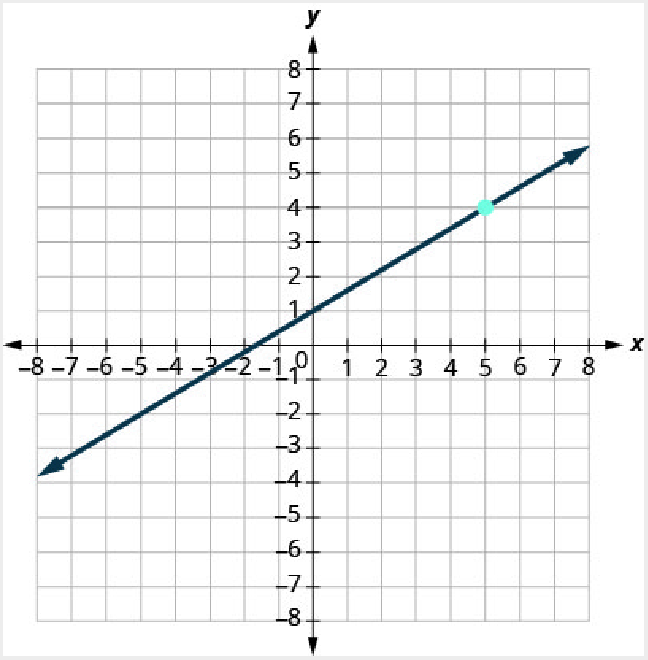
The *y*-intercept is $(0, -4)$ and the graph passes through $(3, -2)$.

Find the slope, by counting the rise and run.	$m = \frac{\text{rise}}{\text{run}}$
	$m = \frac{2}{3}$
Find the *y*-intercept.	*y*-intercept $(0, -4)$

Substitute the values into $y = mx + b$.	$y = mx + b$
	$y = \frac{2}{3}x - 4$

Note:
Exercise:

Problem: Find the equation of the line shown in the graph.

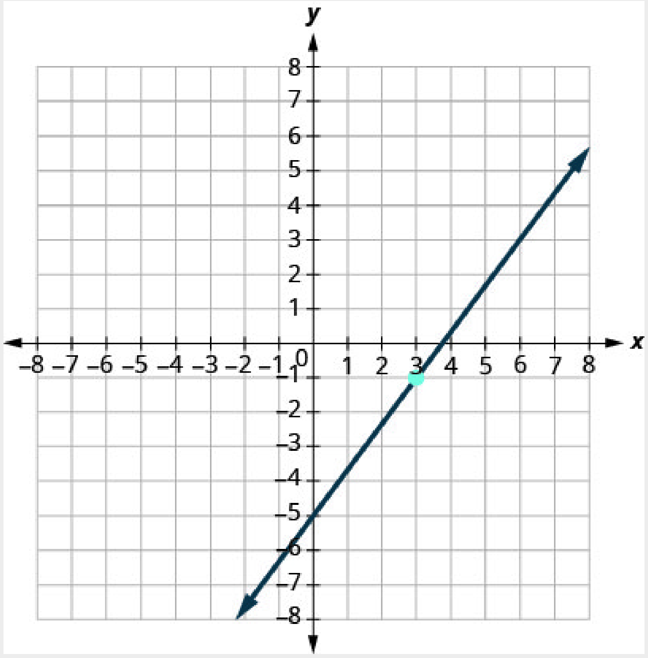


Solution:

$y = \frac{3}{5}x + 1$

Note:
Exercise:

Problem: Find the equation of the line shown in the graph.



Solution:

$y = \frac{4}{3}x - 5$

Find an Equation of the Line Given the Slope and a Point

Finding an equation of a line using the slope-intercept form of the equation works well when you are given the slope and *y*-intercept or when you read them off a graph. But what happens when you have another point instead of the *y*-intercept?

We are going to use the slope formula to derive another form of an equation of the line.

Suppose we have a line that has slope *m* and that contains some specific point (x_1, y_1) and some other point, which we will just call (x, y). We can write the slope of this line and then change it to a different form.

	$m = \frac{y-y_1}{x-x_1}$
Multiply both sides of the equation by $x - x_1$.	$m\left(x - x_1\right) = \left(\frac{y-y_1}{x-x_1}\right)\left(x - x_1\right)$
Simplify.	$m\left(x - x_1\right) = y - y_1$
Rewrite the equation with the y terms on the left.	$y - y_1 = m\left(x - x_1\right)$

This format is called the **point-slope form** of an equation of a line.

Note:
Point-slope Form of an Equation of a Line
The **point-slope form** of an equation of a line with slope m and containing the point (x_1, y_1) is:
Equation:

$$y - y_1 = m(x - x_1)$$

We can use the point-slope form of an equation to find an equation of a line when we know the slope and at least one point. Then, we will rewrite the equation in slope-intercept form. Most applications of linear equations use the the slope-intercept form.

Example:
How to Find an Equation of a Line Given a Point and the Slope
Exercise:

Problem:

Find an equation of a line with slope $m = -\frac{1}{3}$ that contains the point $(6, -4)$. Write the equation in slope-intercept form.

Solution:

Step 1. Identify the slope.	The slope is given.	$m = -\frac{1}{3}$
Step 2. Identify the point.	The point is given.	$\left(\overset{x_1}{6}, \overset{y_1}{-4}\right)$
Step 3. Substitute the values into the point-slope form, $y - y_1 = m(x - x_1)$.	Simplify.	$y - y_1 = m(x - x_1)$ $y - (-4) = -\frac{1}{3}(x - 6)$ $y + 4 = -\frac{1}{3}x + 2$

Step 4. Write the equation in slope–intercept form.		$y = -\frac{1}{3}x - 2$

Note:
Exercise:

Problem:

Find the equation of a line with slope $m = -\frac{2}{5}$ and containing the point $(10, -5)$.

Solution:

$y = -\frac{2}{5}x - 1$

Note:
Exercise:

Problem:

Find the equation of a line with slope $m = -\frac{3}{4}$, and containing the point $(4, -7)$.

Solution:

$y = -\frac{3}{4}x - 4$

We list the steps for easy reference.

Note:
To find an equation of a line given the slope and a point.

Identify the slope.
Identify the point.
Substitute the values into the point-slope form,$y - y_1 = m\,(x - x_1)$.
Write the equation in slope-intercept form.

Example:
Exercise:

Problem:

Find an equation of a horizontal line that contains the point $(-2, -6)$. Write the equation in slope-intercept form.

Solution:

Every horizontal line has slope 0. We can substitute the slope and points into the point-slope form, $y - y_1 = m(x - x_1)$.

Identify the slope.	$m = 0$
Identify the point.	$\begin{pmatrix} x_1, & y_1 \\ -2, & -6 \end{pmatrix}$
Substitute the values into $y - y_1 = m(x - x_1)$.	$y - y_1 = m(x - x_1)$
	$y - (-6) = 0(x - (-2))$
Simplify.	$y + 6 = 0$
	$y = -6$
Write in slope-intercept form.	It is in *y*-form, but could be written $y = 0x - 6$.

Did we end up with the form of a horizontal line, $y = a$?

Note:
Exercise:

Problem: Find the equation of a horizontal line containing the point $(-3, 8)$.

Solution:

$y = 8$

Note:
Exercise:

Problem: Find the equation of a horizontal line containing the point $(-1, 4)$.

Solution:

$y = 4$

Find an Equation of the Line Given Two Points

When real-world data is collected, a linear model can be created from two data points. In the next example we'll see how to find an equation of a line when just two points are given.

So far, we have two options for finding an equation of a line: slope-intercept or point-slope. When we start with two points, it makes more sense to use the point-slope form.

But then we need the slope. Can we find the slope with just two points? Yes. Then, once we have the slope, we can use it and one of the given points to find the equation.

Example:
How to Find the Equation of a Line Given Two Points
Exercise:

Problem:

Find an equation of a line that contains the points $(-3, -1)$ and $(2, -2)$ Write the equation in slope-intercept form.

Solution:

Step 1. Find the slope using the given points.	Find the slope of the line through $(-3, -1)$ and $(2, -2)$.	$m = \frac{y_2 - y_1}{x_2 - x_1}$ $m = \frac{-2-(-1)}{2-(-3)}$ $m = \frac{-1}{5}$ $m = -\frac{1}{5}$
Step 2. Choose one point.	Choose either point.	$(\overset{x_1}{2}, \overset{y_1}{-2})$
Step 3. Substitute the values into the point–slope form, $y - y_1 = m(x - x_1)$.	Simplify.	$y - y_1 = m(x - x_1)$ $y - (-2) = -\frac{1}{5}(x - 2)$ $y + 2 = -\frac{1}{5}x + \frac{2}{5}$
Step 4. Write the equation in slope–intercept form.		$y = -\frac{1}{5}x - \frac{8}{5}$

Note:
Exercise:

Problem: Find the equation of a line containing the points $(-2, -4)$ and $(1, -3)$.

Solution:

$y = \frac{1}{3}x - \frac{10}{3}$

Note:
Exercise:

Problem: Find the equation of a line containing the points $(-4, -3)$ and $(1, -5)$.

Solution:

$y = -\frac{2}{5}x - \frac{23}{5}$

The steps are summarized here.

Note:
To find an equation of a line given two points.

Find the slope using the given points. $m = \frac{y_2 - y_1}{x_2 - x_1}$
Choose one point.
Substitute the values into the point-slope form: $y - y_1 = m\left(x - x_1\right)$.
Write the equation in slope-intercept form.

Example:
Exercise:

Problem:

Find an equation of a line that contains the points $\left(-3, 5\right)$ and $\left(-3, 4\right)$. Write the equation in slope-intercept form.

Solution:

Again, the first step will be to find the slope.

Find the slope of the line through $\left(-3, 5\right)$ and $\left(-3, 4\right)$.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ m &= \frac{4-5}{-3-(-3)} \\ m &= \frac{-1}{0} \end{aligned}$$

The slope is undefined.

This tells us it is a vertical line. Both of our points have an *x*-coordinate of -2. So our equation of the line is $x = -2$. Since there is no *y*, we cannot write it in slope-intercept form.

You may want to sketch a graph using the two given points. Does your graph agree with our conclusion that this is a vertical line?

Note:
Exercise:

Problem: Find the equation of a line containing the points $(5,1)$ and $(5,-4)$.

Solution:

$x=5$

Note:
Exercise:

Problem: Find the equaion of a line containing the points $(-4,4)$ and $(-4,3)$.

Solution:

$x=-4$

We have seen that we can use either the slope-intercept form or the point-slope form to find an equation of a line. Which form we use will depend on the information we are given.

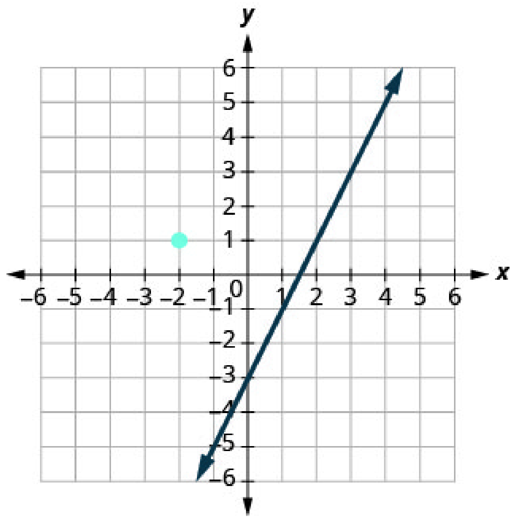
To Write an Equation of a Line		
If given:	**Use:**	**Form:**
Slope and y-intercept	slope-intercept	$y=mx+b$
Slope and a point	point-slope	$y-y_1=m\left(x-x_1\right)$
Two points	point-slope	$y-y_1=m\left(x-x_1\right)$

Find an Equation of a Line Parallel to a Given Line

Suppose we need to find an equation of a line that passes through a specific point and is parallel to a given line. We can use the fact that parallel lines have the same slope. So we will have a point and the slope—just what we need to use the point-slope equation.

First, let’s look at this graphically.

This graph shows $y = 2x - 3$. We want to graph a line parallel to this line and passing through the point $(-2, 1)$.

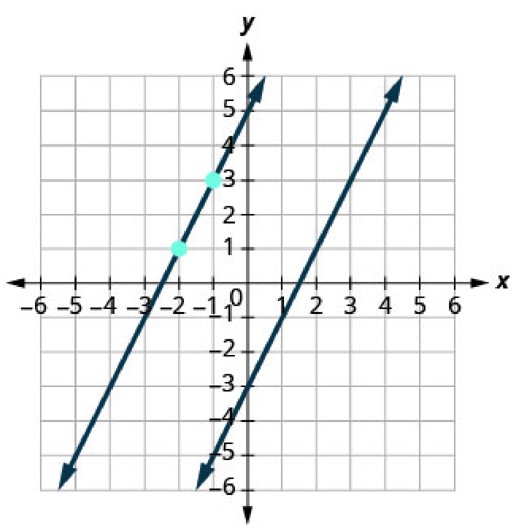


We know that parallel lines have the same slope. So the second line will have the same slope as $y = 2x - 3$. That slope is $m_{\|} = 2$. We’ll use the notation $m_{\|}$ to represent the slope of a line parallel to a line with slope *m*. (Notice that the subscript || looks like two parallel lines.)

The second line will pass through $(-2, 1)$ and have $m = 2$.

To graph the line, we start at$(-2, 1)$ and count out the rise and run.

With $m = 2$ (or $m = \frac{2}{1}$), we count out the rise 2 and the run 1. We draw the line, as shown in the graph.



Do the lines appear parallel? Does the second line pass through$(-2, 1)$?

We were asked to graph the line, now let's see how to do this algebraically.

We can use either the slope-intercept form or the point-slope form to find an equation of a line. Here we know one point and can find the slope. So we will use the point-slope form.

Example:
How to Find the Equation of a Line Parallel to a Given Line and a Point
Exercise:

Problem:

Find an equation of a line parallel to $y = 2x - 3$ that contains the point $(-2, 1)$. Write the equation in slope-intercept form.

Solution:

Step 1. Find the slope of the given line.	The line is in slope–intercept form, $y = 2x - 3$.	$m = 2$
Step 2. Find the slope of the parallel line.	Parallel lines have the same slope.	$m_1 = 2$
Step 3. Identify the point.	The given point is $(-2, 1)$.	$\begin{pmatrix} x_1 & y_1 \\ -2, & 1 \end{pmatrix}$
Step 4. Substitute the values into the point–slope form, $y - y_1 = m(x - x_1)$.	Simplify.	$y - y_1 = m(x - x_1)$ $y - 1 = 2(x - (-2))$ $y - 1 = 2(x + 2)$ $y - 1 = 2x + 4$
Step 5. Write the equation in slope–intercept form.		$y = 2x + 5$

Look at graph with the parallel lines shown previously. Does this equation make sense? What is the *y*-intercept of the line? What is the slope?

Note:
Exercise:

Problem:

Find an equation of a line parallel to the line $y = 3x + 1$ that contains the point $(4, 2)$. Write the equation in slope-intercept form.

Solution:

$y = 3x - 10$

Note:
Exercise:

Problem:

Find an equation of a line parallel to the line $y = \frac{1}{2}x - 3$ that contains the point $(6, 4)$.

Write the equation in slope-intercept form.

Solution:

$y = \frac{1}{2}x + 1$

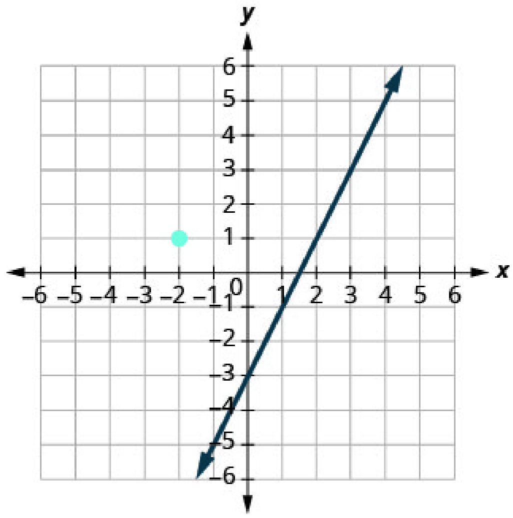
Note:
Find an equation of a line parallel to a given line.

Find the slope of the given line.
Find the slope of the parallel line.
Identify the point.
Substitute the values into the point-slope form:$y - y_1 = m\left(x - x_1\right)$.
Write the equation in slope-intercept form.

Find an Equation of a Line Perpendicular to a Given Line

Now, let's consider perpendicular lines. Suppose we need to find a line passing through a specific point and which is perpendicular to a given line. We can use the fact that perpendicular lines have slopes that are negative reciprocals. We will again use the point-slope equation, like we did with parallel lines.

This graph shows $y = 2x - 3$. Now, we want to graph a line perpendicular to this line and passing through $(-2, 1)$.



We know that perpendicular lines have slopes that are negative reciprocals.

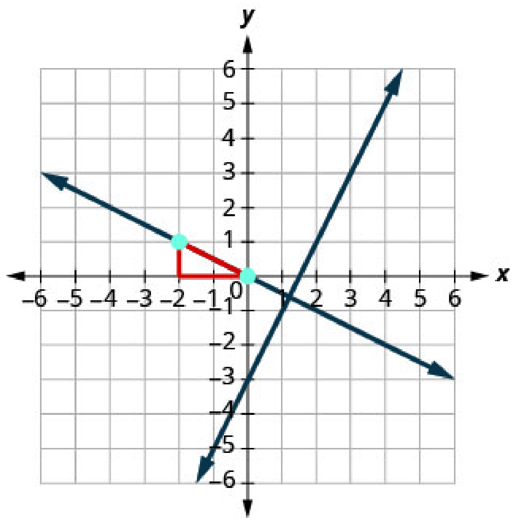
We'll use the notation $m_{\perp}$ to represent the slope of a line perpendicular to a line with slope m. (Notice that the subscript $\perp$ looks like the right angles made by two perpendicular lines.)

Equation:

$$\begin{array}{ll} y = 2x - 3 & \text{perpendicular line} \\ m = 2 & m_{\perp} = -\frac{1}{2} \end{array}$$

We now know the perpendicular line will pass through $(-2, 1)$ with $m_{\perp} = -\frac{1}{2}$.

To graph the line, we will start at $(-2, 1)$ and count out the rise -1 and the run 2. Then we draw the line.



Do the lines appear perpendicular? Does the second line pass through$(-2, 1)$?

We were asked to graph the line, now, let's see how to do this algebraically.

We can use either the slope-intercept form or the point-slope form to find an equation of a line. In this example we know one point, and can find the slope, so we will use the point-slope form.

Example:
How to Find the Equation of a Line Perpendicular to a Given Line and a Point
Exercise:

Problem:

Find an equation of a line perpendicular to $y = 2x - 3$ that contains the point $(-2, 1)$. Write the equation in slope-intercept form.

Solution:

Step 1. Find the slope of the given line.	The line is in slope–intercept form, $y = 2x - 3$.	$m = 2$
Step 2. Find the slope of the perpendicular line.	The slopes of perpendicular lines are negative reciprocals.	$m_{\perp} = -\frac{1}{2}$
Step 3. Identify the point.	The given point is $(-2, 1)$.	$\begin{pmatrix} x_1, & y_1 \\ -2, & 1 \end{pmatrix}$
Step 4. Substitute the values into the point–slope form, $y - y_1 = m(x - x_1)$.	Simplify.	$y - y_1 = m(x - x_1)$ $y - 1 = -\frac{1}{2}(x - (-2))$ $y - 1 = -\frac{1}{2}(x + 2)$ $y - 1 = -\frac{1}{2}x - 1$
Step 5. Write the equation in slope–intercept form.		$y = -\frac{1}{2}x$

Note:
Exercise:

Problem:

Find an equation of a line perpendicular to the line $y = 3x + 1$ that contains the point $(4, 2)$. Write the equation in slope-intercept form.

Solution:

$y = -\frac{1}{3}x + \frac{10}{3}$

Note:
Exercise:

Problem:

Find an equation of a line perpendicular to the line $y = \frac{1}{2}x - 3$ that contains the point $(6, 4)$. Write the equation in slope-intercept form.

Solution:

$y = -2x + 16$

Note:
Find an equation of a line perpendicular to a given line.

Find the slope of the given line.
Find the slope of the perpendicular line.
Identify the point.
Substitute the values into the point-slope form, $y - y_1 = m\left(x - x_1\right)$.
Write the equation in slope-intercept form.

Example:
Exercise:

Problem:

Find an equation of a line perpendicular to $x = 5$ that contains the point $(3, -2)$. Write the equation in slope-intercept form.

Solution:

Again, since we know one point, the point-slope option seems more promising than the slope-intercept option. We need the slope to use this form, and we know the new line will be perpendicular to $x = 5$. This line is vertical, so its perpendicular will be horizontal. This tells us the $m_{\perp} = 0$.

Identify the point.	$(3, -2)$
Identify the slope of the perpendicular line.	$m_{\perp} = 0$
Substitute the values into $y - y_1 = m(x - x_1)$.	$y - y_1 = m\left(x - x_1\right)$
	$y - (-2) = 0\left(x - 3\right)$
Simplify.	$y + 2 = 0$
	$y = -2$

Sketch the graph of both lines. On your graph, do the lines appear to be perpendicular?

Note:
Exercise:

Problem:

Find an equation of a line that is perpendicular to the line $x = 4$ that contains the point $(4, -5)$.. Write the equation in slope-intercept form.

Solution:

$y = -5$

Note:
Exercise:

Problem:

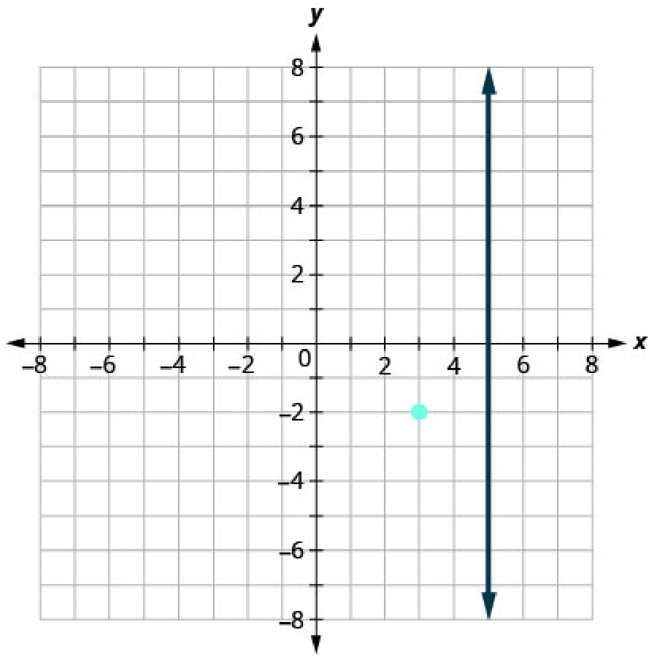
Find an equation of a line that is perpendicular to the line $x = 2$ that contains the point $(2, -1)$. Write the equation in slope-intercept form.

Solution:

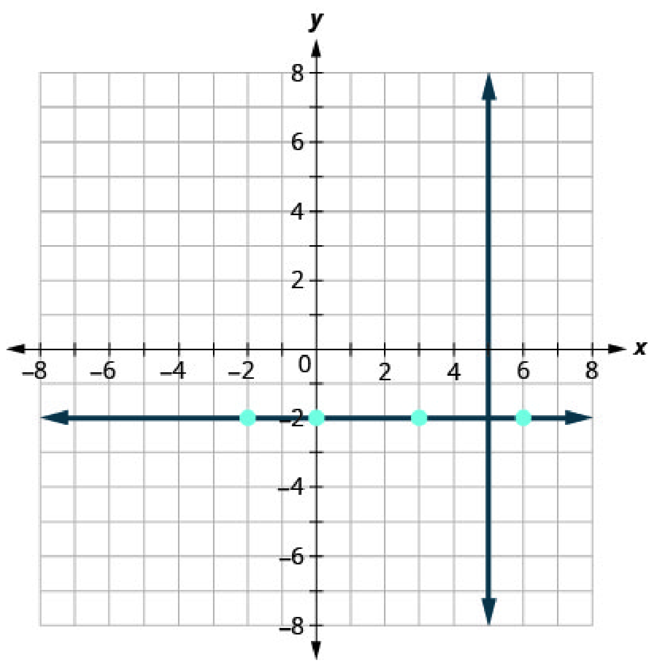
$y = -1$

In [link], we used the point-slope form to find the equation. We could have looked at this in a different way.

We want to find a line that is perpendicular to $x = 5$ that contains the point $(3, -2)$. This graph shows us the line$x = 5$ and the point $(3, -2)$.



We know every line perpendicular to a vertical line is horizontal, so we will sketch the horizontal line through $(3, -2)$.



Do the lines appear perpendicular?

If we look at a few points on this horizontal line, we notice they all have y-coordinates of -2. So, the equation of the line perpendicular to the vertical line $x=5$ is $y=-2$.

Example:
Exercise:

Problem:

Find an equation of a line that is perpendicular to $y=-3$ that contains the point $(-3,5)$. Write the equation in slope-intercept form.

Solution:

The line $y=-3$ is a horizontal line. Any line perpendicular to it must be vertical, in the form $x=a$. Since the perpendicular line is vertical and passes through $(-3,5)$, every point on it has an x-coordinate of -3. The equation of the perpendicular line is $x=-3$

You may want to sketch the lines. Do they appear perpendicular?

Note:
Exercise:

Problem:

Find an equation of a line that is perpendicular to the line $y=1$ that contains the point $(-5,1)$. Write the equation in slope-intercept form.

Solution:

$x=-5$

Note:
Exercise:

Problem:

Find an equation of a line that is perpendicular to the line $y=-5$ that contains the point $(-4,-5)$. Write the equation in slope-intercept form.

Solution:

$x=-4$

Note:
Access these online resources for additional instruction and practice with finding the equation of a line.

- Write an Equation of Line Given its slope and Y-Intercept
- Using Point Slope Form to Write the Equation of a Line, Find the equation given slope and point
- Find the equation given two points
- Find the equation of perpendicular and parallel lines

Key Concepts

- **How to find an equation of a line given the slope and a point.**

  Identify the slope.
  Identify the point.
  Substitute the values into the point-slope form, $y - y_1 = m\left(x - x_1\right)$.
  Write the equation in slope-intercept form.

- **How to find an equation of a line given two points.**

  Find the slope using the given points. $m = \frac{y_2 - y_1}{x_2 - x_1}$
  Choose one point.
  Substitute the values into the point-slope form: $y - y_1 = m\left(x - x_1\right)$.
  Write the equation in slope-
  intercept form.

To Write an Equation of a Line		
If given:	**Use:**	**Form:**
Slope and *y*-intercept	**slope-intercept**	$y = mx + b$
Slope and a point	**point-slope**	$y - y_1 = m\left(x - x_1\right)$
Two points	**point-slope**	$y - y_1 = m\left(x - x_1\right)$

- **How to find an equation of a line parallel to a given line.**

  Find the slope of the given line.
  Find the slope of the parallel line.
  Identify the point.
  Substitute the values into the point-slope form:$y - y_1 = m\left(x - x_1\right)$.
  Write the equation in slope-intercept form

- **How to find an equation of a line perpendicular to a given line.**

  Find the slope of the given line.
  Find the slope of the perpendicular line.
  Identify the point.
  Substitute the values into the point-slope form,$y - y_1 = m\left(x - x_1\right)$
  Write the equation in slope-intercept form.

Practice Makes Perfect

Find an Equation of the Line Given the Slope and *y*-Intercept

In the following exercises, find the equation of a line with given slope and *y*-intercept. Write the equation in slope-intercept form.

Exercise:

Problem: slope 3 and y-intercept $(0, 5)$

Solution:

$y = 3x + 5$

Exercise:

Problem: slope 8 and y-intercept $(0, -6)$

Exercise:

Problem: slope -3 and y-intercept $(0, -1)$

Solution:

$y = -3x - 1$

Exercise:

Problem: slope -1 and y-intercept $(0, 3)$

Exercise:

Problem: slope $\frac{1}{5}$ and y-intercept $(0, -5)$

Solution:

$y = \frac{1}{5}x - 5$

Exercise:

Problem: slope $-\frac{3}{4}$ and y-intercept $(0, -2)$

Exercise:

Problem: slope 0 and y-intercept $(0, -1)$

Solution:

$y = -1$

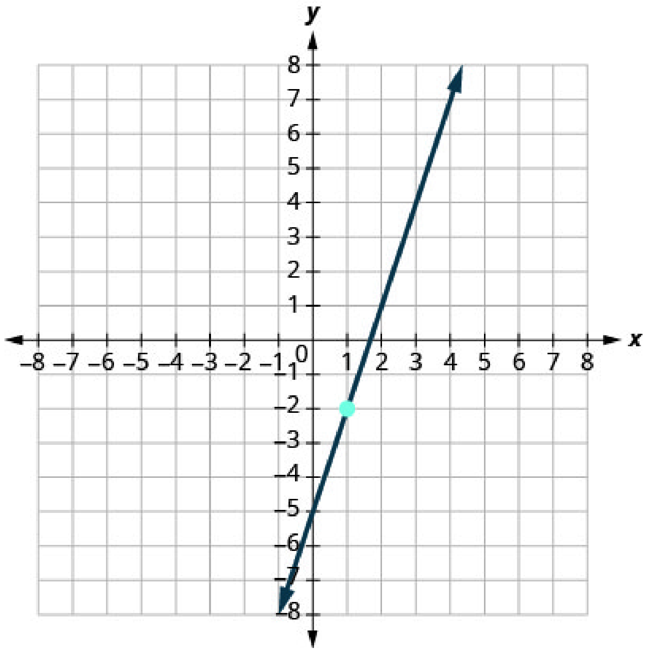
Exercise:

Problem: slope -4 and y-intercept $(0, 0)$

In the following exercises, find the equation of the line shown in each graph. Write the equation in slope-intercept form.

Exercise:

Problem:

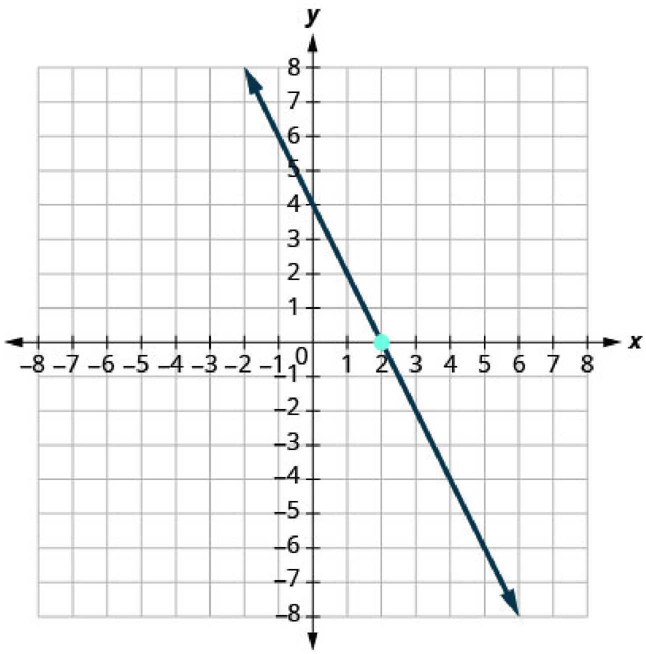


Solution:

$y = 3x - 5$

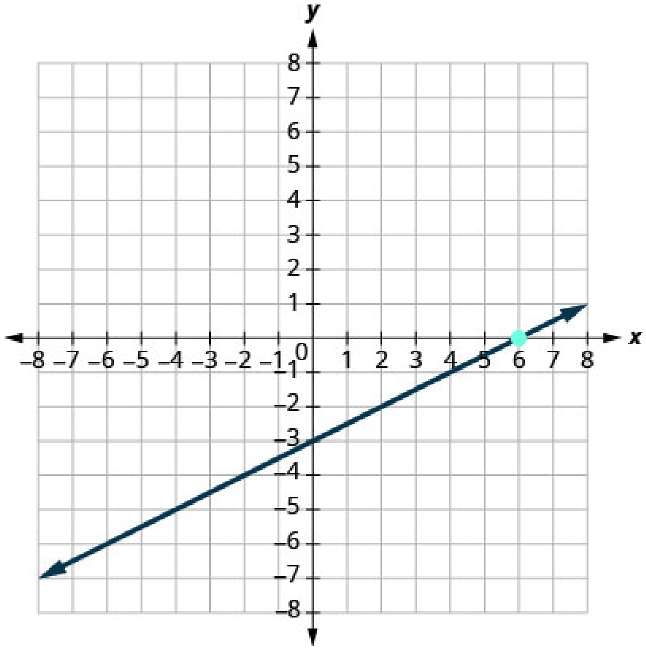
Exercise:

Problem:



Exercise:

Problem:

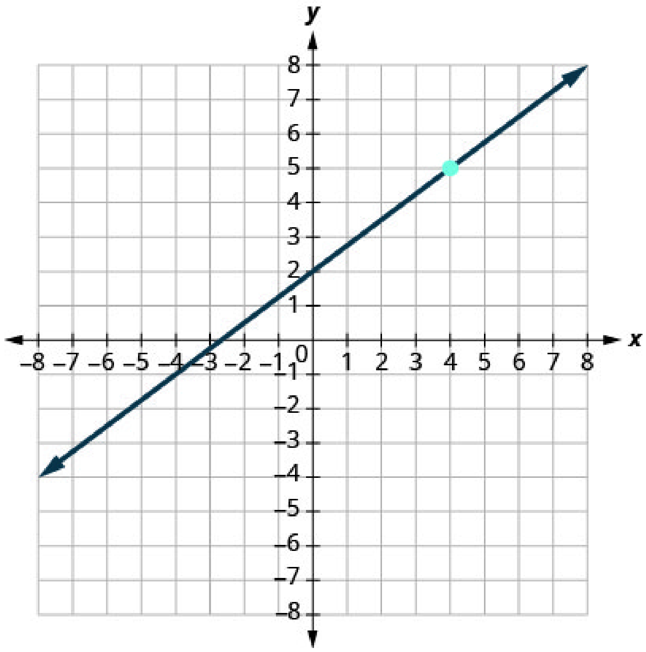


Solution:

$y = \frac{1}{2}x - 3$

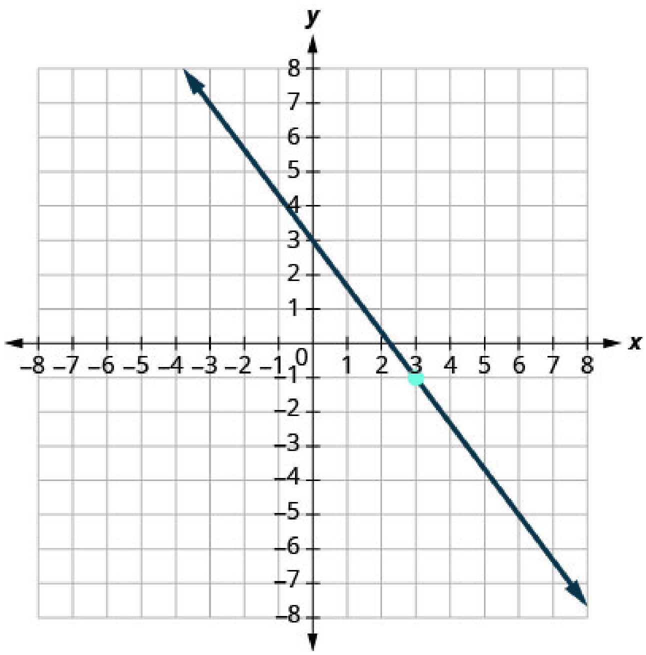
Exercise:

Problem:



Exercise:

Problem:

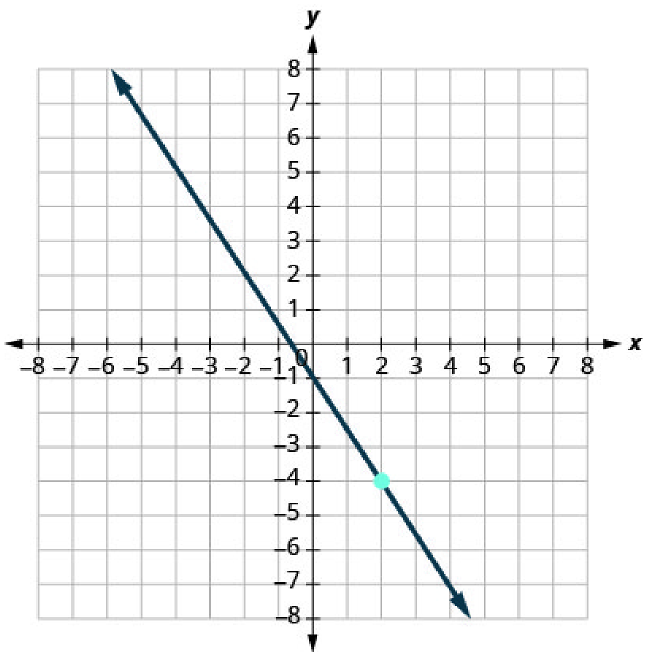


Solution:

$y = -\frac{4}{3}x + 3$

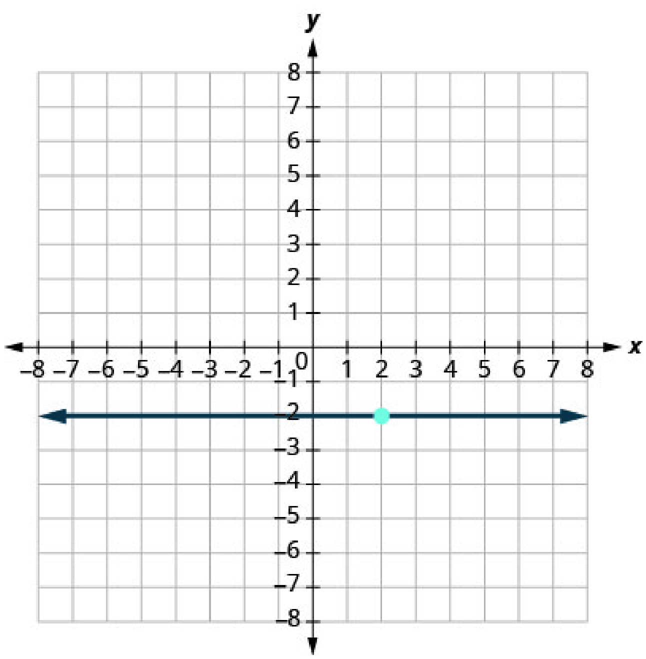
Exercise:

Problem:



Exercise:

Problem:

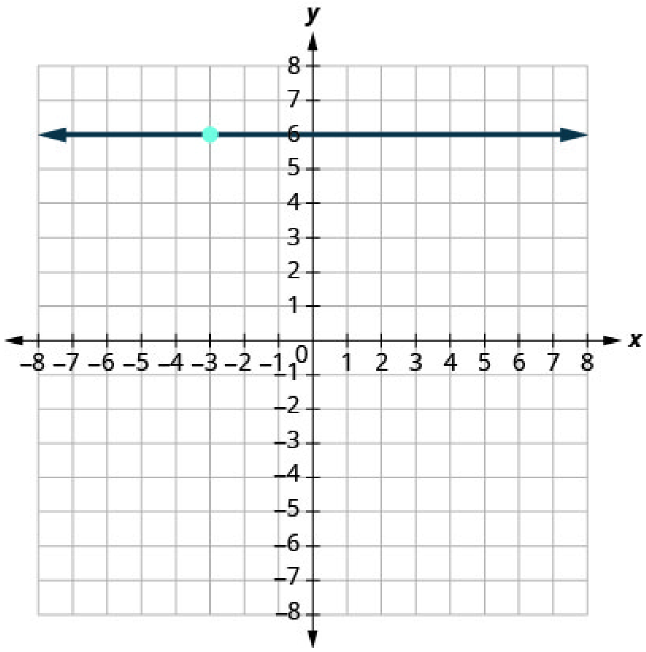


Solution:

$y = -2$

Exercise:

Problem:



Find an Equation of the Line Given the Slope and a Point

In the following exercises, find the equation of a line with given slope and containing the given point. Write the equation in slope-intercept form.

Exercise:

Problem: $m = \frac{5}{8}$, point $(8, 3)$

Solution:

$y = \frac{5}{8}x - 2$

Exercise:

Problem: $m = \frac{5}{6}$, point $(6, 7)$

Exercise:

Problem: $m = -\frac{3}{5}$, point $(10, -5)$

Solution:

$y = -\frac{3}{5}x + 1$

Exercise:

Problem: $m = -\frac{3}{4}$, point $(8, -5)$

Exercise:

Problem: $m = -\frac{3}{2}$, point $(-4, -3)$

Solution:

$y = -\frac{3}{2}x + 9$

Exercise:

Problem: $m = -\frac{5}{2}$, point $(-8, -2)$

Exercise:

Problem: $m = -7$, point $(-1, -3)$

Solution:

$y = -7x - 10$

Exercise:

Problem: $m = -4$, point $(-2, -3)$

Exercise:

Problem: Horizontal line containing $(-2, 5)$

Solution:

$y = 5$

Exercise:

Problem: Horizontal line containing $(-2, -3)$

Exercise:

Problem: Horizontal line containing $(-1, -7)$

Solution:

$y = -7$

Exercise:

Problem: Horizontal line containing $(4, -8)$

Find an Equation of the Line Given Two Points

In the following exercises, find the equation of a line containing the given points. Write the equation in slope-intercept form.

Exercise:

Problem: $(2, 6)$ and $(5, 3)$

Solution:

$y = -x + 8$

Exercise:

Problem: $(4, 3)$ and $(8, 1)$

Exercise:

Problem: $(-3, -4)$ and $(5 - 2)$.

Solution:

$y = \frac{1}{4}x - \frac{13}{4}$

Exercise:

Problem: $(-5, -3)$ and $(4, -6)$.

Exercise:

Problem: $(-1, 3)$ and $(-6, -7)$.

Solution:

$y = 2x + 5$

Exercise:

Problem: $(-2, 8)$ and $(-4, -6)$.

Exercise:

Problem: $(0,4)$ and $(2,-3)$.

Solution:

$y=-\frac{7}{2}x+4$

Exercise:

Problem: $(0,-2)$ and $(-5,-3)$.

Exercise:

Problem: $(7,2)$ and $(7,-2)$.

Solution:

$x=7$

Exercise:

Problem: $(-2,1)$ and $(-2,-4)$.

Exercise:

Problem: $(3,-4)$ and $(5,-4)$.

Solution:

$y=-4$

Exercise:

Problem: $(-6,-3)$ and $(-1,-3)$

Find an Equation of a Line Parallel to a Given Line

In the following exercises, find an equation of a line parallel to the given line and contains the given point. Write the equation in slope-intercept form.

Exercise:

Problem: line $y=4x+2$, point $(1,2)$

Solution:

$y = 4x - 2$

Exercise:

Problem: line $y = -3x - 1$, point $(2, -3)$.

Exercise:

Problem: line $2x - y = 6$, point $(3, 0)$.

Solution:

$y = 2x - 6$

Exercise:

Problem: line $2x + 3y = 6$, point $(0, 5)$.

Exercise:

Problem: line $x = -4$, point $(-3, -5)$.

Solution:

$x = -3$

Exercise:

Problem: line $x - 2 = 0$, point $(1, -2)$

Exercise:

Problem: line $y = 5$, point $(2, -2)$

Solution:

$y = -2$

Exercise:

Problem: line $y + 2 = 0$, point $(3, -3)$

Find an Equation of a Line Perpendicular to a Given Line

In the following exercises, find an equation of a line perpendicular to the given line and contains the given point. Write the equation in slope-intercept form.

Exercise:

Problem: line $y = -2x + 3$, point $(2, 2)$

Solution:

$y = \frac{1}{2}x + 1$

Exercise:

Problem: line $y = -x + 5$, point $(3, 3)$

Exercise:

Problem: line $y = \frac{3}{4}x - 2$, point $(-3, 4)$

Solution:

$y = -\frac{4}{3}x$

Exercise:

Problem: line $y = \frac{2}{3}x - 4$, point $(2, -4)$

Exercise:

Problem: line $2x - 3y = 8$, point $(4, -1)$

Solution:

$y = -\frac{3}{2}x + 5$

Exercise:

Problem: line $4x - 3y = 5$, point $(-3, 2)$

Exercise:

Problem: line $2x+5y=6$, point $(0,0)$

Solution:

$y=\frac{5}{2}x$

Exercise:

Problem: line $4x+5y=-3$, point $(0,0)$

Exercise:

Problem: line $x=3$, point $(3,4)$

Solution:

$y=4$

Exercise:

Problem: line $x=-5$, point $(1,-2)$

Exercise:

Problem: line $x=7$, point $(-3,-4)$

Solution:

$y=-4$

Exercise:

Problem: line $x=-1$, point $(-4,0)$

Exercise:

Problem: line $y-3=0$, point $(-2,-4)$

Solution:

$x = -2$

Exercise:

Problem: line $y - 6 = 0$, point $(-5, -3)$

Exercise:

Problem: line *y*-axis, point $(3, 4)$

Solution:

$y = 4$

Exercise:

Problem: line *y*-axis, point $(2, 1)$

Mixed Practice

In the following exercises, find the equation of each line. Write the equation in slope-intercept form.

Exercise:

Problem: Containing the points $(4, 3)$ and $(8, 1)$

Solution:

$y = -\frac{1}{2}x + 5$

Exercise:

Problem: Containing the points $(-2, 0)$ and $(-3, -2)$

Exercise:

Problem: $m = \frac{1}{6}$, containing point $(6, 1)$

Solution:

$y = \frac{1}{6}x$

Exercise:

Problem: $m = \frac{5}{6}$, containing point $(6, 7)$

Exercise:

Problem: Parallel to the line $4x + 3y = 6$, containing point $(0, -3)$

Solution:

$y = -\frac{4}{3}x - 3$

Exercise:

Problem: Parallel to the line $2x + 3y = 6$, containing point $(0, 5)$

Exercise:

Problem: $m = -\frac{3}{4}$, containing point $(8, -5)$

Solution:

$y = -\frac{3}{4}x + 1$

Exercise:

Problem: $m = -\frac{3}{5}$, containing point $(10, -5)$

Exercise:

Problem: Perpendicular to the line $y - 1 = 0$, point $(-2, 6)$

Solution:

$x = -2$

Exercise:

Problem: Perpendicular to the line *y*-axis, point $(-6, 2)$

Exercise:

Problem: Parallel to the line $x = -3$, containing point $(-2, -1)$

Solution:

$x = -2$

Exercise:

Problem: Parallel to the line $x = -4$, containing point $(-3, -5)$

Exercise:

Problem: Containing the points $(-3, -4)$ and $(2, -5)$

Solution:

$y = -\frac{1}{5}x - \frac{23}{5}$

Exercise:

Problem: Containing the points $(-5, -3)$ and $(4, -6)$

Exercise:

Problem: Perpendicular to the line $x - 2y = 5$, point $(-2, 2)$

Solution:

$y = -2x - 2$

Exercise:

Problem: Perpendicular to the line $4x + 3y = 1$, point $(0, 0)$

Writing Exercises

Exercise:

Problem: Why are all horizontal lines parallel?

Solution:

Answers will vary.

Exercise:

Problem:

Explain in your own words why the slopes of two perpendicular lines must have opposite signs.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
find the equation of the line given the slope and y-intercept.			
find an equation of the line given the slope and a point.			
find an equation of the line given two points.			
find an equation of a line parallel to a given line.			
find an equation of a line perpendicular to a given line.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Glossary

point-slope form
: The point-slope form of an equation of a line with slope m and containing the point (x_1, y_1) is $y - y_1 = m\left(x - x_1\right)$.

Scatterplots and Linear Models

By the end of this section, you will be able to:

- Distinguish between dependent and independent variables.
- Visually recognize positive versus negative relationships.
- Visually recognize outliers.
- Visually recognize strong versus weak versus no correlation between variables on a scatterplot.
- Graphically determine a Line of Best Fit from a scatterplot.
- Distinguish between interpolation and extrapolation.
- Use a Line of Best Fit for interpolation and extrapolation.

One way to display the relation between two variables x and y is with a table. A more visual way is to graph the ordered pairs. A **scatterplot** is the name **statisticians** give to this type of graph. The following examples illustrates a table and its corresponding scatterplot.

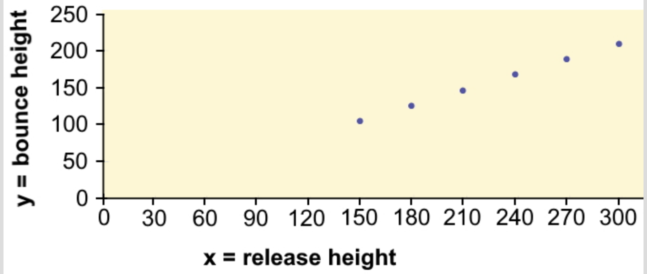
Example:
Dropped Ball

A ball is dropped and how high off the ground it is when it is dropped and how high it bounces are measured. Those values are an ordered pair: (release height, bounce height). If the ball is dropped at various heights there will be many ordered pairs. The bounce height depends on the release height. We say that the bounce height is the **dependent variable** because its value depends on the release height which is the **independent variable**. When we make a scatterplot it is normal to put the independent variable on the X-axis and the dependent variable on the Y-axis.

Table showing bounce height (in cm).

Scatterplot showing bounce height (in cm).

Release Height	Bounce Height
300 cm	210 cm
270 cm	189 cm
240 cm	168 cm
210 cm	147 cm
180 cm	126 cm
150 cm	105 cm



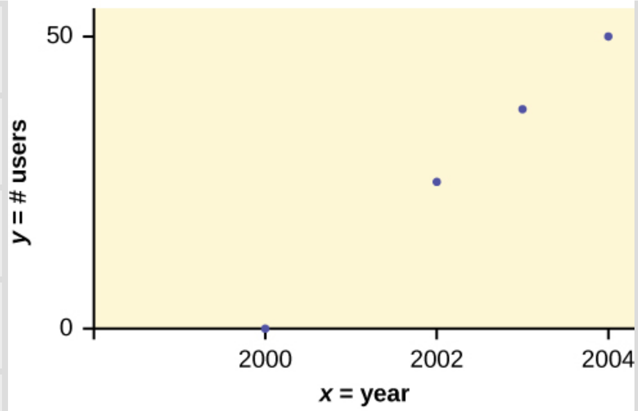
Dropped Ball Height

Example:
Mobile Commerce
In Europe and Asia, m-commerce is popular. M-commerce users have special mobile phones that work like electronic wallets as well as provide phone and Internet services. Users can do everything from paying for parking to buying a TV set or soda from a machine to banking to checking sports scores on the Internet. For the years 2000 through 2004, was there a relationship between the year and the number of m-commerce users? Construct a scatter plot. Let x = the year and let y = the number of m-commerce users, in millions.

Table showing the number of m-commerce users (in millions) by year.

Scatterplot showing the number of m-commerce users (in millions) by year.

x (year)	y (# of users)
2000	0.5
2002	20.0
2003	33.0
2004	47.0



Note:
Try It
Exercise:

Problem:

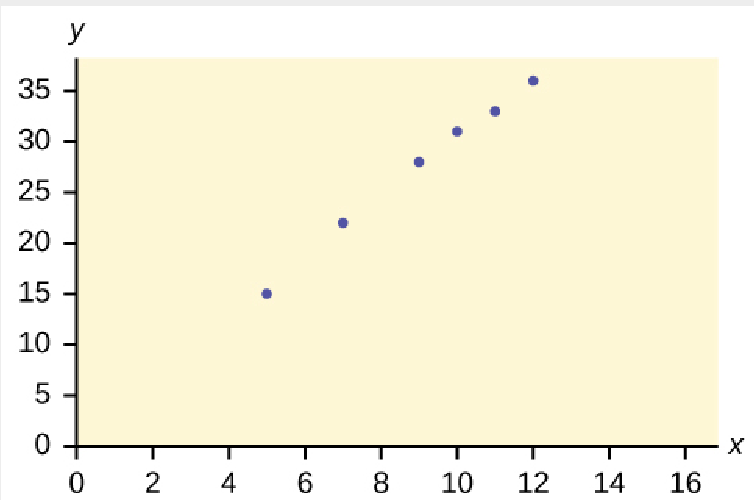
Amelia plays basketball for her high school. She wants to improve to play at the college level. She notices that the number of points she scores in a game goes up in response to the number of hours she practices her jump shot each week. She records the following data:

X (hours practicing jump shot)	Y (points scored in a game)
5	15
7	22
9	28

X (hours practicing jump shot)	*Y* (points scored in a game)
10	31
11	33
12	36

Construct a scatterplot and state if what Amelia thinks appears to be true.

Solution:



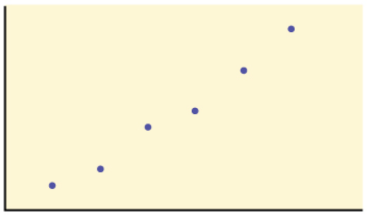
Yes, Amelia's assumption appears to be correct. The number of points Amelia scores per game goes up when she practices her jump shot more.

A scatterplot shows the **direction** of a relationship between the variables. A clear direction happens when there is either:

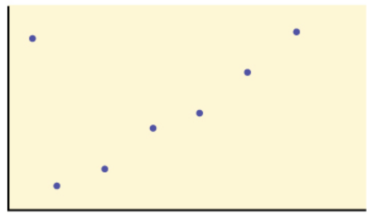
- High values of one variable occurring with high values of the other variable or low values of one variable occurring with low values of the

other variable.
- High values of one variable occurring with low values of the other variable.

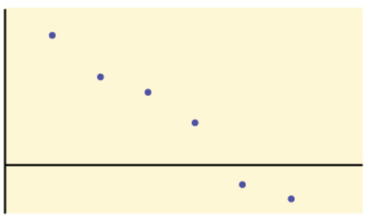
When you look at a scatterplot, you want to notice the **overall pattern** and any **deviations** from the pattern. The following scatterplot examples illustrate these concepts.



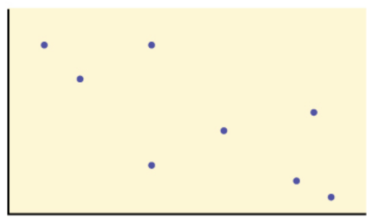
(a) Positive linear pattern (strong)



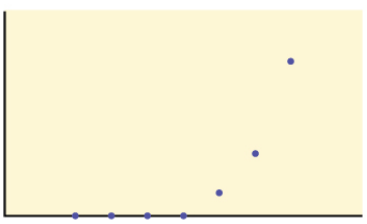
(b) Linear pattern w/ one deviation



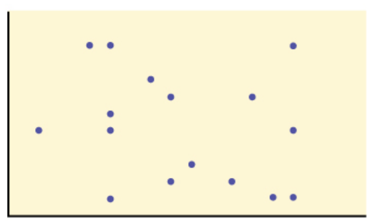
(a) Negative linear pattern (strong)



(b) Negative linear pattern (weak)



(a) Exponential growth pattern



(b) No pattern

The scatterplot in figure 5a above is an exponential growth pattern. It is an example of a pattern that is not linear.

Linear patterns are quite common. The linear relationship is strong if the points are close to a straight line, except in the case of a horizontal line where there is no relationship. If we think that the points show a linear relationship, we can draw a line on the scatterplot that is reasonable close to almost all of the points. In a more advanced class, one would use an exact technique called **regression** to determine the formula for this line. The line is called the **regression line** or the **Line of Best Fit**. This line can be used to make predictions for the dependent variable based on the value of the independent variable.

Interpolation and Extrapolation
Once a line has been drawn showing the linear relationship, the equation for the line can be determined using techniques explained earlier in this chapter. Rarely will the line be a perfect fit and go through all of the points. The line is a **model** that should give a close estimate of the dependent variable for values of the independent variable that may not be present in the data.

When a value of the independent variable lies between two values of the independent variable already in the data this is called **interpolation.** When the value of the independent variable is less than the least of the independent variable already in the data or greater than the greatest of the independent variable already in the data this is called **extrapolation.**

Models aren't guaranteed to accurately predict the dependent variable. As a practical matter, interpolated values are much more accurate than extrapolated values.

Dropped Ball part 2
After drawing a line through the data it was determined that the equation for the line was y = 0.7x. Use this equation to predict the height of the bounce when the drop height is 250 cm. Is this interpolation or extrapolation?

y = 0.7 (250 cm) = 175 cm. This is interpolation because 250 cm is between 270 cm and 240 cm.
Notice that 175 cm is between the bounce height for 270 cm (189 cm) and 240 cm (168 cm) so this makes sense.

Section Review

Scatterplots are particularly helpful graphs when we want to see if there is a linear relationship among data points. They indicate both the direction of the relationship between the x variables and the y variables, and the strength of the relationship.

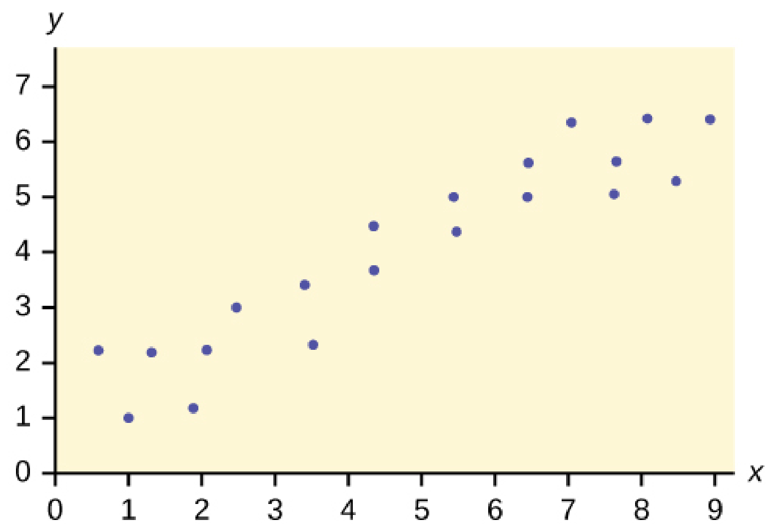
If a linear relationship exists between the independent and dependent

variables then it is appropriate to find a linear equation that relates the two variables. This equation can be used for interpolation and extrapolation.

Exercise:

Problem:

Does the scatterplot appear linear? Strong or weak? Positive or negative?
Draw a line through the data and estimate the equation. Based on your equation, what value should you expect for y when x = 5? Does this make sense in relation to other points on the graph?



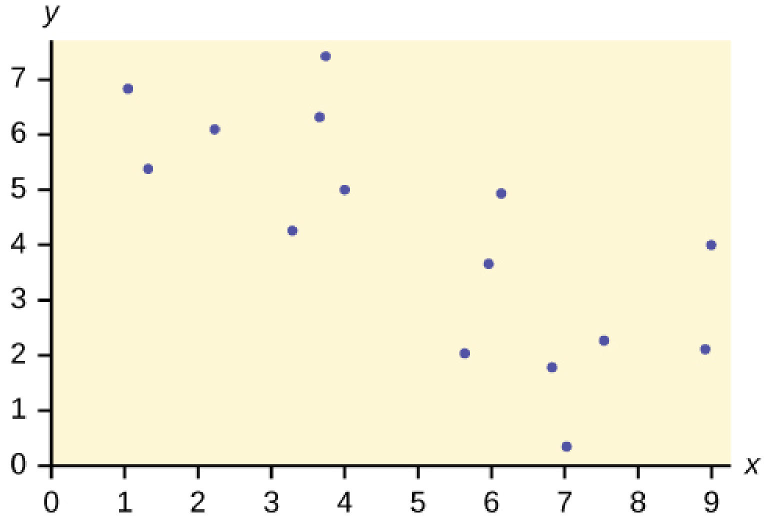
Solution:

The data appear to be linear with a strong, positive correlation.
The answers for the equation may differ since we are not doing this exactly. One possible equation is y = 0.75x +0.5.
When x = 5, y = 0.75 * 5 + 0.5. y = 4.25.

Exercise:

Problem:

Does the scatterplot appear linear? Strong or weak? Positive or negative?
If appropriate, draw a line through the data and estimate the equation.
Based on your equation, what value should you expect for y when x = 5? Does this make sense in relation to other points on the graph?



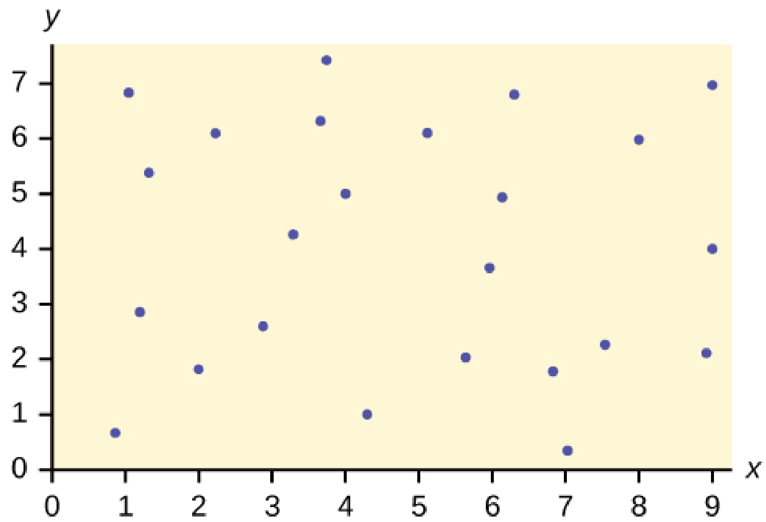
Solution:

The data appear to be linear with a weak, negative correlation. The slope of the line will be negative and the y-intercept will be greater than 7. Answer may vary.

Exercise:

Problem:

Does the scatterplot appear linear? Strong or weak? Positive or negative?



Solution:

The data appear to have no correlation so drawing in a line and obtaining an equation is not appropriate.

Media Resources

https://www.mathsisfun.com/data/scatter-xy-plots.html

Homework

Exercise:

Problem:

The Gross Domestic Product Purchasing Power Parity is an indication of a country’s currency value compared to another country. The following table shows the GDP PPP of Cuba as compared to US dollars. Construct a scatterplot of the data.
If appropriate, draw a line through the data and estimate the equation.

Year	Cuba’s PPP	Year	Cuba’s PPP
1999	1,700	2006	4,000
2000	1,700	2007	11,000
2002	2,300	2008	9,500
2003	2,900	2009	9,700
2004	3,000	2010	9,900
2005	3,500		

Solution:

Solutions may vary. 2007 may be considered an outlier. If so, it would be excluded from the data. The results below are calculated using a spreadsheet program. Drawn solutions will not have this level of accuracy.

Without excluding 2007, the regression line is : y = 919x -1965.
Excluding 2007, the regression line is y = 849.52x - 1936.
Notice that with the outlier excluded the Line of Best Fit is less steep and the Y-intercept is closer to the origin.

Exercise:

Problem:

The following table shows the poverty rates and cell phone usage in the United States. Construct a scatterplot of the data.
If appropriate, draw a line through the data and estimate the equation.
What value does your equation estimate for years 2006 and 2012?
Which of these estimates is more likely to be accurate?

Year	Poverty Rate	Cellular Usage per Capita
2003	12.7	54.67
2005	12.6	74.19
2007	12	84.86
2009	12	90.82

Solution:

Assume that Poverty Rate is the independent variable and that Cellular Usage per Capita is the dependent variable.

Using a spreadsheet the Line of Best Fit is $y = -38x + 543$. This tells us that as the poverty rate decreases cell usage increases. This is a correlation and doesn't tell us if one of these variables is causing the other.

Graph Linear Inequalities in Two Variables: ASE
By the end of this section, you will be able to:

- Verify solutions to an inequality in two variables.
- Recognize the relation between the solutions of an inequality and its graph.
- Graph linear inequalities in two variables
- Solve applications using linear inequalities in two variables

Verify Solutions to an Inequality in Two Variables

Previously we learned to solve inequalities with only one variable. We will now learn about inequalities containing two variables. In particular we will look at **linear inequalities** in two variables which are very similar to linear equations in two variables.

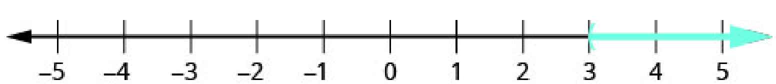
Linear inequalities in two variables have many applications. If you ran a business, for example, you would want your revenue to be greater than your costs—so that your business made a profit.

Note:
Linear Inequality
A **linear inequality** is an inequality that can be written in one of the following forms:
Equation:

$$Ax + By > C \qquad Ax + By \geq C \qquad Ax + By < C \qquad Ax + By \leq C$$

Where *A* and *B* are not both zero.

Recall that an inequality with one variable had many solutions. For example, the solution to the inequality $x > 3$ is any number greater than 3. We showed this on the number line by shading in the number line to the right of 3, and putting an open parenthesis at 3. See [link].



Similarly, linear inequalities in two variables have many solutions. Any ordered pair (x, y) that makes an inequality true when we substitute in the values is a **solution to a linear inequality**.

Note:
Solution to a Linear Inequality

An ordered pair (x, y) is a **solution to a linear inequality** if the inequality is true when we substitute the values of *x* and *y*.

Example:
Exercise:

Problem: Determine whether each ordered pair is a solution to the inequality $y > x + 4$:

ⓐ $(0, 0)$ ⓑ $(1, 6)$ ⓒ $(2, 6)$ ⓓ $(-5, -15)$ ⓔ $(-8, 12)$

Solution:

ⓐ

$(0, 0)$	$y > x + 4$
Substitute 0 for *x* and 0 for *y*.	$0 \overset{?}{>} 0 + 4$
Simplify.	$0 \not> 4$
	So, $(0, 0)$ is not a solution to $y > x + 4$.

ⓑ

$(1, 6)$	$y > x + 4$
Substitute 1 for *x* and 6 for *y*.	$6 \overset{?}{>} 1 + 4$

Simplify.	$6 > 5$
	So, $(1, 6)$ is a solution to $y > x + 4$.

ⓒ

$(2, 6)$	$y > x + 4$
Substitute 2 for x and 6 for y.	$6 \overset{?}{>} 2 + 4$
Simplify.	$6 \not> 6$
	So, $(2, 6)$ is not a solution to $y > x + 4$.

ⓓ

$(-5, -15)$	$y > x + 4$
Substitute -5 for x and -15 for y.	$-15 \overset{?}{>} -5 + 4$
Simplify.	$-15 \not> -1$
	So, $(-5, -15)$ is not a solution to $y > x + 4$.

ⓔ

$(-8, 12)$	$y > x + 4$
Substitute -8 for x and 12 for y.	$12 \overset{?}{>} -8 + 4$
Simplify.	$12 > -4$
	So, $(-8, 12)$ is a solution to $y > x + 4$.

Note:
Exercise:

Problem: Determine whether each ordered pair is a solution to the inequality $y > x - 3$:

ⓐ $(0, 0)$ ⓑ $(4, 9)$ ⓒ $(-2, 1)$ ⓓ $(-5, -3)$ ⓔ $(5, 1)$

Solution:

ⓐ yes ⓑ yes ⓒ yes ⓓ yes ⓔ no

Note:
Exercise:

Problem: Determine whether each ordered pair is a solution to the inequality $y < x + 1$:

ⓐ $(0, 0)$ ⓑ $(8, 6)$ ⓒ $(-2, -1)$ ⓓ $(3, 4)$ ⓔ $(-1, -4)$

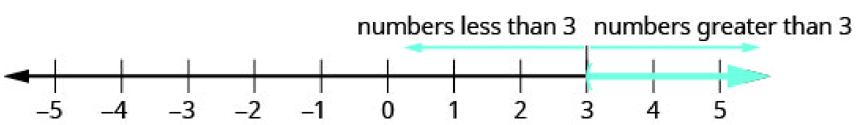
Solution:

ⓐ yes ⓑ yes ⓒ no ⓓ no
ⓔ yes

Recognize the Relation Between the Solutions of an Inequality and its Graph

Now, we will look at how the solutions of an inequality relate to its graph.

Let's think about the number line in shown previously again. The point $x = 3$ separated that number line into two parts. On one side of 3 are all the numbers less than 3. On the other side of 3 all the numbers are greater than 3. See [link].

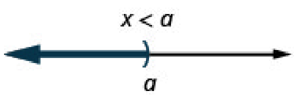


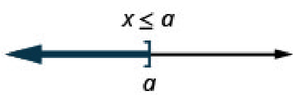
The solution to $x > 3$ is the shaded part of the number line to the right of $x = 3$.

Similarly, the line $y = x + 4$ separates the plane into two regions. On one side of the line are points with $y < x + 4$. On the other side of the line are the points with $y > x + 4$. We call the line $y = x + 4$ a **boundary line**.

Note:
Boundary Line
The line with equation $Ax + By = C$ is the **boundary line** that separates the region where $Ax + By > C$ from the region where $Ax + By < C$.

For an inequality in one variable, the endpoint is shown with a parenthesis or a bracket depending on whether or not *a* is included in the solution:

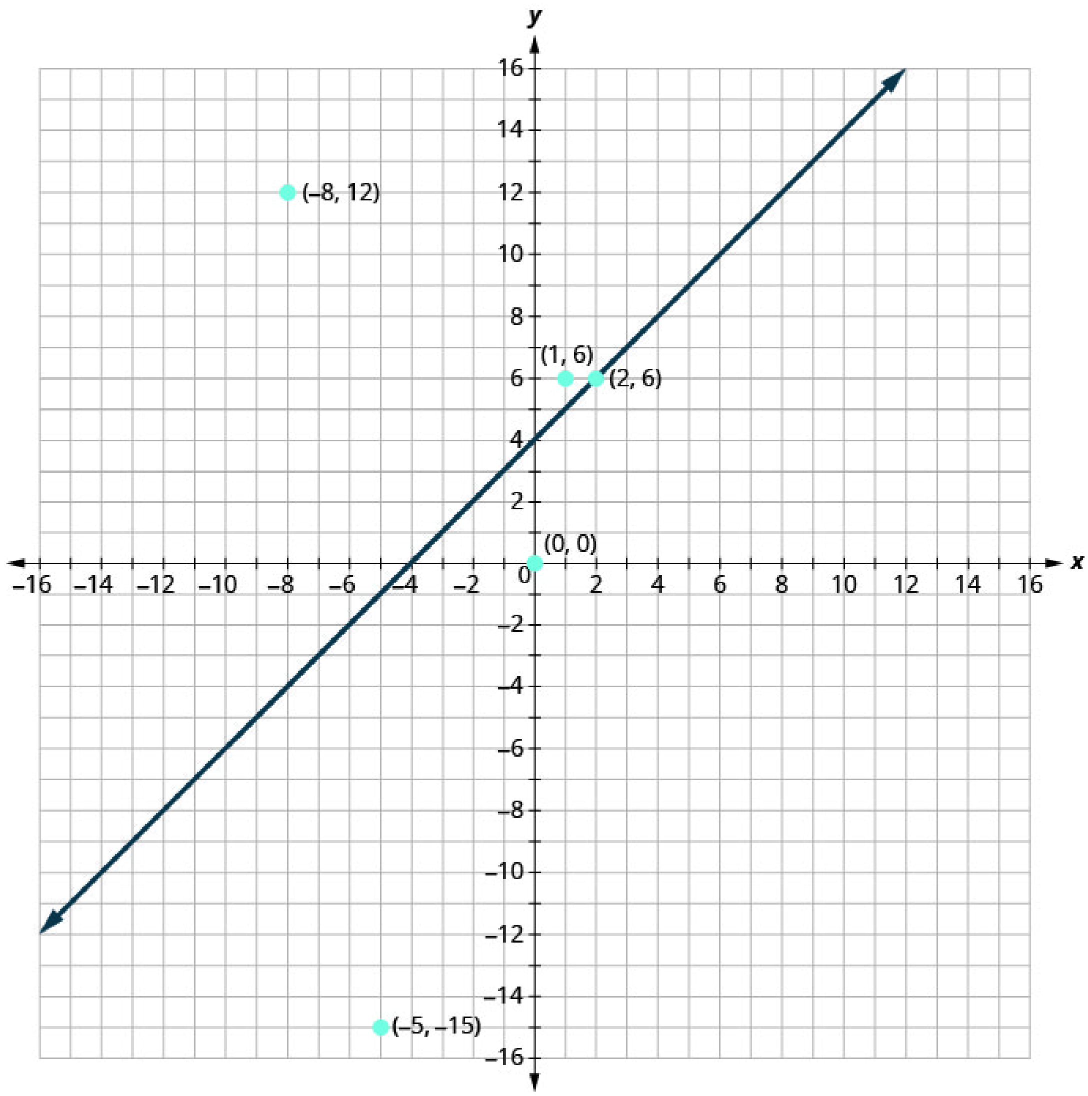




Similarly, for an inequality in two variables, the boundary line is shown with a solid or dashed line to show whether or not it the line is included in the solution.
Equation:

$Ax + By < C$	$Ax + By \leq C$
$Ax + By > C$	$Ax + By \geq C$
Boundary line is $Ax + By = C$	Boundary line is $Ax + By = C$
Boundary line is not included in solution.	Boundary line is included in solution.
Boundary line is dashed.	**Boundary line is solid.**

Now, let's take a look at what we found in [link]. We'll start by graphing the line $y = x + 4$, and then we'll plot the five points we tested, as shown in the graph. See [link].



In [link] we found that some of the points were solutions to the inequality $y > x + 4$ and some were not.

Which of the points we plotted are solutions to the inequality $y > x + 4$?

The points $(1,6)$ and $(-8,12)$ are solutions to the inequality $y > x + 4$. Notice that they are both on the same side of the boundary line $y = x + 4$.

The two points $(0,0)$ and $(-5,-15)$ are on the other side of the boundary line $y = x + 4$, and they are not solutions to the inequality $y > x + 4$. For those two points, $y < x + 4$.

What about the point $(2,6)$? Because $6 = 2 + 4$, the point is a solution to the equation $y = x + 4$, but not a solution to the inequality $y > x + 4$. So the point $(2,6)$ is on the boundary line.

Let’s take another point above the boundary line and test whether or not it is a solution to the inequality $y > x + 4$. The point $(0,10)$ clearly looks to above the boundary line, doesn’t it? Is it a solution to the inequality?

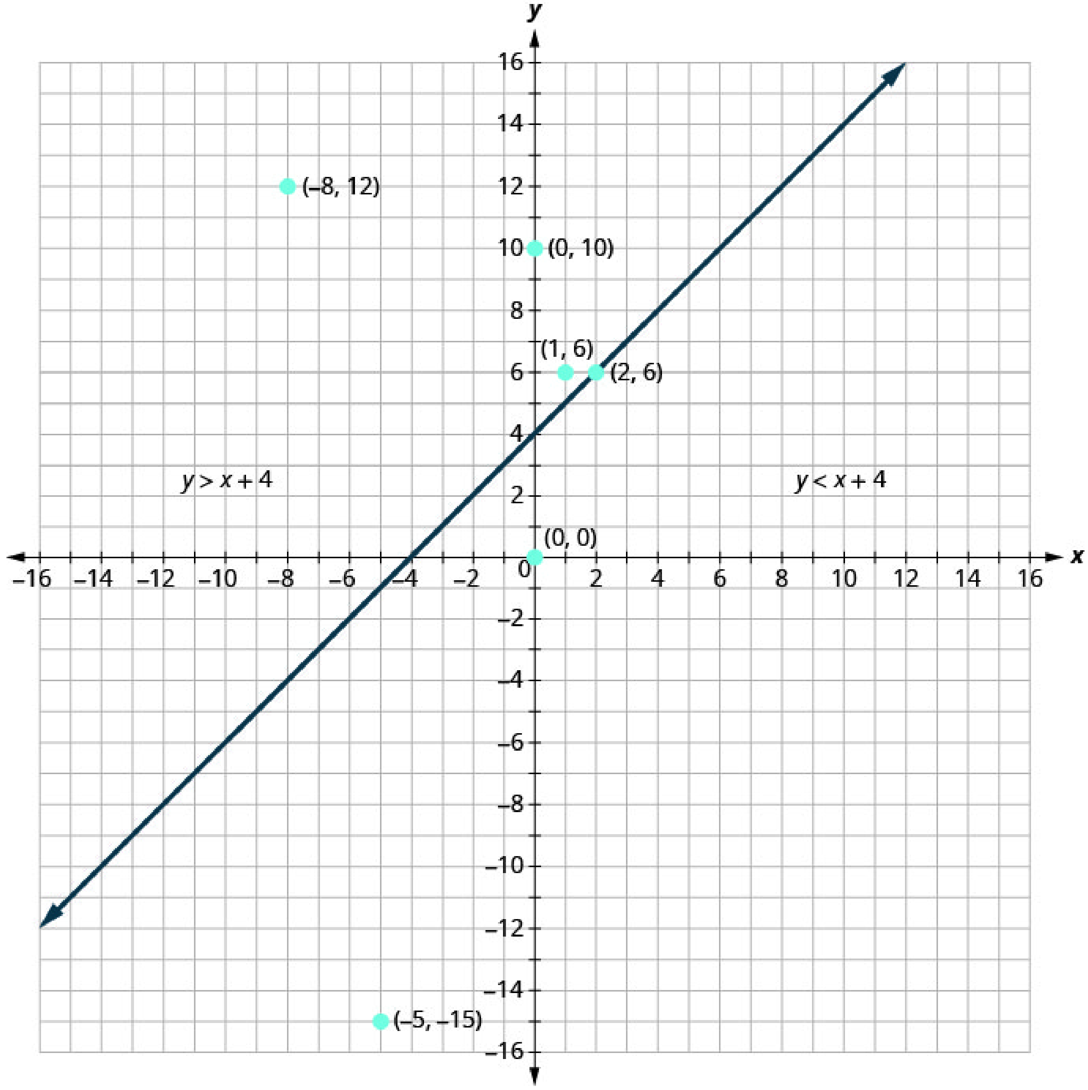
Equation:

$$\begin{array}{rcl} y & > & x+4 \\ 10 & \overset{?}{>} & 0+4 \\ 10 & > & 4 \end{array}$$

So, $(0,10)$ is a solution to $y > x + 4$.

Any point you choose above the boundary line is a solution to the inequality $y > x + 4$. All points above the boundary line are solutions.

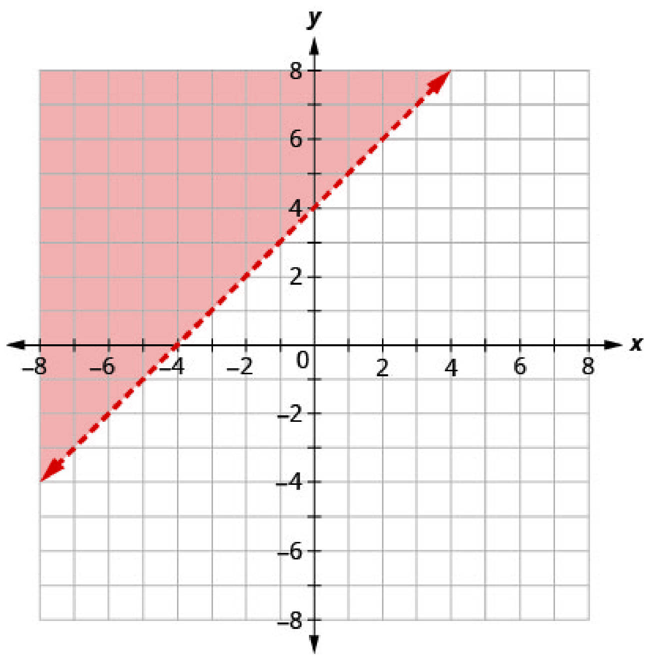
Similarly, all points below the boundary line, the side with $(0,0)$ and $(-5,-15)$, are not solutions to $y > x + 4$, as shown in [link].



The graph of the inequality $y > x + 4$ is shown in below.

The line $y = x + 4$ divides the plane into two regions. The shaded side shows the solutions to the inequality $y > x + 4$.

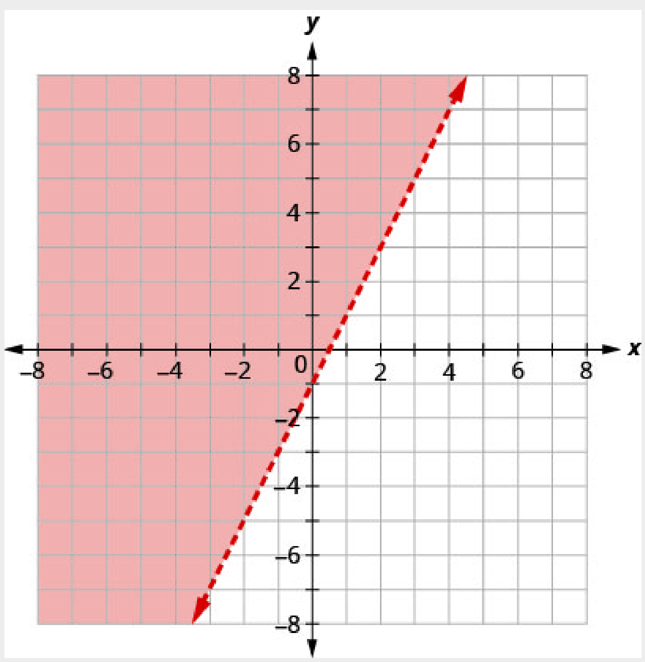
The points on the boundary line, those where $y = x + 4$, are not solutions to the inequality $y > x + 4$, so the line itself is not part of the solution. We show that by making the line dashed, not solid.



Example:
Exercise:

Problem:

The boundary line shown in this graph is $y = 2x - 1$. Write the inequality shown by the graph.



Solution:

The line $y = 2x - 1$ is the boundary line. On one side of the line are the points with $y > 2x - 1$ and on the other side of the line are the points with $y < 2x - 1$.

Let's test the point $(0, 0)$ and see which inequality describes its position relative to the boundary line.

At $(0, 0)$, which inequality is true: $y > 2x - 1$ or $y < 2x - 1$?

Equation:

$$\begin{array}{ll} y > 2x - 1 & y < 2x - 1 \\ 0 \overset{?}{>} 2 \cdot 0 - 1 & 0 \overset{?}{<} 2 \cdot 0 - 1 \\ 0 > -1 \text{ True} & 0 < -1 \text{ False} \end{array}$$

Since, $y > 2x - 1$ is true, the side of the line with $(0, 0)$, is the solution. The shaded region shows the solution of the inequality $y > 2x - 1$.

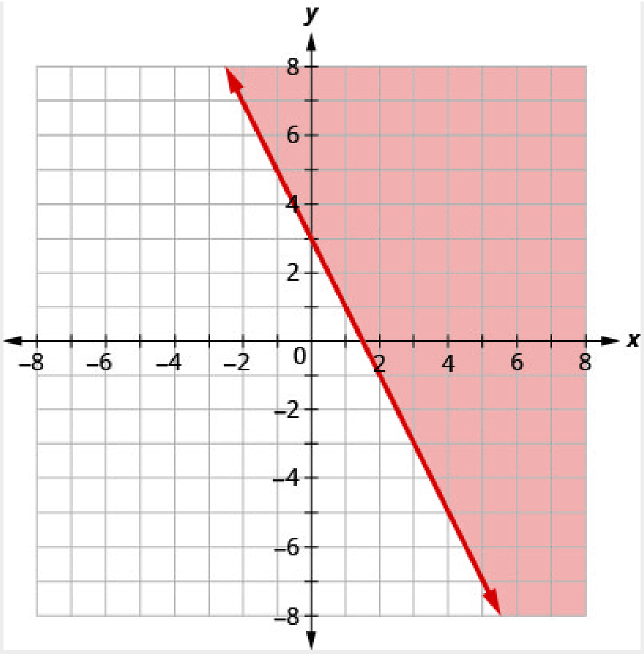
Since the boundary line is graphed with a solid line, the inequality includes the equal sign.

The graph shows the inequality $y \geq 2x - 1$.

We could use any point as a test point, provided it is not on the line. Why did we choose $(0, 0)$? Because it's the easiest to evaluate. You may want to pick a point on the other side of the boundary line and check that $y < 2x - 1$.

Note:
Exercise:

Problem: Write the inequality shown by the graph with the boundary line $y = -2x + 3$.

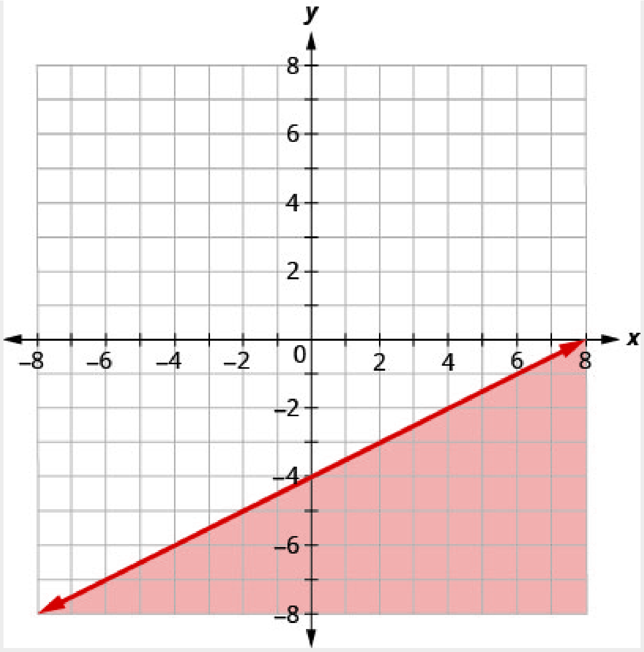


Solution:

$y \geq -2x + 3$

Note:
Exercise:

Problem: Write the inequality shown by the graph with the boundary line $y = \frac{1}{2}x - 4$.



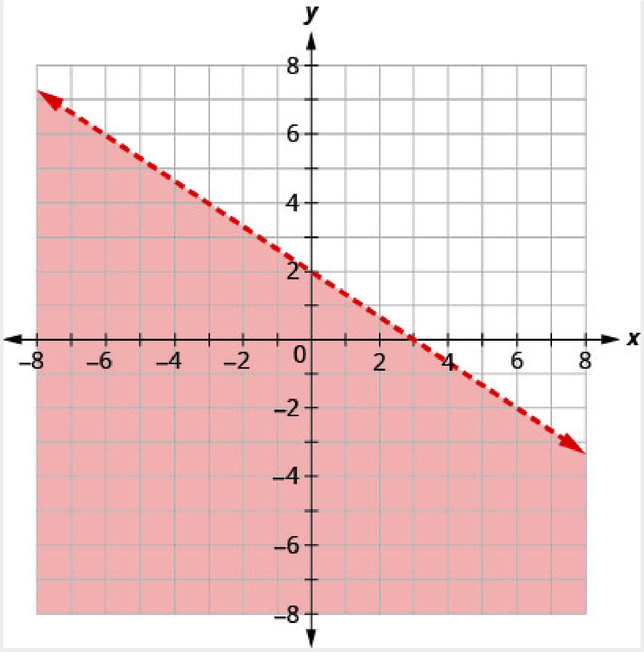
Solution:

$y \leq \frac{1}{2}x - 4$

Example:
Exercise:

Problem:

The boundary line shown in this graph is $2x + 3y = 6$. Write the inequality shown by the graph.



Solution:

The line $2x + 3y = 6$ is the boundary line. On one side of the line are the points with $2x + 3y > 6$ and on the other side of the line are the points with $2x + 3y < 6$.

Let’s test the point $(0, 0)$ and see which inequality describes its side of the boundary line.

At $(0, 0)$, which inequality is true: $2x + 3y > 6$ or $2x + 3y < 6$?

Equation:

$$\begin{aligned} 2x + 3y &> 6 \\ 2(0) + 3(0) &\overset{?}{>} 6 \\ 0 &> 6 \text{ False} \end{aligned} \qquad \begin{aligned} 2x + 3y &< 6 \\ 2(0) + 3(0) &\overset{?}{<} 6 \\ 0 &< 6 \text{ True} \end{aligned}$$

So the side with $(0, 0)$ is the side where $2x + 3y < 6$.

(You may want to pick a point on the other side of the boundary line and check that $2x + 3y > 6$.)

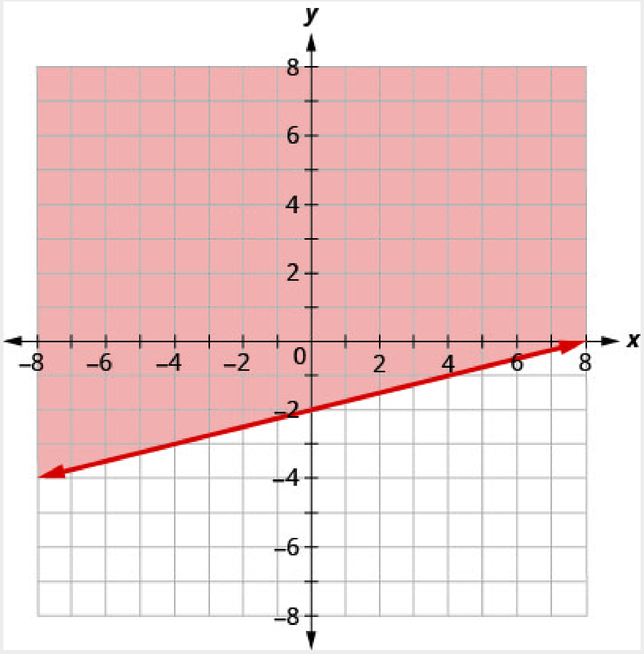
Since the boundary line is graphed as a dashed line, the inequality does not include an equal sign.

The shaded region shows the solution to the inequality $2x + 3y < 6$.

Note:
Exercise:

Problem:

Write the inequality shown by the shaded region in the graph with the boundary line $x - 4y = 8$.



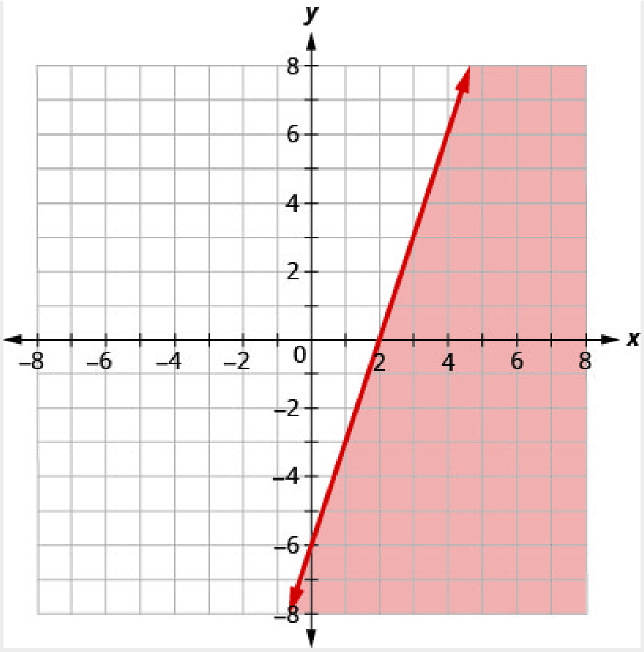
Solution:

$x - 4y \leq 8$

Note:
Exercise:

Problem:

Write the inequality shown by the shaded region in the graph with the boundary line $3x - y = 6$.



Solution:

$3x - y \geq 6$

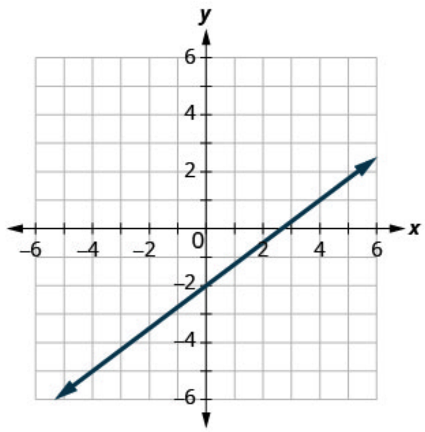
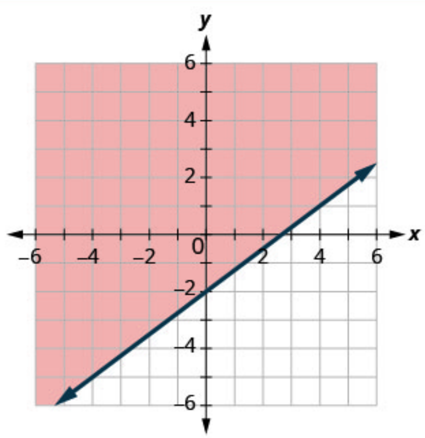
Graph Linear Inequalities in Two Variables

Now that we know what the graph of a linear inequality looks like and how it relates to a boundary equation we can use this knowledge to graph a given linear inequality.

Example:
How to Graph a Linear Equation in Two Variables
Exercise:

Problem: Graph the linear inequality $y \geq \frac{3}{4}x - 2$.

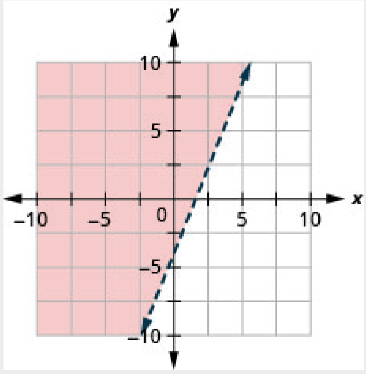
Solution:

Step 1. Identify and graph the boundary line. • If the inequality is $\le$ or $\ge$, the boundary line is solid. • If the inequality is $<$ or $>$, the boundary line is dashed.	Replace the inequality sign with an equal sign to find the boundary line. Graph the boundary line $y = \frac{3}{4}x - 2$. The inequality sign is $\ge$, so we draw a solid line.	
Step 2. Test a point that is not on the boundary line. Is it a solution of the inequality?	We'll test (0, 0). Is it a solution of the inequality?	At (0, 0), is $y \ge \frac{3}{4}x - 2$? $0 \overset{?}{\ge} \frac{3}{4}(0) - 2$ $0 \ge -2$ So, (0, 0) is a solution.
Step 3. Shade in one side of the boundary line. • If the test point is a solution, shade in the side that includes the point. • If the test point is not a solution, shade in the opposite side.	The test point (0, 0) is a solution to $y \ge \frac{3}{4}x - 2$. So we shade in that side.	 All points in the shaded region and on the boundary line represent the solutions to $y \ge \frac{3}{4}x - 2$.

Note:
Exercise:

Problem: Graph the linear inequality $y > \frac{5}{2}x - 4$.

Solution:

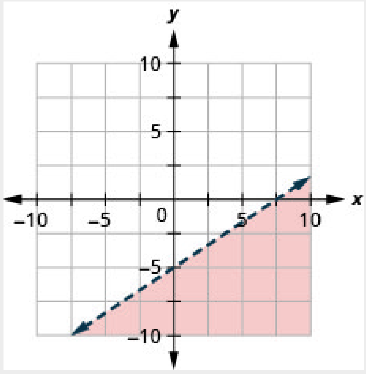


All points in the shaded region and on the boundary line, represent the solutions to $y > \frac{5}{2}x - 4$.

Note:
Exercise:

Problem: Graph the linear inequality $y < \frac{2}{3}x - 5$.

Solution:



All points in the shaded region, but not those on the boundary line, represent the solutions to $y < \frac{2}{3}x - 5$.

The steps we take to graph a linear inequality are summarized here.

Note:
Graph a linear inequality in two variables.

Identify and graph the boundary line.

- If the inequality is $\leq$ or $\geq$, the boundary line is solid.
- If the inequality is $<$ or $>$, the boundary line is dashed.

Test a point that is not on the boundary line. Is it a solution of the inequality?

Shade in one side of the boundary line.

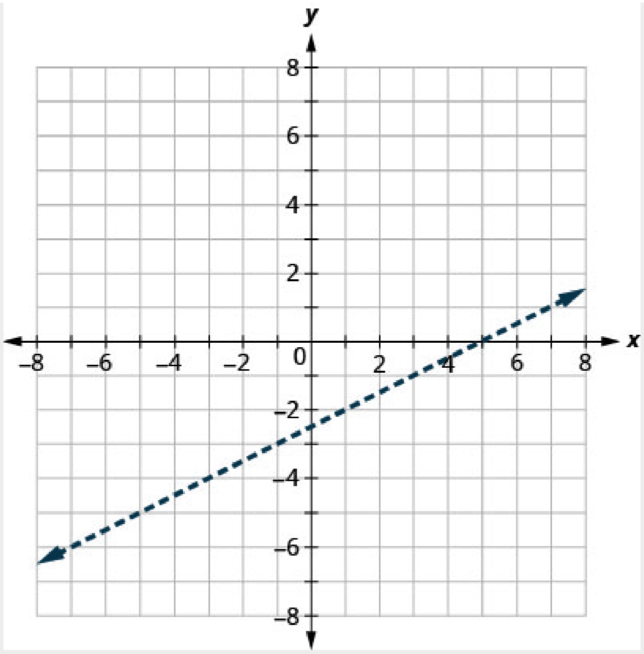
- If the test point is a solution, shade in the side that includes the point.
- If the test point is not a solution, shade in the opposite side.

Example:
Exercise:

Problem: Graph the linear inequality $x - 2y < 5$.

Solution:

First, we graph the boundary line $x - 2y = 5$. The inequality is $<$ so we draw a dashed line.

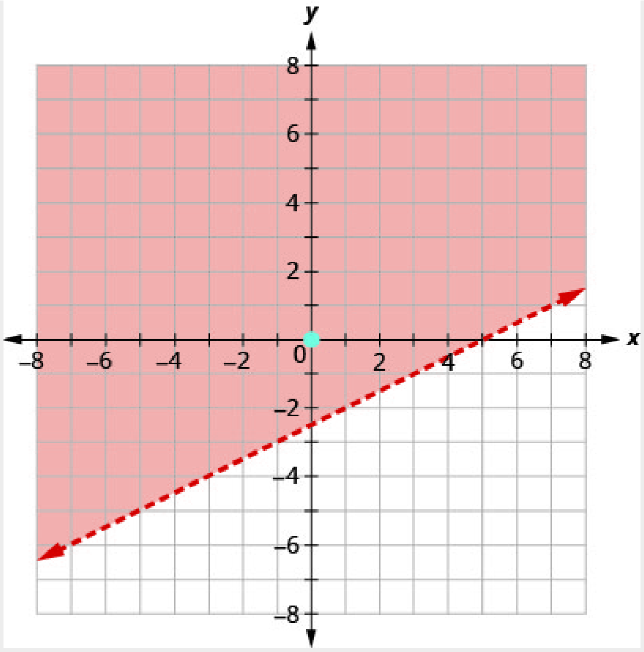


Then, we test a point. We'll use $(0, 0)$ again because it is easy to evaluate and it is not on the boundary line.

Is $(0, 0)$ a solution of $x - 2y < 5$?

$$0 - 2(0) \overset{?}{<} 5$$
$$0 - 0 \overset{?}{<} 5$$
$$0 < 5$$

The point $(0, 0)$ is a solution of $x - 2y < 5$, so we shade in that side of the boundary line.

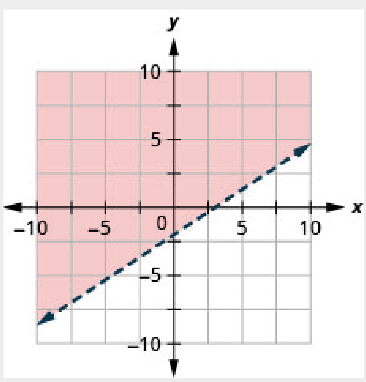


All points in the shaded region, but not those on the boundary line, represent the solutions to $x - 2y < 5$.

Note:
Exercise:

Problem: Graph the linear inequality: $2x - 3y < 6$.

Solution:

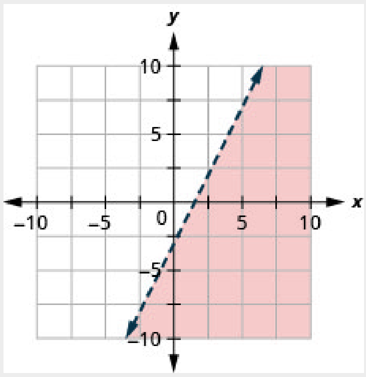


All points in the shaded region, but not those on the boundary line, represent the solutions to $2x - 3y < 6$.

Note:
Exercise:

Problem: Graph the linear inequality: $2x - y > 3$.

Solution:



All points in the shaded region, but not those on the boundary line, represent the solutions to $2x - y > 3$.

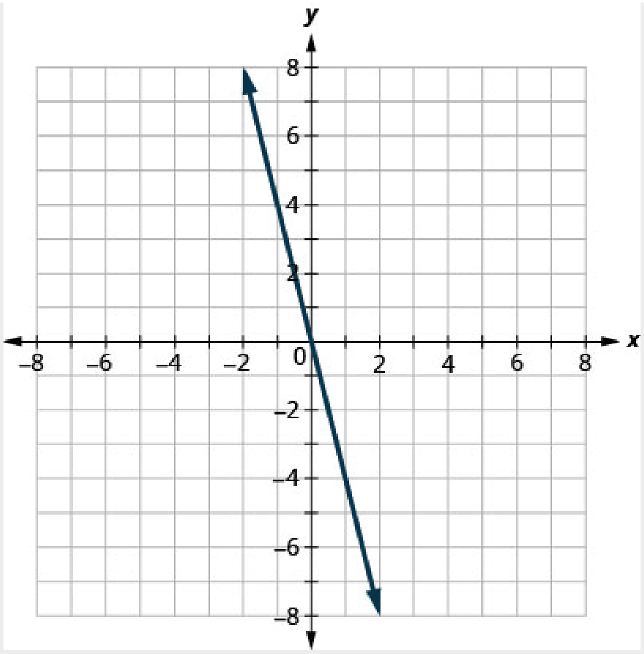
What if the boundary line goes through the origin? Then, we won't be able to use $(0, 0)$ as a test point. No problem—we'll just choose some other point that is not on the boundary line.

Example:
Exercise:

Problem: Graph the linear inequality: $y \leq -4x$.

Solution:

First, we graph the boundary line $y = -4x$. It is in slope–intercept form, with $m = -4$ and $b = 0$. The inequality is $\leq$ so we draw a solid line.



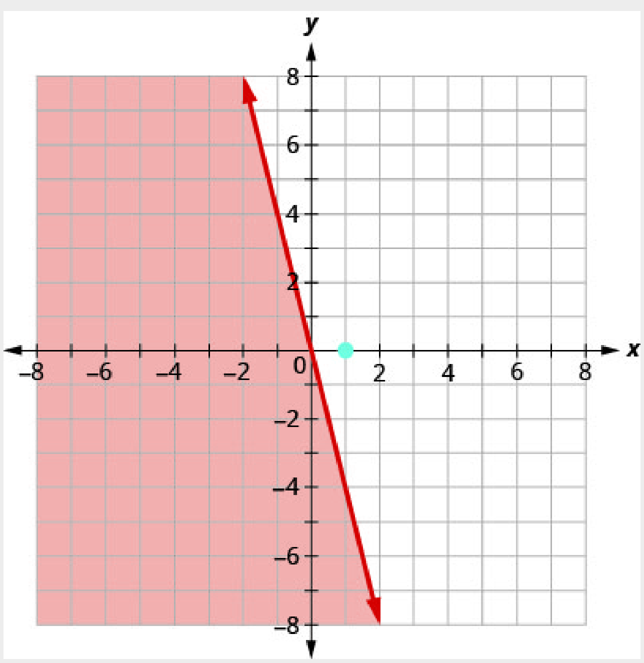
Now we need a test point. We can see that the point $(1, 0)$ is not on the boundary line.

Is $(1, 0)$ a solution of $y \leq -4x$?

$$0 \overset{?}{\leq} -4(1)$$

$$0 \nleq -4$$

The point $(1, 0)$ is not a solution to $y \leq\ -4x$, so we shade in the opposite side of the boundary line.

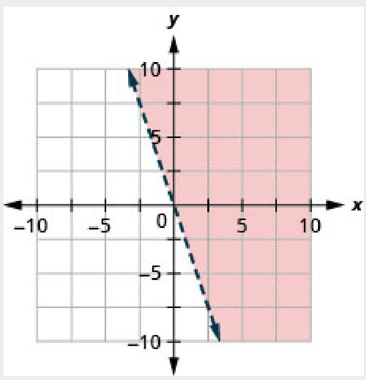


All points in the shaded region and on the boundary line represent the solutions to $y \leq\ -4x$.

Note:
Exercise:

Problem: Graph the linear inequality: $y > -3x$.

Solution:

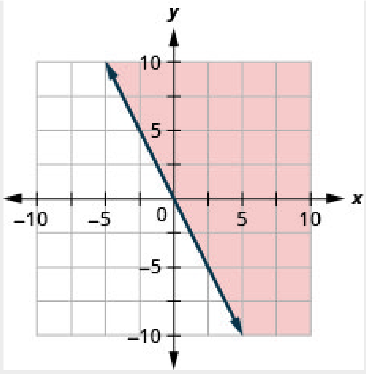


All points in the shaded region, but not those on the boundary line, represent the solutions to $y > -3x$.

Note:
Exercise:

Problem: Graph the linear inequality: $y \geq -2x$.

Solution:



All points in the shaded region and on the boundary line, represent the solutions to $y \geq -2x$.

Some linear inequalities have only one variable. They may have an *x* but no *y*, or a *y* but no *x*. In these cases, the boundary line will be either a vertical or a horizontal line.

Recall that:
Equation:

$$\begin{array}{ll} x = a & \text{vertical line} \\ y = b & \text{horizontal line} \end{array}$$

Example:
Exercise:

Problem: Graph the linear inequality: $y > 3$.

Solution:

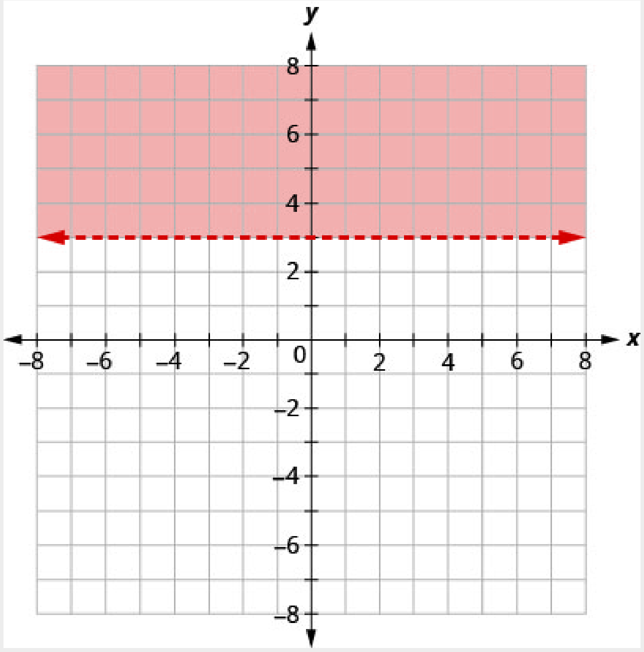
First, we graph the boundary line $y = 3$. It is a horizontal line. The inequality is $>$ so we draw a dashed line.

We test the point $(0, 0)$.
Equation:

$$\begin{array}{l} y > 3 \\ 0 \not> 3 \end{array}$$

So, $(0, 0)$ is not a solution to $y > 3$.

So we shade the side that does not include $(0, 0)$ as shown in this graph.

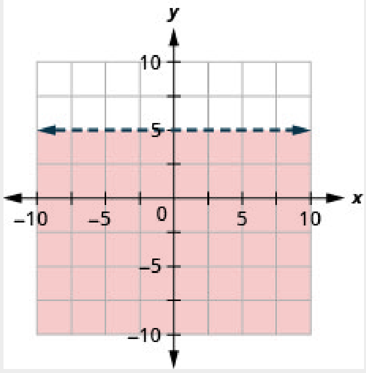


All points in the shaded region, but not those on the boundary line, represent the solutions to $y > 3$.

Note:
Exercise:

Problem: Graph the linear inequality: $y < 5$.

Solution:

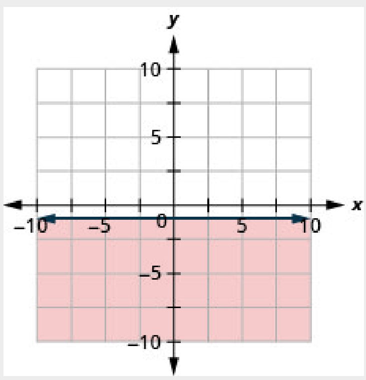


All points in the shaded region, but not those on the boundary line, represent the solutions to $y < 5$.

Note:
Exercise:

Problem: Graph the linear inequality: $y \leq -1$.

Solution:



All points in the shaded region and on the boundary line represent the solutions to $y \leq -1$.

Solve Applications using Linear Inequalities in Two Variables

Many fields use linear inequalities to model a problem. While our examples may be about simple situations, they give us an opportunity to build our skills and to get a feel for how thay might be used.

Example:
Exercise:

Problem:

Hilaria works two part time jobs in order to earn enough money to meet her obligations of at least $240 a week. Her job in food service pays $10 an hour and her tutoring job on campus pays $15 an hour. How many hours does Hilaria need to work at each job to earn at least $240?

ⓐ Let x be the number of hours she works at the job in food service and let y be the number of hours she works tutoring. Write an inequality that would model this situation.

ⓑ Graph the inequality.

ⓒ Find three ordered pairs (x, y) that would be solutions to the inequality. Then, explain what that means for Hilaria.

Solution:

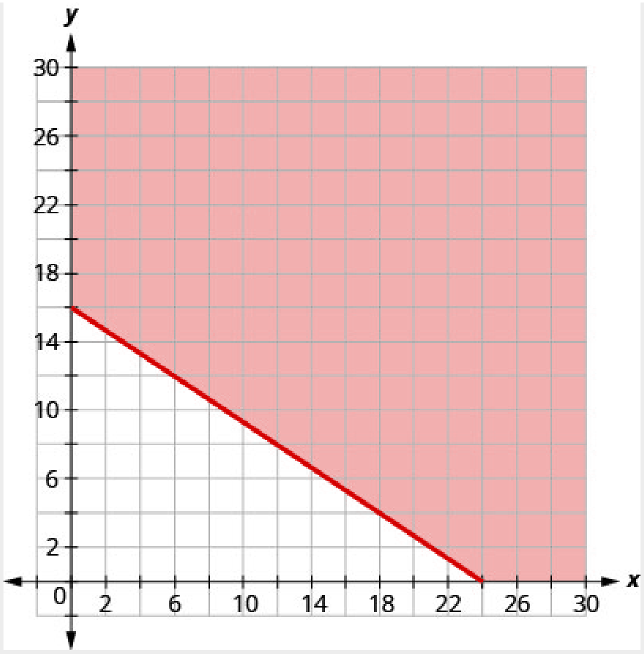
ⓐ We let x be the number of hours she works at the job in food service and let y be the number of hours she works tutoring.

She earns $10 per hour at the job in food service and $15 an hour tutoring. At each job, the number of hours multiplied by the hourly wage will gives the amount earned at that job.

$$\underbrace{\text{Amount earned at the food service job}}_{10x}\ \underset{+}{\text{plus}}\ \underbrace{\text{the amount earned tutoring}}_{15y}\ \underbrace{\text{is at least \$240}}_{\geq 240}$$

ⓑ To graph the inequality, we put it in slope–intercept form.
Equation:

$$\begin{aligned} 10x + 15y &\geq 240 \\ 15y &\geq -10x + 240 \\ y &\geq -\frac{2}{3}x + 16 \end{aligned}$$



ⓒ From the graph, we see that the ordered pairs $(15, 10)$, $(0, 16)$, $(24, 0)$ represent three of infinitely many solutions. Check the values in the inequality.

(15, 10)	(0, 16)	(24, 0)
$10x + 15y \geq 240$	$10x + 15y \geq 240$	$10x + 15y \geq 240$
$10(15) + 15(10) \overset{?}{\geq} 240$	$10(0) + 15(16) \overset{?}{\geq} 240$	$10(24) + 15(0) \overset{?}{\geq} 240$
$300 \geq 240$ True	$240 \geq 240$ True	$240 \geq 240$ True

For Hilaria, it means that to earn at least \$240, she can work 15 hours tutoring and 10 hours at her fast-food job, earn all her money tutoring for 16 hours, or earn all her money while working 24 hours at the job in food service.

Note:
Exercise:

Problem:

Hugh works two part time jobs. One at a grocery store that pays \$10 an hour and the other is babysitting for \$13 hour. Between the two jobs, Hugh wants to earn at least \$260 a week. How many hours does Hugh need to work at each job to earn at least \$260?

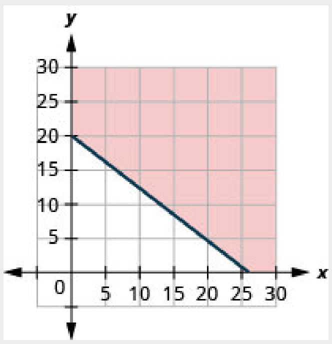
ⓐ Let *x* be the number of hours he works at the grocery store and let *y* be the number of hours he works babysitting. Write an inequality that would model this situation.

ⓑ Graph the inequality.

ⓒ Find three ordered pairs (x, y) that would be solutions to the inequality. Then, explain what that means for Hugh.

Solution:

ⓐ $10x + 13y \geq 260$
ⓑ



ⓒ Answers will vary.

Note:
Exercise:

Problem:

Veronica works two part time jobs in order to earn enough money to meet her obligations of at least $280 a week. Her job at the day spa pays $10 an hour and her administrative assistant job on campus pays $17.50 an hour. How many hours does Veronica need to work at each job to earn at least $280?

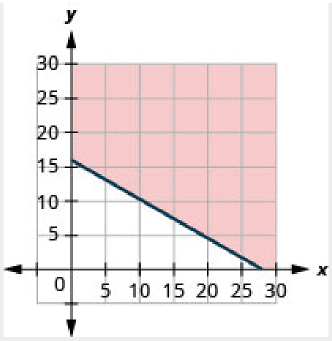
ⓐ Let x be the number of hours she works at the day spa and let y be the number of hours she works as administrative assistant. Write an inequality that would model this situation.

ⓑ Graph the inequality.

ⓒ Find three ordered pairs (x, y) that would be solutions to the inequality. Then, explain what that means for Veronica

Solution:

ⓐ $10x + 17.5y \geq 280$
ⓑ



ⓒ Answers will vary.

Note:
Access this online resource for additional instruction and practice with graphing linear inequalities in two variables.

- Graphing Linear Inequalities in Two Variables

Key Concepts

- **How to graph a linear inequality in two variables.**

Identify and graph the boundary line.	If the inequality is $\leq$ or $\geq$, the boundary line is solid.	If the inequality is $<$ or $>$, the boundary line is dashed.
Test a point that is not on the boundary line. Is it a solution of the inequality?		
Shade in one side of the boundary line.	If the test point is a solution, shade in the side that includes the point.	If the test point is not a solution, shade in the opposite side.

Practice Makes Perfect

Verify Solutions to an Inequality in Two Variables

In the following exercises, determine whether each ordered pair is a solution to the given inequality.
Exercise:

Problem: Determine whether each ordered pair is a solution to the inequality $y > x - 1$:

ⓐ $(0, 1)$
ⓑ $(-4, -1)$
ⓒ $(4, 2)$
ⓓ $(3, 0)$
ⓔ $(-2, -3)$

Solution:

ⓐ yes ⓑ yes ⓒ no ⓓ no ⓔ no

Exercise:

Problem: Determine whether each ordered pair is a solution to the inequality $y > x - 3$:

ⓐ $(0, 0)$
ⓑ $(2, 1)$
ⓒ $(-1, -5)$
ⓓ $(-6, -3)$
ⓔ $(1, 0)$

Exercise:

Problem: Determine whether each ordered pair is a solution to the inequality $y < 3x + 2$:

ⓐ $(0, 3)$
ⓑ $(-3, -2)$
ⓒ $(-2, 0)$
ⓓ $(0, 0)$
ⓔ $(-1, 4)$

Solution:

ⓐ no ⓑ no ⓒ yes ⓓ yes ⓔ no

Exercise:

Problem:

Determine whether each ordered pair is a solution to the inequality $y < -2x + 5$:

ⓐ $(-3, 0)$
ⓑ $(1, 6)$
ⓒ $(-6, -2)$
ⓓ $(0, 1)$
ⓔ $(5, -4)$

Exercise:

Problem:

Determine whether each ordered pair is a solution to the inequality $3x - 4y > 4$:

ⓐ $(5, 1)$
ⓑ $(-2, 6)$
ⓒ $(3, 2)$
ⓓ $(10, -5)$
ⓔ $(0, 0)$

Solution:

ⓐ yes ⓑ no ⓒ no ⓓ no ⓔ no

Exercise:

Problem:

Determine whether each ordered pair is a solution to the inequality $2x + 3y > 2$:

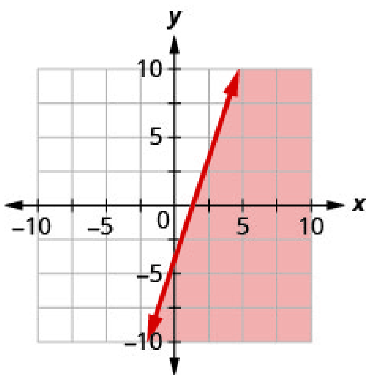
ⓐ $(1, 1)$
ⓑ $(4, -3)$
ⓒ $(0, 0)$
ⓓ $(-8, 12)$
ⓔ $(3, 0)$

Recognize the Relation Between the Solutions of an Inequality and its Graph

In the following exercises, write the inequality shown by the shaded region.

Exercise:

Problem: Write the inequality shown by the graph with the boundary line $y = 3x - 4$.

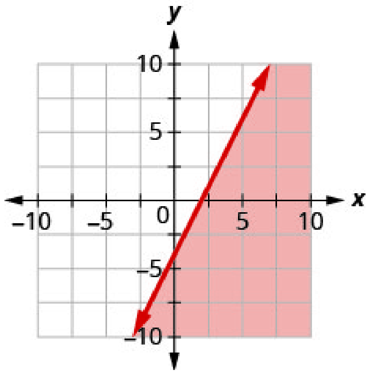


Solution:

$y \leq 3x - 4$

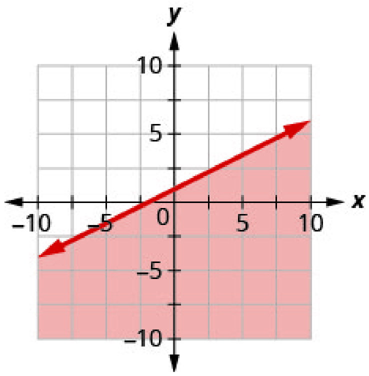
Exercise:

Problem: Write the inequality shown by the graph with the boundary line $y = 2x - 4$.



Exercise:

Problem: Write the inequality shown by the graph with the boundary line $y = -\frac{1}{2}x + 1$.

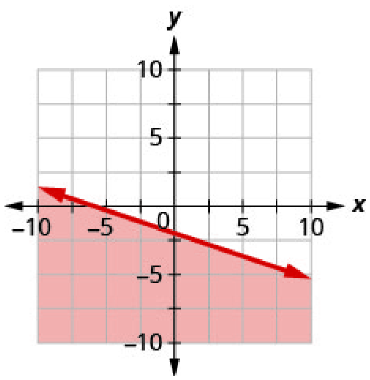


Solution:

$y \leq -\frac{1}{2}x + 1$

Exercise:

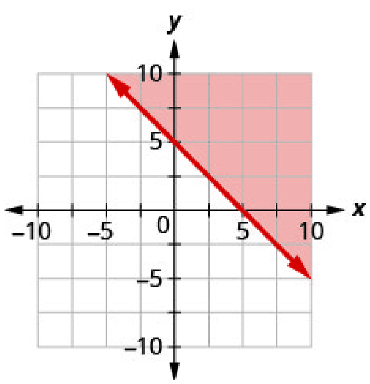
Problem: Write the inequality shown by the graph with the boundary line $y = -\frac{1}{3}x - 2$.



Exercise:

Problem:

Write the inequality shown by the shaded region in the graph with the boundary line $x + y = 5$.



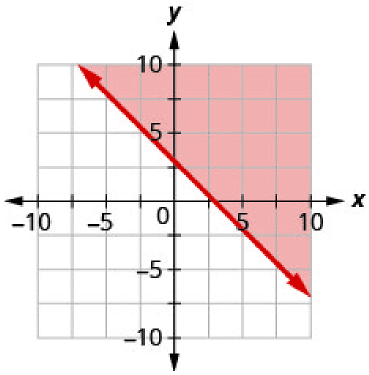
Solution:

$x + y \geq 5$

Exercise:

Problem:

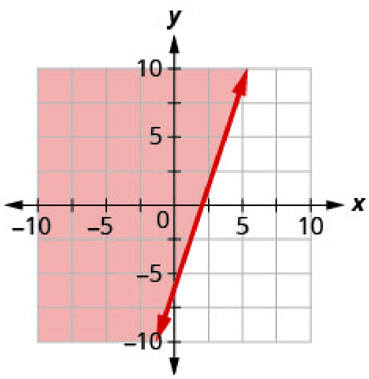
Write the inequality shown by the shaded region in the graph with the boundary line $x + y = 3$.



Exercise:

Problem:

Write the inequality shown by the shaded region in the graph with the boundary line $3x - y = 6$.



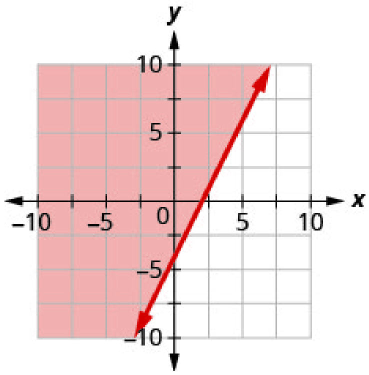
Solution:

$3x - y \leq 6$

Exercise:

Problem:

Write the inequality shown by the shaded region in the graph with the boundary line $2x - y = 4$.



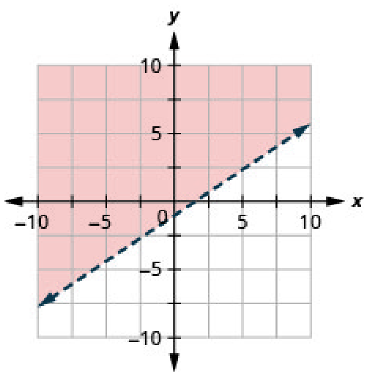
Graph Linear Inequalities in Two Variables

In the following exercises, graph each linear inequality.

Exercise:

Problem: Graph the linear inequality: $y > \frac{2}{3}x - 1$.

Solution:



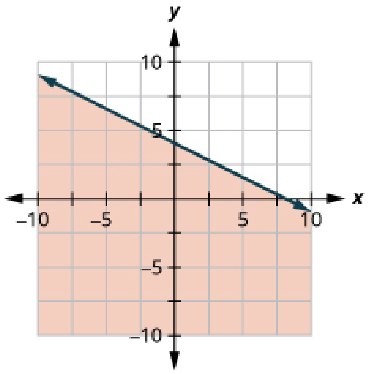
Exercise:

Problem: Graph the linear inequality: $y < \frac{3}{5}x + 2$.

Exercise:

Problem: Graph the linear inequality: $y \leq -\frac{1}{2}x + 4$.

Solution:



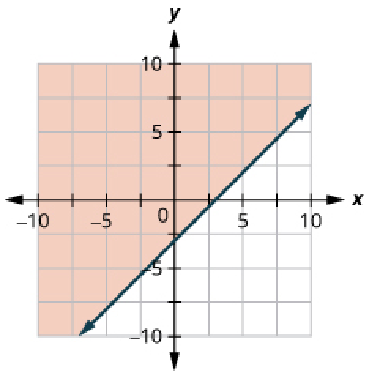
Exercise:

Problem: Graph the linear inequality: $y \geq -\frac{1}{3}x - 2$.

Exercise:

Problem: Graph the linear inequality: $x - y \leq 3$.

Solution:



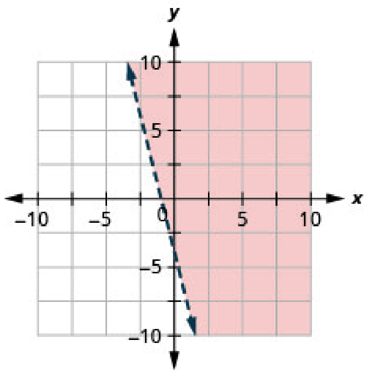
Exercise:

Problem: Graph the linear inequality: $x - y \geq -2$.

Exercise:

Problem: Graph the linear inequality: $4x + y > -4$.

Solution:



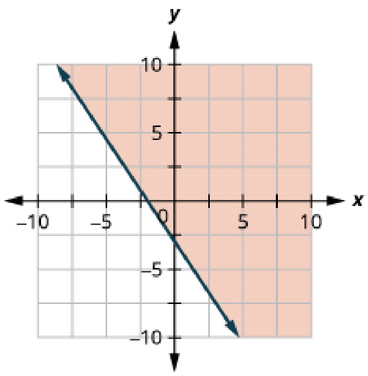
Exercise:

Problem: Graph the linear inequality: $x + 5y < -5$.

Exercise:

Problem: Graph the linear inequality: $3x + 2y \geq -6$.

Solution:



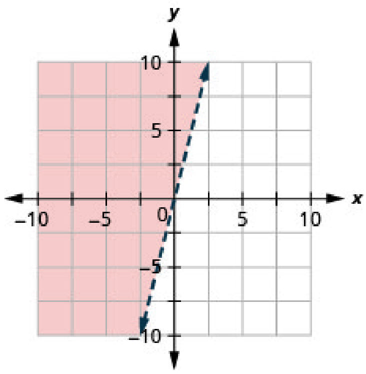
Exercise:

Problem: Graph the linear inequality: $4x + 2y \geq -8$.

Exercise:

Problem: Graph the linear inequality: $y > 4x$.

Solution:



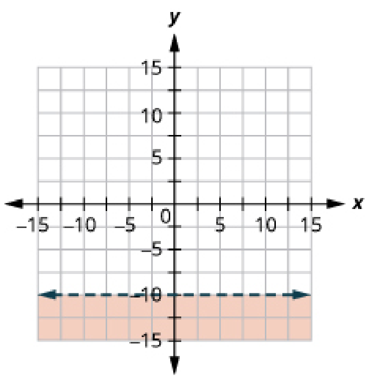
Exercise:

Problem: Graph the linear inequality: $y \leq -3x$.

Exercise:

Problem: Graph the linear inequality: $y < -10$.

Solution:



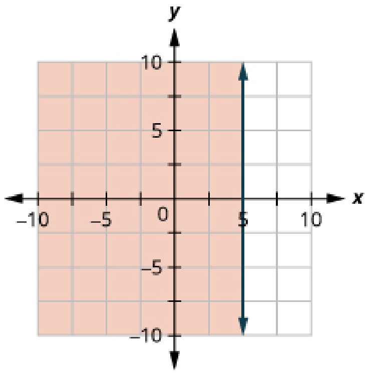
Exercise:

Problem: Graph the linear inequality: $y \geq 2$.

Exercise:

Problem: Graph the linear inequality: $x \leq 5$.

Solution:



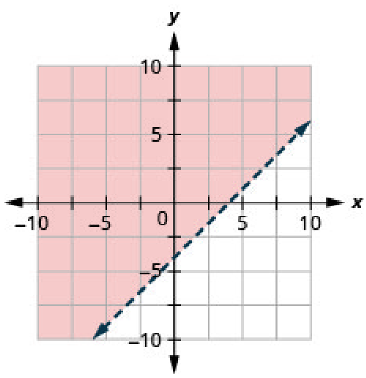
Exercise:

Problem: Graph the linear inequality: $x \geq 0$.

Exercise:

Problem: Graph the linear inequality: $x - y < 4$.

Solution:



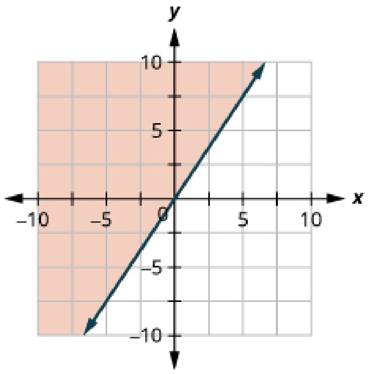
Exercise:

Problem: Graph the linear inequality: $x - y < -3$.

Exercise:

Problem: Graph the linear inequality: $y \geq \frac{3}{2}x$.

Solution:



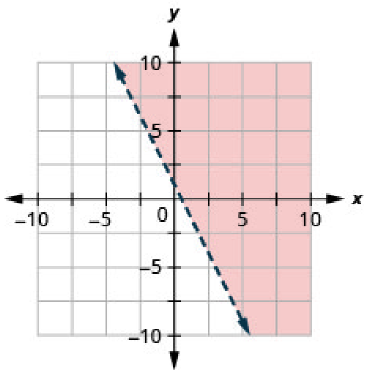
Exercise:

Problem: Graph the linear inequality: $y \leq \frac{5}{4}x$.

Exercise:

Problem: Graph the linear inequality: $y > -2x + 1$.

Solution:



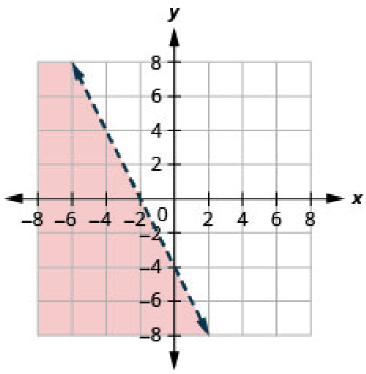
Exercise:

Problem: Graph the linear inequality: $y < -3x - 4$.

Exercise:

Problem: Graph the linear inequality: $2x + y \geq -4$.

Solution:



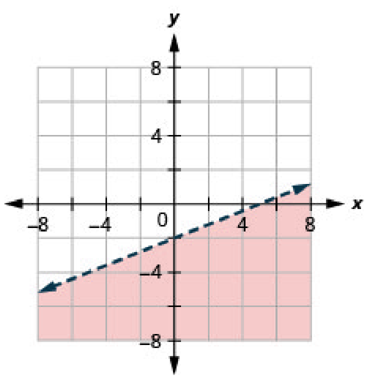
Exercise:

Problem: Graph the linear inequality: $x + 2y \leq -2$.

Exercise:

Problem: Graph the linear inequality: $2x - 5y > 10$.

Solution:



Exercise:

Problem: Graph the linear inequality: $4x - 3y > 12$.

Solve Applications using Linear Inequalities in Two Variables

Exercise:

Problem:

Harrison works two part time jobs. One at a gas station that pays $11 an hour and the other is IT troubleshooting for $16.50 an hour. Between the two jobs, Harrison wants to earn at least $330 a week. How many hours does Harrison need to work at each job to earn at least $330?

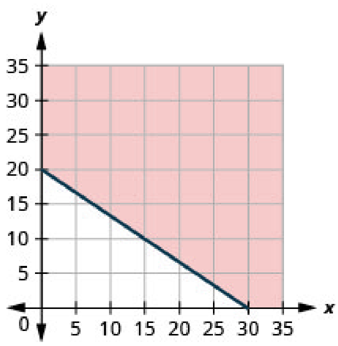
ⓐ Let *x* be the number of hours he works at the gas station and let *y* be the number of (hours he works troubleshooting. Write an inequality that would model this situation.

ⓑ Graph the inequality.

ⓒ Find three ordered pairs (x, y) that would be solutions to the inequality. Then, explain what that means for Harrison.

Solution:

ⓐ $11x + 16.5y \geq 330$
ⓑ



ⓒ Answers will vary.

Exercise:

Problem:

Elena needs to earn at least $450 a week during her summer break to pay for college. She works two jobs. One as a swimming instructor that pays $9 an hour and the other as an intern in a genetics lab for $22.50 per hour. How many hours does Elena need to work at each job to earn at least $450 per week?

ⓐ Let *x* be the number of hours she works teaching swimming and let *y* be the number of hours she works as an intern. Write an inequality that would model this situation.

ⓑ Graph the inequality.

ⓒ Find three ordered pairs (x, y) that would be solutions to the inequality. Then, explain what that means for Elena.

Exercise:

Problem:

The doctor tells Laura she needs to exercise enough to burn 500 calories each day. She prefers to either run or bike and burns 15 calories per minute while running and 10 calories a minute while biking.

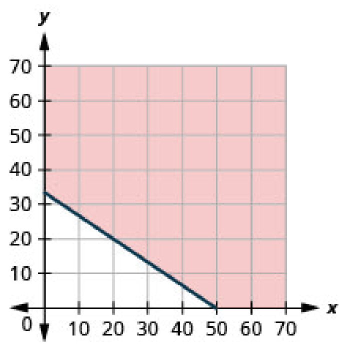
ⓐ If x is the number of minutes that Laura runs and y is the number minutes she bikes, find the inequality that models the situation.

ⓑ Graph the inequality.

ⓒ List three solutions to the inequality. What options do the solutions provide Laura?

Solution:

ⓐ $15x+10y\ge 500$
ⓑ



ⓒ Answers will vary.

Exercise:

Problem:

Armando’s workouts consist of kickboxing and swimming. While kickboxing, he burns 10 calories per minute and he burns 7 calories a minute while swimming. He wants to burn 600 calories each day.

ⓐ If x is the number of minutes that Armando will kickbox and y is the number minutes he will swim, find the inequality that will help Armando create a workout for today.

ⓑ Graph the inequality.

ⓒ List three solutions to the inequality. What options do the solutions provide Armando?

Writing Exercises

Exercise:

Problem:

Lester thinks that the solution of any inequality with a $>$ sign is the region above the line and the solution of any inequality with a $<$ sign is the region below the line. Is Lester correct? Explain why or why not.

Solution:

Answers will vary.

Exercise:

Problem:

Explain why, in some graphs of linear inequalities, the boundary line is solid but in other graphs it is dashed.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
verify solutions to an inequality in two variables.			
recognize the relation between the solutions of an inequality and its graph.			
graph linear inequalities.			

ⓑ On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Glossary

boundary line
: The line with equation $Ax+By=C$ is the boundary line that separates the region where $Ax+By>C$ from the region where $Ax+By<C$.

linear inequality
: A linear inequality is an inequality that can be written in one of the following forms: $Ax+By>C,\ \ Ax+By\ge C,\ \ Ax+By<C$, or $Ax+By\le C$, where A and B are not both zero.

solution to a linear inequality

An ordered pair (x, y) is a solution to a linear inequality if the inequality is true when we substitute the values of x and y.

Relations and Functions: ASE

By the end of this section, you will be able to:

- Find the domain and range of a relation
- Determine if a relation is a function
- Find the value of a function
- Determine a rule for a visual pattern

Related Ideas

Part of what makes mathematics so powerful and interesting is that many of the ideas are closely related and build on each other. Sometimes different people at different times in different locations think of similar ideas that are related. Often a newer, more general idea helps us understand older ideas more completely or in a different context. This is the case with relations and functions.

When you go through this section you should think that you have seen some of these ideas before. For example, a function is similar to a formula and a scatterplot is a relation.

Find the Domain and Range of a Relation

As we go about our daily lives, we have many data items or quantities that are paired to our names. Our social security number, student ID number, email address, phone number and our birthday are matched to our name. There is a relationship between our name and each of those items.

When your professor gets her class roster, the names of all the students in the class are listed in one column and then the student ID number is likely to be in the next column. If we think of the correspondence as a set of ordered pairs, where the first element is a student name and the second element is that student's ID number, we call this a **relation**.

Equation:

$$(\text{Student name, Student ID }\#)$$

The set of all the names of the students in the class is called the **domain** of the relation and the set of all student ID numbers paired with these students is the range of the relation.

There are many similar situations where one variable is paired or matched with another. The set of ordered pairs that records this matching is a relation.

Note:

Relation

A **relation** is any set of ordered pairs, (x, y). All the *x*-values in the ordered pairs together make up the **domain**. All the *y*-values in the ordered pairs together make up the **range**.

Set Notation

One way to indicate a set is to list all of its elements or members of the set. This is done by listing the elements one after the other separated by commas and the entire list inside curly brackets. It does not matter what order the elements of a set are listed in.

Example: Show the set of even whole numbers less than 10.
Answer: {2, 4, 6, 8}

Example: Show the set of order pairs where the x value is 0 or 1 and the y value is 0 or 1.
Answer: {(0,0), (0,1), (1,0), (1,1)}

Example:
Exercise:

Problem: For the relation $\{(1,1),(2,4),(3,9),(4,16),(5,25)\}$:

ⓐ Find the domain of the relation.

ⓑ Find the range of the relation.

Solution:

$$\{(1,1),(2,4),(3,9),(4,16),(5,25)\}$$

ⓐ The domain is the set of all *x*-values of the relation. $\{1,2,3,4,5\}$

ⓑ The range is the set of all *y*-values of the relation. $\{1,4,9,16,25\}$

Note:
Exercise:

Problem: For the relation $\{(1,1),(2,8),(3,27),(4,64),(5,125)\}$:

ⓐ Find the domain of the relation.

ⓑ Find the range of the relation.

Solution:

ⓐ $\{1,2,3,4,5\}$
ⓑ $\{1,8,27,64,125\}$

Note:
Exercise:

Problem: For the relation $\{(1,3),(2,6),(3,9),(4,12),(5,15)\}$:

a Find the domain of the relation.

b Find the range of the relation.

Solution:

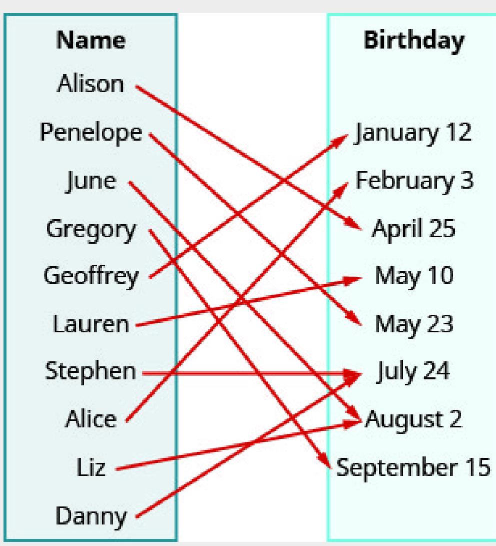
a $\{1,2,3,4,5\}$
b $\{3,6,9,12,15\}$

Note:
Mapping
A **mapping** is sometimes used to show a relation. The arrows show the pairing of the elements of the domain with the elements of the range.

Example:
Exercise:

Problem:

Use the **mapping** of the relation shown to a list the ordered pairs of the relation, b find the domain of the relation, and c find the range of the relation.



Solution:

ⓐ The arrow shows the matching of the person to their birthday. We create ordered pairs with the person's name as the *x*-value and their birthday as the *y*-value.

{(Alison, April 25), (Penelope, May 23), (June, August 2), (Gregory, September 15), (Geoffrey, January 12), (Lauren, May 10), (Stephen, July 24), (Alice, February 3), (Liz, August 2), (Danny, July 24)}

ⓑ The domain is the set of all *x*-values of the relation.

{Alison, Penelope, June, Gregory, Geoffrey, Lauren, Stephen, Alice, Liz, Danny}

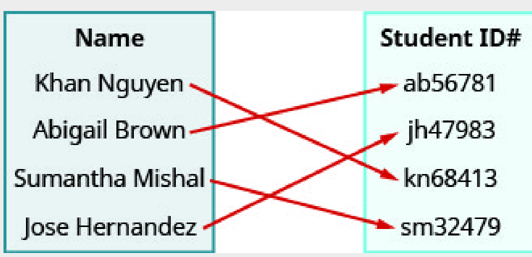
ⓒ The range is the set of all *y*-values of the relation.

{January 12, February 3, April 25, May 10, May 23, July 24, August 2, September 15}

Note:
Exercise:

Problem:

Use the mapping of the relation shown to ⓐ list the ordered pairs of the relation ⓑ find the domain of the relation ⓒ find the range of the relation.



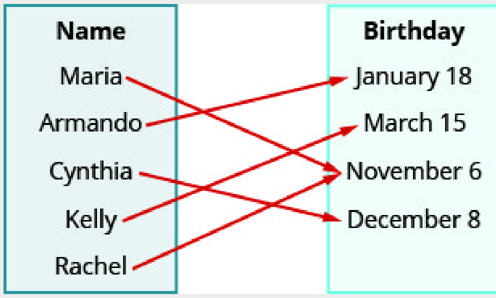
Solution:

ⓐ (Khanh Nguyen, kn68413), (Abigail Brown, ab56781), (Sumantha Mishal, sm32479), (Jose Hern and ez, jh47983) ⓑ {Khanh Nguyen, Abigail Brown, Sumantha Mishal, Jose Hern and ez} ⓒ {kn68413, ab56781, sm32479, jh47983}

Note:
Exercise:

Problem:

Use the mapping of the relation shown to ⓐ list the ordered pairs of the relation ⓑ find the domain of the relation ⓒ find the range of the relation.



Solution:

ⓐ (Maria, November 6), (Arm and o, January 18), (Cynthia, December 8), (Kelly, March 15), (Rachel, November 6) ⓑ {Maria, Arm and o, Cynthia, Kelly, Rachel} ⓒ {November 6, January 18, December 8, March 15}

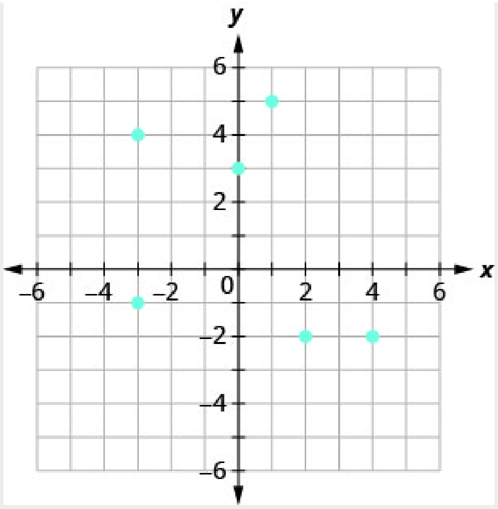
A graph is yet another way that a relation can be represented. The set of ordered pairs of all the points plotted is the relation. The set of all *x*-coordinates is the domain of the relation and the set of all *y*-coordinates is the range. Generally we write the numbers in ascending order for both the domain and range.

Notice that the graph of a relation is a scatterplot.

Example:
Exercise:

Problem:

Use the graph of the relation to ⓐ list the ordered pairs of the relation ⓑ find the domain of the relation ⓒ find the range of the relation.



Solution:

ⓐ The ordered pairs of the relation are:

$$\{(1,5),(-3,-1),(4,-2),(0,3),(2,-2),(-3,4)\}.$$

ⓑ The domain is the set of all *x*-values of the relation: $\{-3,0,1,2,4\}$.

Notice that while -3 repeats, it is only listed once.

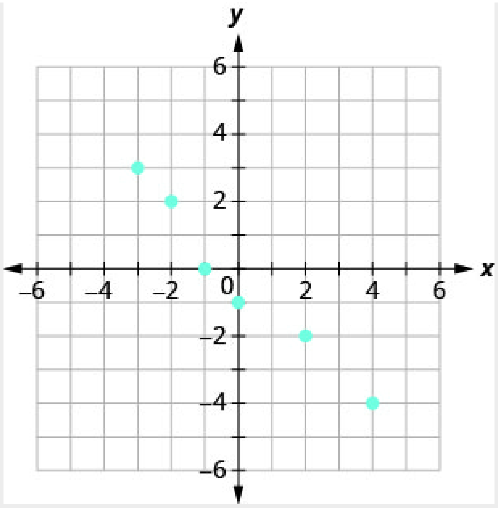
ⓒ The range is the set of all *y*-values of the relation: $\{-2,-1,3,4,5\}$.

Notice that while -2 repeats, it is only listed once.

Note:
Exercise:

Problem:

Use the graph of the relation to ⓐ list the ordered pairs of the relation ⓑ find the domain of the relation ⓒ find the range of the relation.



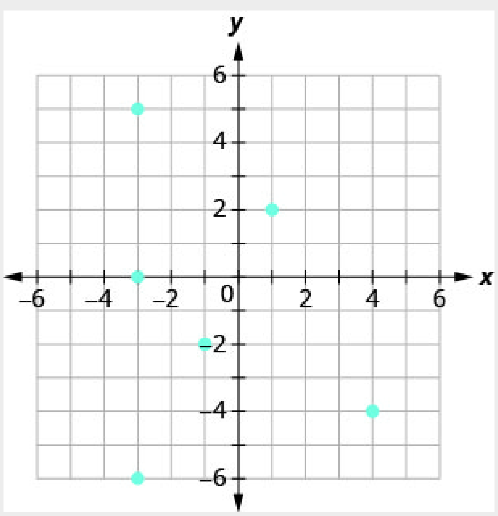
Solution:

ⓐ $(-3,3),(-2,2),(-1,0),$
$(0,-1),(2,-2),(4,-4)$
ⓑ $\{-3,-2,-1,0,2,4\}$
ⓒ$\{3,2,0,-1,-2,-4\}$

Note:
Exercise:

Problem:

Use the graph of the relation to ⓐ list the ordered pairs of the relation ⓑ find the domain of the relation ⓒ find the range of the relation.



Solution:

ⓐ $(-3,0),(-3,5),(-3,-6),$
$(-1,-2),(1,2),(4,-4)$
ⓑ $\{-3,-1,1,4\}$
ⓒ $\{-6,0,5,-2,2,-4\}$

Determine if a Relation is a Function

A special type of relation, called a **function**, occurs extensively in mathematics. A function is a relation that assigns to each element in its domain exactly one element in the range. For each ordered pair in the relation, each *x*-value is matched with only one *y*-value.

Note:
Function
A **function** is a relation that assigns to each element in its domain exactly one element in the range.

The birthday example from [link] helps us understand this definition. Every person has a birthday but no one has two birthdays. It is okay for two people to share a birthday. It is okay that Danny and Stephen share July 24th as their birthday and that June and Liz share August 2nd. Since each person has exactly one birthday, the relation in [link] is a function.

The relation shown by the graph in [link] includes the ordered pairs $(-3,-1)$ and $(-3,4)$. Is that okay in a function? No, as this is like one person having two different birthdays.

Example:
Exercise:

Problem:

Use the set of ordered pairs to (i) determine whether the relation is a function (ii) find the domain of the relation (iii) find the range of the relation.

ⓐ $\{(-3,27),(-2,8),(-1,1),(0,0),(1,1),(2,8),(3,27)\}$

ⓑ $\{(9,-3),(4,-2),(1,-1),(0,0),(1,1),(4,2),(9,3)\}$

Solution:

ⓐ $\{(-3,27),(-2,8),(-1,1),(0,0),(1,1),(2,8),(3,27)\}$

(i) Each *x*-value is matched with only one *y*-value. So this relation is a function.

(ii) The domain is the set of all *x*-values in the relation.
The domain is: $\{-3,-2,-1,0,1,2,3\}$.

(iii) The range is the set of all *y*-values in the relation. Notice we do not list range values twice.
The range is: $\{27,8,1,0\}$.

ⓑ $\{(9,-3),(4,-2),(1,-1),(0,0),(1,1),(4,2),(9,3)\}$

(i) The *x*-value 9 is matched with two *y*-values, both 3 and -3. So this relation is not a function.

(ii) The domain is the set of all *x*-values in the relation. Notice we do not list domain values twice.
The domain is: $\{0,1,2,4,9\}$.

(iii) The range is the set of all *y*-values in the relation.
The range is: $\{-3,-2,-1,0,1,2,3\}$.

Note:
Exercise:

Problem:

Use the set of ordered pairs to (i) determine whether the relation is a function (ii) find the domain of the relation (iii) find the range of the function.

ⓐ $\{(-3,-6),(-2,-4),(-1,-2),(0,0),(1,2),(2,4),(3,6)\}$

ⓑ $\{(8,-4),(4,-2),(2,-1),(0,0),(2,1),(4,2),(8,4)\}$

Solution:

ⓐ Yes; $\{-3,-2,-1,0,1,2,3\}$;
$\{-6,-4,-2,0,2,4,6\}$
ⓑ No; $\{0,2,4,8\}$;
$\{-4,-2,-1,0,1,2,4\}$

Note:

Exercise:

Problem:

Use the set of ordered pairs to (i) determine whether the relation is a function (ii) find the domain of the relation (iii) find the range of the relation.

ⓐ $\{(27,-3),(8,-2),(1,-1),(0,0),(1,1),(8,2),(27,3)\}$

ⓑ $\{(7,-3),(-5,-4),(8,-0),(0,0),(-6,4),(-2,2),(-1,3)\}$

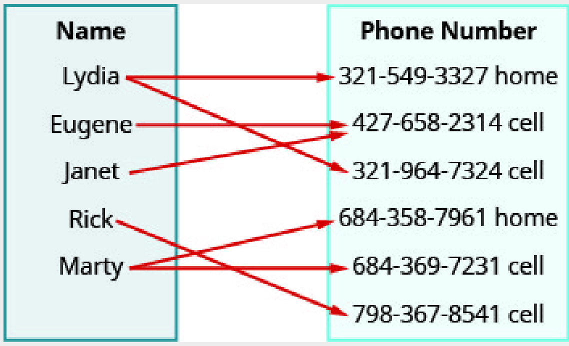
Solution:

ⓐ No; $\{0,1,8,27\}$;
$\{-3,-2,-1,0,2,2,3\}$
ⓑ Yes; $\{7,-5,8,0,-6,-2,-1\}$;
$\{-3,-4,0,4,2,3\}$

Example:
Exercise:

Problem:

Use the mapping to ⓐ determine whether the relation is a function ⓑ find the domain of the relation ⓒ find the range of the relation.



Solution:

ⓐ Both Lydia and Marty have two phone numbers. So each *x*-value is not matched with only one *y*-value. So this relation is not a function.

ⓑ The domain is the set of all *x*-values in the relation. The domain is: {Lydia, Eugene, Janet, Rick, Marty}

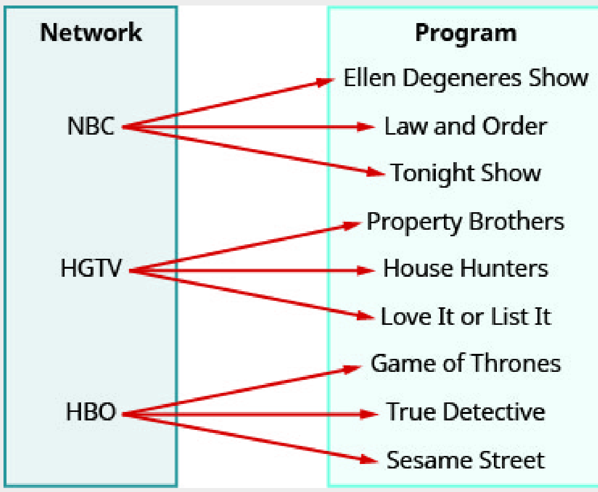
ⓒ The range is the set of all *y*-values in the relation. The range is:

{321-549-3327, 427-658-2314, 321-964-7324, 684-358-7961, 684-369-7231, 798-367-8541}

Note:
Exercise:

Problem:

Use the mapping to ⓐ determine whether the relation is a function ⓑ find the domain of the relation ⓒ find the range of the relation.



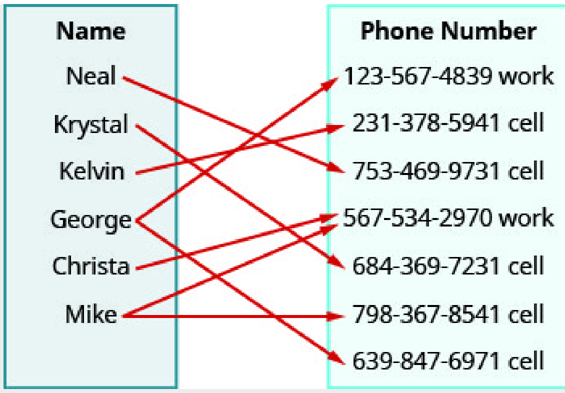
Solution:

ⓐ no ⓑ {NBC, HGTV, HBO} ⓒ {Ellen Degeneres Show, Law and Order, Tonight Show, Property Brothers, House Hunters, Love it or List it, Game of Thrones, True Detective, Sesame Street}

Note:
Exercise:

Problem:

Use the mapping to ⓐ determine whether the relation is a function ⓑ find the domain of the relation ⓒ find the range of the relation.



Solution:

ⓐ No ⓑ {Neal, Krystal, Kelvin, George, Christa, Mike} ⓒ {123-567-4839 work, 231-378-5941 cell, 743-469-9731 cell, 567-534-2970 work, 684-369-7231 cell, 798-367-8541 cell, 639-847-6971 cell}

In algebra, more often than not, functions will be represented by an equation. It is easiest to see if the equation is a function when it is solved for *y*. If each value of the domain *x* results in only one value of the range *y*, then the equation defines a function.

Example:
Exercise:

Problem: Determine whether each equation is a function.

ⓐ $2x + y = 7$ ⓑ $y = x^2 + 1$ ⓒ$x + y^2 = 3$

Solution:

ⓐ $2x + y = 7$

For each value of *x*, we multiply it by -2 and then add 7 to get the *y*-value

	$y = -2x + 7$

For example, if $x = 3$:	$y = -2 \cdot 3 + 7$
	$y = 1$

We have that when $x = 3$, then $y = 1$. It would work similarly for any value of x. Since each value of x, corresponds to only one value of y the equation defines a function.

ⓑ $y = x^2 + 1$

For each value of x, we square it and then add 1 to get the y-value.

	$y = x^2 + 1$
For example, if $x = 2$:	$y = 2^2 + 1$
	$y = 5$

We have that when $x = 2$, then $y = 5$. It would work similarly for any value of x. Since each value of x, corresponds to only one value of y the equation defines a function.

ⓒ

	$x + y^2 = 3$
Isolate the y term.	$y^2 = -x + 3$

Let’s substitute $x = 2$.	$y^2 = -2 + 3$
	$y^2 = 1$
This give us two values for *y*.	$y = 1$ $y = -1$

We have shown that when $x = 2$, then $y = 1$ and $y = -1$. It would work similarly for any value of *x* less than 3. Since at least one value of *x* does not corresponds to only one value of *y* the equation does not define a function.

Note:
Exercise:

Problem: Determine whether each equation is a function.

ⓐ $4x + y = -3$ ⓑ $x + y^2 = 1$ ⓒ $y - x^2 = 2$

Solution:

ⓐ yes ⓑ no ⓒ yes

Note:
Exercise:

Problem: Determine whether each equation is a function.

ⓐ $x + y^2 = 4$ ⓑ $y = x^2 - 7$ ⓒ $y = 5x - 4$

Solution:

ⓐ no ⓑ yes ⓒ yes

Find the Value of a Function

It is very convenient to name a function and most often we name it *f*, *g*, *h*, *F*, *G*, or *H*. In any function, for each *x*-value from the domain we get a corresponding *y*-value in the range. For the function *f*, we write this range value *y* as $f\left(x\right)$. This is called function notation and is read *f* of *x* or the value of *f* at *x*. In this case the parentheses does not indicate multiplication.

Note:
Function Notation
For the function $y=f(x)$
Equation:

$$\begin{array}{l} f \text{ is the name of the function} \\ x \text{ is the domain value} \\ f(x) \text{ is the range value } y \text{ corresponding to the value } x \end{array}$$

We read $f(x)$ as *f* of *x* or the value of *f* at *x*.

We call *x* the independent variable as it can be any value in the domain. We call *y* the dependent variable as its value depends on *x*.

Note:
Independent and Dependent Variables
For the function $y=f(x)$,
Equation:

$$\begin{array}{l} x \text{ is the independent variable as it can be any value in the domain} \\ y \text{ the dependent variable as its value depends on } x \end{array}$$

Much as when you first encountered the variable *x*, function notation may be rather unsettling. It seems strange because it is new. You will feel more comfortable with the notation as you use it.

Let's look at the equation $y=4x-5$. To find the value of *y* when $x=2$, we know to substitute $x=2$ into the equation and then simplify.

	$y = 4x - 5$
Let $x = 2$.	$y = 4 \cdot 2 - 5$
	$y = 3$

The value of the function at $x = 2$ is 3.

We do the same thing using function notation, the equation $y = 4x - 5$ can be written as $f(x) = 4x - 5$. To find the value when $x = 2$, we write:

	$f(x) = 4x - 5$
Let $x = 2$.	$f(2) = 4 \cdot 2 - 5$
	$f(2) = 3$

The value of the function at $x = 2$ is 3.

This process of finding the value of $f(x)$ for a given value of *x* is called *evaluating the function.*

Example:
Exercise:

Problem: For the function $f(x) = 2x^2 + 3x - 1$, evaluate the function.

ⓐ $f(3)$ ⓑ $f(-2)$ ⓒ $f(a)$

Solution:

ⓐ

	$f(x) = 2x^2 + 3x - 1$
To evaluate $f(3)$, substitute 3 for *x*.	$f(3) = 2(3)^2 + 3 \cdot 3 - 1$
Simplify.	$f(3) = 2 \cdot 9 + 3 \cdot 3 - 1$
	$f(3) = 18 + 9 - 1$
	$f(3) = 26$

ⓑ

		$f(x) = 2x^2 + 3x - 1$
To evaluate $f(-2)$, substitute -2 for *x*.		$f(-2) = 2(-2)^2 + 3(-2) - 1$
Simplify.		$f(-2) = 2 \cdot 4 + (-6) - 1$
		$f(-2) = 8 + (-6) - 1$

		$f(-2) = 1$

ⓒ

	$f(x) = 2x^2 + 3x - 1$
To evaluate $f(a)$, substitute a for x.	$f(a) = 2(a)^2 + 3 \cdot a - 1$
Simplify.	$f(a) = 2a^2 + 3a - 1$

Note:
Exercise:

Problem: For the function $f(x) = 3x^2 - 2x + 1$, evaluate the function.

ⓐ $f(3)$ ⓑ $f(-1)$ ⓒ $f(t)$

Solution:

ⓐ $f(3) = 22$ ⓑ $f(-1) = 6$ ⓒ $f(t) = 3t^2 - 2t - 1$

Note:
Exercise:

Problem: For the function $f(x) = 2x^2 + 4x - 3$, evaluate the function.

ⓐ $f(2)$ ⓑ $f(-3)$ ⓒ $f(h)$

Solution:

ⓐ $(2) = 13$ ⓑ $f(-3) = 3$
ⓒ $f(h) = 2h^2 + 4h - 3$

In the last example, we found $f(x)$ for a constant value of x. In the next example, we are asked to find $g(x)$ with values of x that are variables. We still follow the same procedure and substitute the variables in for the x.

Example:
Exercise:

Problem: For the function $g(x) = 3x - 5$, evaluate the function.

ⓐ $g(h^2)$ ⓑ $g(x+2)$ ⓒ $g(x) + g(2)$

Solution:

ⓐ

			$g(x) = 3x - 5$
To evaluate $g\left(h^2\right)$, substitute h^2 for x.			$g(h^2) = 3h^2 - 5$
			$g(h^2) = 3h^2 - 5$

ⓑ

	$g(x) = 3x - 5$
To evaluate $g(x+2)$, substitute $x+2$ for x.	$g(x+2) = 3(x+2) - 5$
Simplify.	$g(x+2) = 3x + 6 - 5$
	$g(x+2) = 3x + 1$

ⓒ

		$g(x) = 3x - 5$
To evaluate $g(x) + g(2)$, first find $g(2)$.		$g(2) = 3 \cdot 2 - 5$
		$g(2) = 1$
Now find $g(x) + g(2)$		$g(x) + g(2) = \underbrace{3x - 5}_{g(x)} + \underbrace{1}_{g(2)}$
Simplify.		$g(x) + g(2) = 3x - 5 + 1$
		$g(x) + g(2) = 3x - 4$

Notice the difference between part ⓑ and ⓒ. We get $g(x+2) = 3x+1$ and $g(x) + g(2) = 3x - 4$. So we see that $g(x+2) \neq g(x) + g(2)$.

Note:
Exercise:

Problem: For the function $g(x) = 4x - 7$, evaluate the function.

ⓐ $g(m^2)$ ⓑ $g(x-3)$ ⓒ $g(x) - g(3)$

Solution:

ⓐ $4m^2 - 7$ ⓑ $4x - 19$
ⓒ $x - 12$

Note:
Exercise:

Problem: For the function $h(x) = 2x + 1$, evaluate the function.

ⓐ $h(k^2)$ ⓑ $h(x+1)$ ⓒ $h(x) + h(1)$

Solution:

ⓐ $2k^2 + 1$ ⓑ $2x + 3$
ⓒ $2x + 4$

Many everyday situations can be modeled using functions.

Example:
Exercise:

Problem:

The number of unread emails in Sylvia's account is 75. This number grows by 10 unread emails a day. The function $N(t) = 75 + 10t$ represents the relation between the number of emails, *N*, and the time, *t*, measured in days.

ⓐ Determine the independent and dependent variable.

ⓑ Find $N(5)$. Explain what this result means.

Solution:

ⓐ The number of unread emails is a function of the number of days. The number of unread emails, N, depends on the number of days, t. Therefore, the variable N, is the dependent variable and the variable t is the independent variable.

ⓑ Find $N(5)$. Explain what this result means.

	$N(t) = 75 + 10t$
Substitute in $t = 5$.	$N(5) = 75 + 10 \cdot 5$
Simplify.	$N(5) = 75 + 50$
	$N(5) = 125$

Since 5 is the number of days, $N(5)$, is the number of unread emails after 5 days. After 5 days, there are 125 unread emails in the account.

Note:
Exercise:

Problem:

The number of unread emails in Bryan's account is 100. This number grows by 15 unread emails a day. The function $N(t) = 100 + 15t$ represents the relation between the number of emails, N, and the time, t, measured in days.

ⓐ Determine the independent and dependent variable.

ⓑ Find $N(7)$. Explain what this result means.

Solution:

ⓐ *t* IND; *N* DEP ⓑ 205; the number of unread emails in Bryan's account on the seventh day.

Note:
Exercise:

Problem:

The number of unread emails in Anthony's account is 110. This number grows by 25 unread emails a day. The function $N(t) = 110 + 25t$ represents the relation between the number of emails, *N*, and the time, *t*, measured in days.

ⓐ Determine the independent and dependent variable.

ⓑ Find $N(14)$. Explain what this result means.

Solution:

ⓐ *t* IND; *N* DEP ⓑ 460; the number of unread emails in Anthony's account on the fourteenth day

Functions and the Line of Best Fit

A key feature of functions is that for every value of the independent variable there is exactly one value of the dependent variable. When we worked with scatterplots, often there was more than one value of the range for some of the values of the domain. A Line of Best Fit is a simplification of the data that results in approximating a relation with a linear function. This function can then be evaluated to predict the value of the dependent variable for a value of the independent variable.

Exercise:

Weight and Height

Problem:

People are all different shapes and sizes. Yet, it is true that taller people tend to weigh more than shorter people. Assume that (weight, height) order pairs have been obtained and a Line of Best Fit has been determined to be:
w = -186 + 4.8h where h is the height in inches and w is the weight in pounds.
Use this equation to predict the weight of somebody 72 inches tall.

Solution:

Using the Line of Best Fit, w = -186 + 4.8(72).
w = -186 + 345.6
w = 159.6
We expect the person will weigh 159.6 pounds.

Note:
Access this online resource for additional instruction and practice with relations and functions.

- Introduction to Functions

Sequences and Visual Patterns

Sequences
A **sequence** is a function where the domain is a set of consecutive natural numbers usually starting with either 0 or 1. We will start our sequences with 1. The range of the sequence is all of the values that form ordered pairs with the values in the domain. Normally, we just list the values of the range in order separated by commas. Each value of the range is called a **term**. The individual terms are often referred to using a subscript indicating their place in the sequence rather than by using ordered pair notation.

Example:
The first four odd numbers
The first four odd numbers is the sequence: 1,3,5,7.
Written using ordered pair notation this is: (1,1), (2,3),(3,5),(4,7).
The domain is {1,2,3,4} and the range is {1,3,5,7}.

Visual Patterns
Working with Visual Patterns is a fun way of becoming more comfortable with functions. The idea is to look at the first few terms of the sequence and try to figure out the rule (function) that calculates the size of any term of the sequence.

Example:
A Simple Visual Pattern

, **, ******, ********, ...
Notice that there is first 2 *, then 4 *, then 6 *, then 8 *.
What do you think probably is the next term it the sequence? ********** or 10 *.
What about the 12th term in the sequence? 24 *.
The rule is twice the term number. Written as a function: f(n) = 2n.
This is nearly identical to y = 2x, the difference being the notation and the domain for the sequence is natural numbers while the domain for y = 2x is normally assumed to be all real numbers. In either instance the pattern of growth is linear since we know that y = 2x is a line with slope 2 and y-intercept 0.

Practice Visual Patterns

www.visualpatterns.org has many visual patterns for you to practice finding rules. The patterns get a little more difficult as the problem number gets larger so you probably want to start at the beginning.

Key Concepts

- **Function Notation:** For the function $y = f(x)$
  - *f* is the name of the function
  - *x* is the domain value
  - $f(x)$ is the range value *y* corresponding to the value *x*
    We read $f(x)$ as *f* of *x* or the value of *f* at *x*.
- **Independent and Dependent Variables:** For the function $y = f(x)$,
  - *x* is the independent variable as it can be any value in the domain
  - *y* is the dependent variable as its value depends on *x*

Practice Makes Perfect

Find the Domain and Range of a Relation

In the following exercises, for each relation ⓐ find the domain of the relation ⓑ find the range of the relation.

Exercise:

Problem: $\{(1,4),(2,8),(3,12),(4,16),(5,20)\}$

Solution:

ⓐ {1, 2, 3, 4, 5} ⓑ {4, 8, 12, 16, 20}

Exercise:

Problem: $\{(1,-2),(2,-4),(3,-6),(4,-8),(5,-10)\}$

Exercise:

Problem: $\{(1,7),(5,3),(7,9),(-2,-3),(-2,8)\}$

Solution:

ⓐ {1, 5, 7, −2} ⓑ {7, 3, 9, −3, 8}

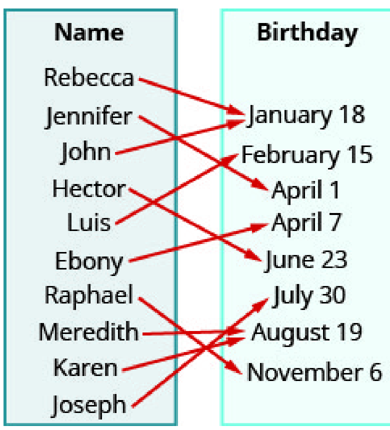
Exercise:

Problem: $\{(11,3),(-2,-7),(4,-8),(4,17),(-6,9)\}$

In the following exercises, use the mapping of the relation to ⓐ list the ordered pairs of the relation, ⓑ find the domain of the relation, and ⓒ find the range of the relation.

Exercise:

Problem:



Solution:

ⓐ (Rebecca, January 18), (Jennifer, April 1), (John, January 18), (Hector, June 23), (Luis, February 15), (Ebony, April 7), (Raphael, November 6), (Meredith, August 19), (Karen, August 19), (Joseph, July 30)
ⓑ {Rebecca, Jennifer, John, Hector, Luis, Ebony, Raphael, Meredith, Karen, Joseph}
ⓒ {January 18, April 1, June 23, February 15, April 7, November 6, August 19, July 30}

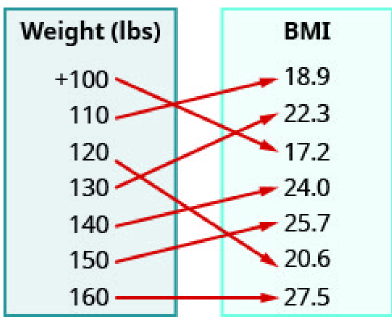
Exercise:

Problem:

Exercise:

Problem:

For a woman of height 5′4″ the mapping below shows the corresponding Body Mass Index (BMI). The body mass index is a measurement of body fat based on height and weight. A BMI of 18.5– 24.9 is considered healthy.



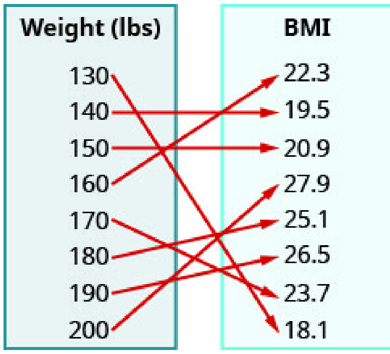
Solution:

ⓐ (+100, 17. 2), (110, 18.9), (120, 20.6), (130, 22.3), (140, 24.0), (150, 25.7), (160, 27.5)
ⓑ {+100, 110, 120, 130, 140, 150, 160,} ⓒ {17.2, 18.9, 20.6, 22.3, 24.0, 25.7, 27.5}

Exercise:

Problem:

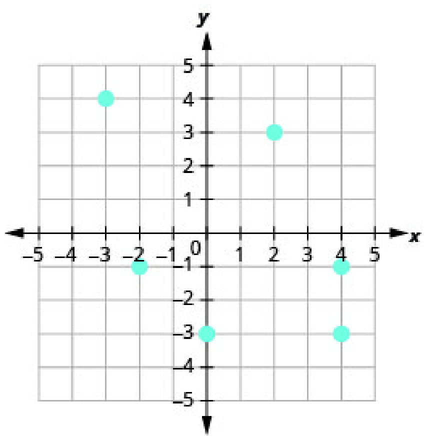
For a man of height 5′11″ the mapping below shows the corresponding Body Mass Index (BMI). The body mass index is a measurement of body fat based on height and weight. A BMI of 18.5– 24.9 is considered healthy.



In the following exercises, use the graph of the relation to ⓐ list the ordered pairs of the relation ⓑ find the domain of the relation ⓒ find the range of the relation.

Exercise:

Problem:

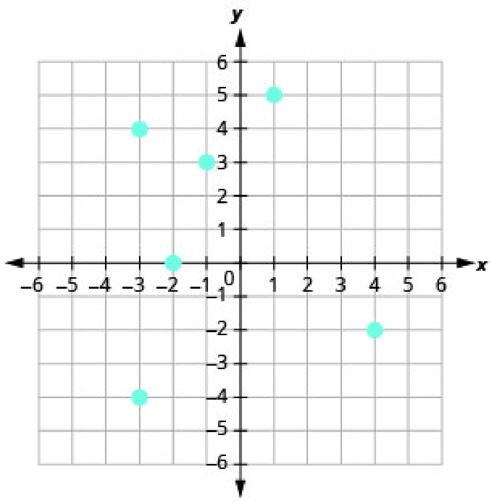


Solution:

ⓐ (2, 3), (4, −3), (−2, −1), (−3, 4), (4, −1), (0, −3) ⓑ {−3, −2, 0, 2, 4}
ⓒ {−3, −1, 3, 4}

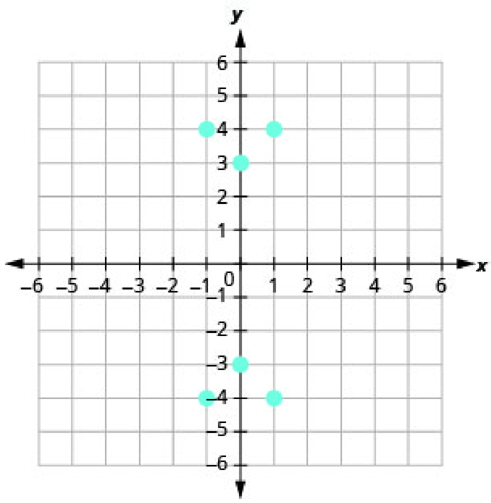
Exercise:

Problem:



Exercise:

Problem:

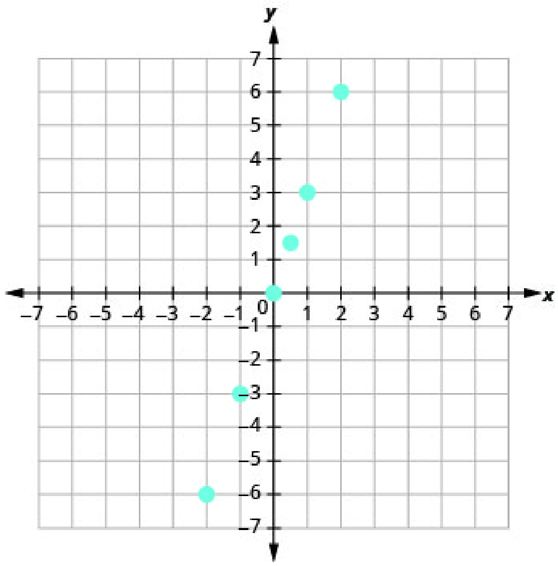


Solution:

ⓐ (1, 4), (1, −4), (−1, 4), (−1, −4), (0, 3), (0, −3) ⓑ {−1, 0, 1} ⓒ {−4, −3, 3,4}

Exercise:

Problem:



Determine if a Relation is a Function

In the following exercises, use the set of ordered pairs to ⓐ determine whether the relation is a function, ⓑ find the domain of the relation, and ⓒ find the range of the relation.

Exercise:

Problem: $\{(-3,9),(-2,4),(-1,1),$
$(0,0),(1,1),(2,4),(3,9)\}$

Solution:

ⓐ yes ⓑ {−3, −2, −1, 0, 1, 2, 3} ⓒ {9, 4, 1, 0}

Exercise:

Problem: $\{(9,-3),(4,-2),(1,-1),$
$(0,0),(1,1),(4,2),(9,3)\}$

Exercise:

Problem: $\{(-3,27),(-2,8),(-1,1),$
$(0,0),(1,1),(2,8),(3,27)\}$

Solution:

ⓐ yes ⓑ {−3, −2, −1, 0, 1, 2, 3} ⓒ 0, 1, 8, 27}

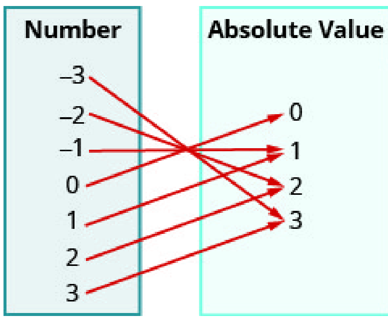
Exercise:

Problem: $\{(-3,-27),(-2,-8),(-1,-1),$
$(0,0),(1,1),(2,8),(3,27)\}$

In the following exercises, use the mapping to ⓐ determine whether the relation is a function, ⓑ find the domain of the function, and ⓒ find the range of the function.

Exercise:

Problem:

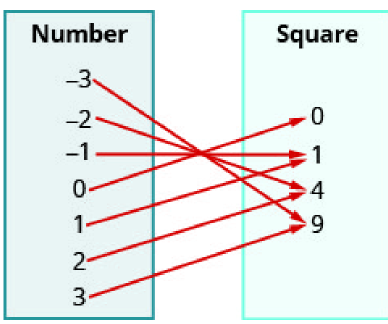


Solution:

ⓐ yes ⓑ {−3, −2, −1, 0, 1, 2, 3} ⓒ {0, 1, 2, 3}

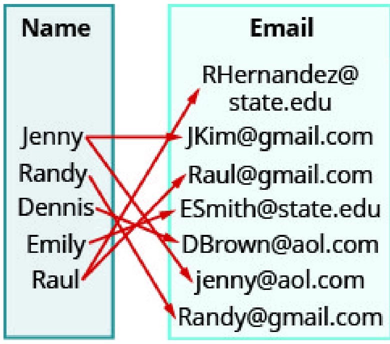
Exercise:

Problem:



Exercise:

Problem:

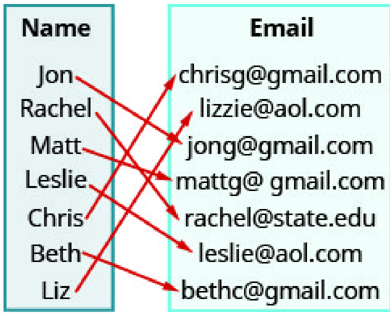


Solution:

ⓐ no ⓑ {Jenny, R and y, Dennis, Emily, Raul} ⓒ {RHern and ez@state.edu, JKim@gmail.com, Raul@gmail.com, ESmith@state.edu, DBroen@aol.com, jenny@aol.cvom, R and y@gmail.com}

Exercise:

Problem:



In the following exercises, determine whether each equation is a function.

Exercise:

Problem: ⓐ $2x + y = -3$
ⓑ $y = x^2$
ⓒ $x + y^2 = -5$

Solution:

ⓐ yes ⓑ yes ⓒ no

Exercise:

ⓐ $y = 3x - 5$
ⓑ $y = x^3$
Problem: ⓒ$2x + y^2 = 4$

Exercise:

ⓐ $y - 3x^3 = 2$
ⓑ $x + y^2 = 3$
Problem: ⓒ$3x - 2y = 6$

Solution:

ⓐ yes ⓑ no ⓒ yes

Exercise:

ⓐ $2x - 4y = 8$
ⓑ $-4 = x^2 - y$
Problem: ⓒ$y^2 = -x + 5$

Find the Value of a Function

In the following exercises, evaluate the function: ⓐ $f(2)$ ⓑ $f(-1)$ ⓒ $f(a)$.

Exercise:

Problem: $f(x) = 5x - 3$

Solution:

ⓐ $f(2) = 7$ ⓑ $f(-1) = -8$ ⓒ $f(a) = 5a - 3$

Exercise:

Problem: $f(x) = 3x + 4$

Exercise:

Problem: $f(x) = -4x + 2$

Solution:

ⓐ $f(2) = -6$ ⓑ $f(-1) = 6$ ⓒ $f(a) = -4a + 2$

Exercise:

Problem: $f(x) = -6x - 3$

Exercise:

Problem: $f(x) = x^2 - x + 3$

Solution:

ⓐ $f\left(2\right) = 5$ ⓑ $f\left(-1\right) = 5$
ⓒ $f(a) = a^2 - a + 3$

Exercise:

Problem: $f(x) = x^2 + x - 2$

Exercise:

Problem: $f(x) = 2x^2 - x + 3$

Solution:

ⓐ $f\left(2\right) = 9$ ⓑ $f\left(-1\right) = 6$
ⓒ $f(a) = 2a^2 - a + 3$

Exercise:

Problem: $f(x) = 3x^2 + x - 2$

In the following exercises, evaluate the function: ⓐ $g(h^2)$ ⓑ $g(x+2)$ ⓒ $g\left(x\right) + g\left(2\right)$.

Exercise:

Problem: $g(x) = 2x + 1$

Solution:

ⓐ $g(h^2) = 2h^2 + 1$
ⓑ $g\left(x+2\right) = 2x + 5$
ⓒ $g\left(x\right) + g\left(2\right) = 2x + 6$

Exercise:

Problem: $g(x) = 5x - 8$

Exercise:

Problem: $g(x) = -3x - 2$

Solution:

ⓐ $g(h^2) = -3h^2 - 2$
ⓑ $g\left(x+2\right) = -3x - 8$
ⓒ $g\left(x\right) + g\left(2\right) = -3x - 10$

Exercise:

Problem: $g(x) = -8x + 2$

Exercise:

Problem: $g(x) = 3 - x$

Solution:

ⓐ $g(h^2) = 3 - h^2$
ⓑ $g\left(x+2\right) = 1 - x$
ⓒ $g\left(x\right) + g\left(2\right) = 4 - x$

Exercise:

Problem: $g(x) = 7 - 5x$

In the following exercises, evaluate the function.

Exercise:

Problem: $f(x) = 3x^2 - 5x;\ f(2)$

Solution:

2

Exercise:

Problem: $g(x) = 4x^2 - 3x;\ g(3)$

Exercise:

Problem: $F(x) = 2x^2 - 3x + 1;$ $F(-1)$

Solution:

6

Exercise:

Problem: $G(x) = 3x^2 - 5x + 2;$ $G(-2)$

Exercise:

Problem: $h(t) = 2\,|t - 5| + 4;$ $f(-4)$

Solution:

22

Exercise:

Problem: $h(y) = 3\,|y - 1| - 3;$ $h(-4)$

Exercise:

Problem: $f(x) = \frac{x+2}{x-1};$ $f(2)$

Solution:

4

Exercise:

Problem: $g(x) = \frac{x-2}{x+2};$ $g(4)$

In the following exercises, solve.

Exercise:

Problem:

The number of unwatched shows in Sylvia's DVR is 85. This number grows by 20 unwatched shows per week. The function $N\,(t) = 85 + 20t$ represents the relation between the number of unwatched shows, *N*, and the time, *t*, measured in weeks.

ⓐ Determine the independent and dependent variable.

ⓑ Find $N\,(4)$. Explain what this result means

Solution:

ⓐ *t* IND; *N* DEP
ⓑ $N\,(4) = 165$ the number of unwatched shows in Sylvia's DVR at the fourth week.

Exercise:

Problem:

Every day a new puzzle is downloaded into Ken's account. Right now he has 43 puzzles in his account. The function $N(t) = 43 + t$ represents the relation between the number of puzzles, *N*, and the time, *t*, measured in days.

ⓐ Determine the independent and dependent variable.

ⓑ Find $N(30)$. Explain what this result means.

Exercise:

Problem:

The daily cost to the printing company to print a book is modeled by the function $C(x) = 3.25x + 1500$ where *C* is the total daily cost and *x* is the number of books printed.

ⓐ Determine the independent and dependent variable.

ⓑ Find $N(0)$. Explain what this result means.

ⓒ Find $N(1000)$. Explain what this result means.

Solution:

ⓐ *x* IND; *C* DEP
ⓑ $N(0) = 1500$ the daily cost if no books are printed
ⓒ $N(1000) = 4750$ the daily cost of printing 1000 books

Exercise:

Problem:

The daily cost to the manufacturing company is modeled by the function $C(x) = 7.25x + 2500$ where $C(x)$ is the total daily cost and *x* is the number of items manufactured.

ⓐ Determine the independent and dependent variable.

ⓑ Find $C(0)$. Explain what this result means.

ⓒ Find $C(1000)$. Explain what this result means.

Writing Exercises

Exercise:

Problem: In your own words, explain the difference between a relation and a function.

Exercise:

Problem: In your own words, explain what is meant by domain and range.

Exercise:

Problem: Is every relation a function? Is every function a relation?

Exercise:

Problem: How do you find the value of a function?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
find the domain and range of a relation.			
determine if a relation is a function.			
find the value of a function.			

ⓑ After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

Glossary

domain of a relation
: The domain of a relation is all the *x*-values in the ordered pairs of the relation.

function
: A function is a relation that assigns to each element in its domain exactly one element in the range.

mapping
: A mapping is sometimes used to show a relation. The arrows show the pairing of the elements of the domain with the elements of the range.

range of a relation

The range of a relation is all the *y*-values in the ordered pairs of the relation.

relation

A relation is any set of ordered pairs, (x, y). All the *x*-values in the ordered pairs together make up the domain. All the *y*-values in the ordered pairs together make up the range.

Graphs of Functions: ASE
By the end of this section, you will be able to:

- Use the vertical line test
- Identify graphs of basic functions
- Read information from a graph of a function

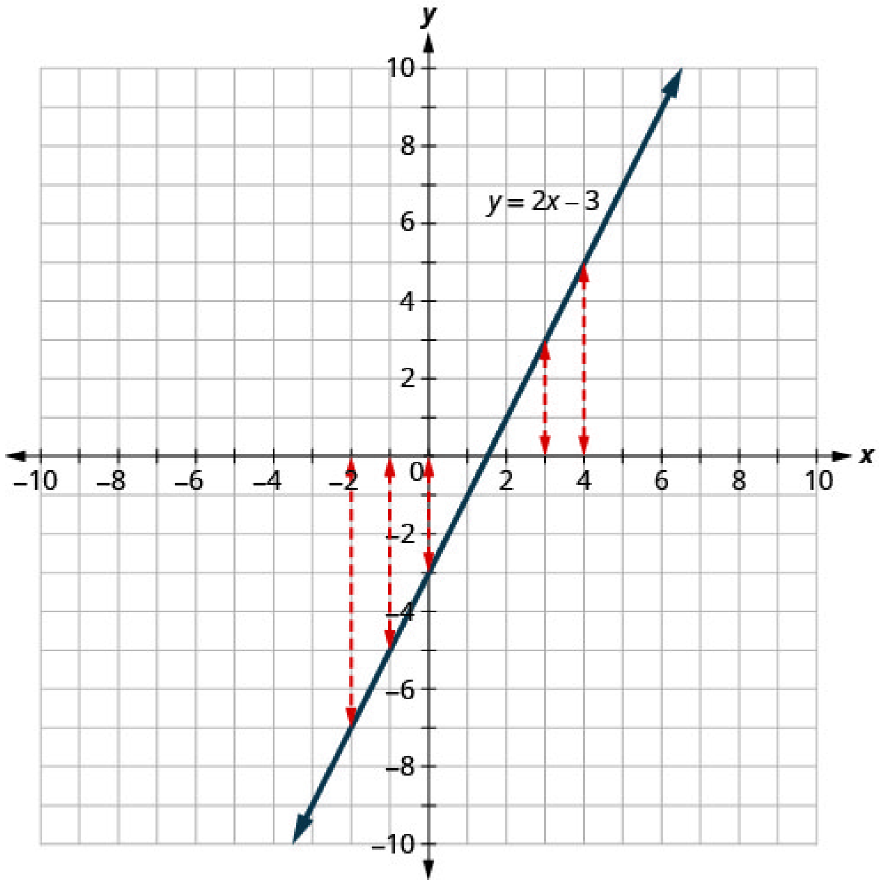
Use the Vertical Line Test

In the last section we learned how to determine if a relation is a function. The relations we looked at were expressed as a set of ordered pairs, a mapping or an equation. We will now look at how to tell if a graph is that of a function.

An ordered pair (x, y) is a solution of a linear equation, if the equation is a true statement when the *x*- and *y*-values of the ordered pair are substituted into the equation.

The graph of a linear equation is a straight line where every point on the line is a solution of the equation and every solution of this equation is a point on this line.

In [link], we can see that, in graph of the equation $y = 2x - 3$, for every *x*-value there is only one *y*-value, as shown in the accompanying table.



$y = 2x - 3$		
x	y	(x, y)
–2	–7	(–2, –7)
–1	–5	(–1, –5)
0	–3	(0, –3)
3	3	(3, 3)
4	5	(4, 5)

A relation is a function if every element of the domain has exactly one value in the range. So the relation defined by the equation $y = 2x - 3$ is a function.

If we look at the graph, each vertical dashed line only intersects the line at one point. This makes sense as in a function, for every *x*-value there is only one *y*-value.

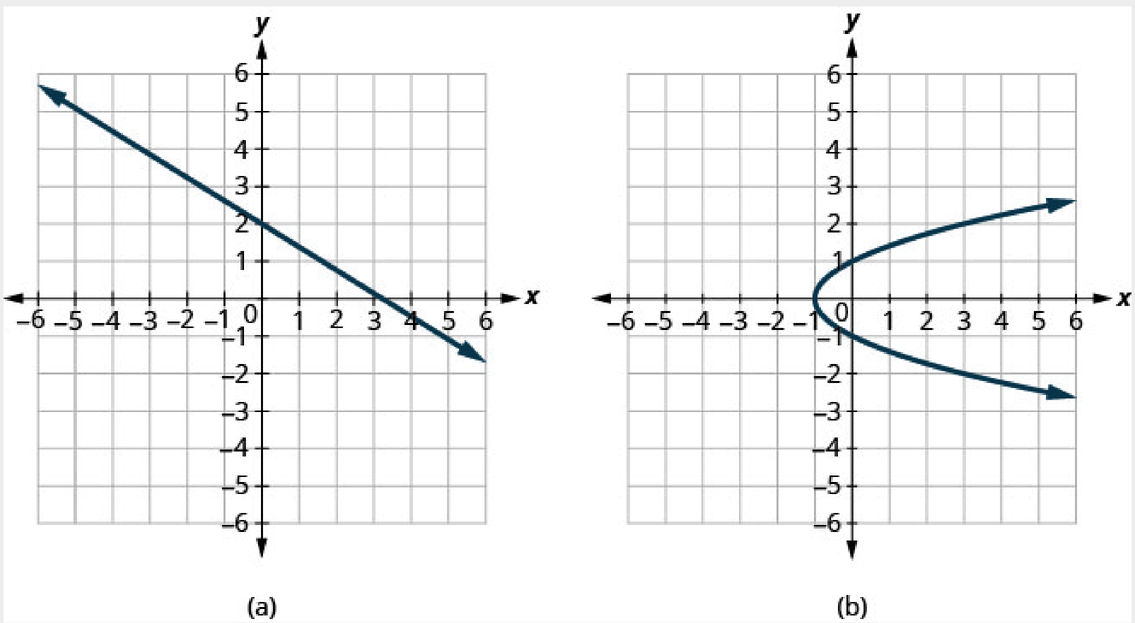
If the vertical line hit the graph twice, the *x*-value would be mapped to two *y*-values, and so the graph would not represent a function.

This leads us to the vertical line test. A set of points in a rectangular coordinate system is the graph of a function if every vertical line intersects the graph in at most one point. If any vertical line intersects the graph in more than one point, the graph does not represent a function.

Note:
Vertical Line Test
A set of points in a rectangular coordinate system is the graph of a function if every vertical line intersects the graph in at most one point.
If any vertical line intersects the graph in more than one point, the graph does not represent a function.

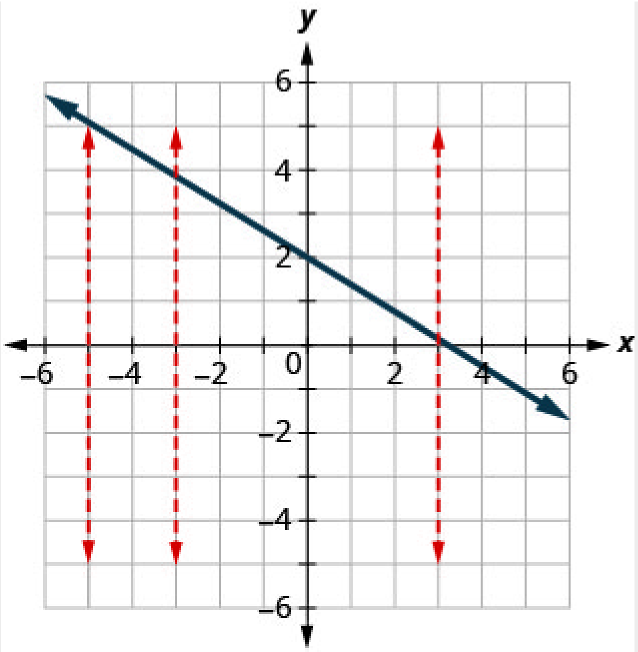
Example:
Exercise:

Problem: Determine whether each graph is the graph of a function.

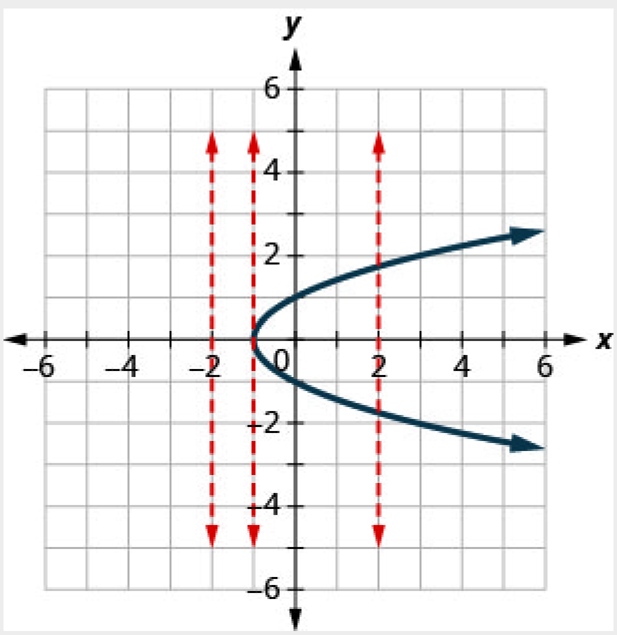


Solution:

ⓐ Since any vertical line intersects the graph in at most one point, the graph is the graph of a function.

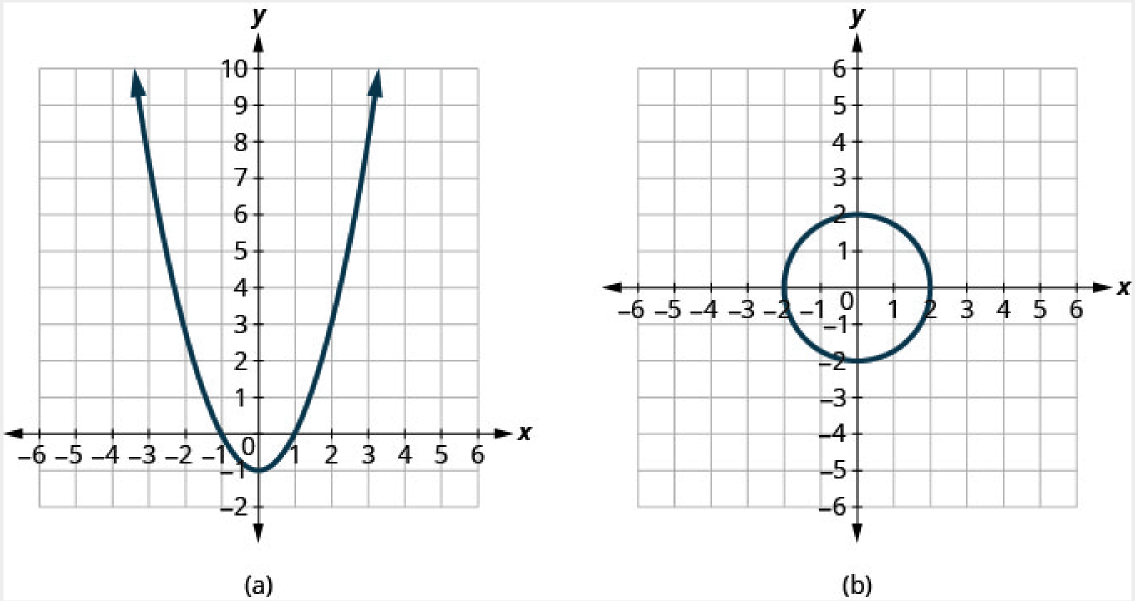


ⓑ One of the vertical lines shown on the graph, intersects it in two points. This graph does not represent a function.



Note:
Exercise:

Problem: Determine whether each graph is the graph of a function.

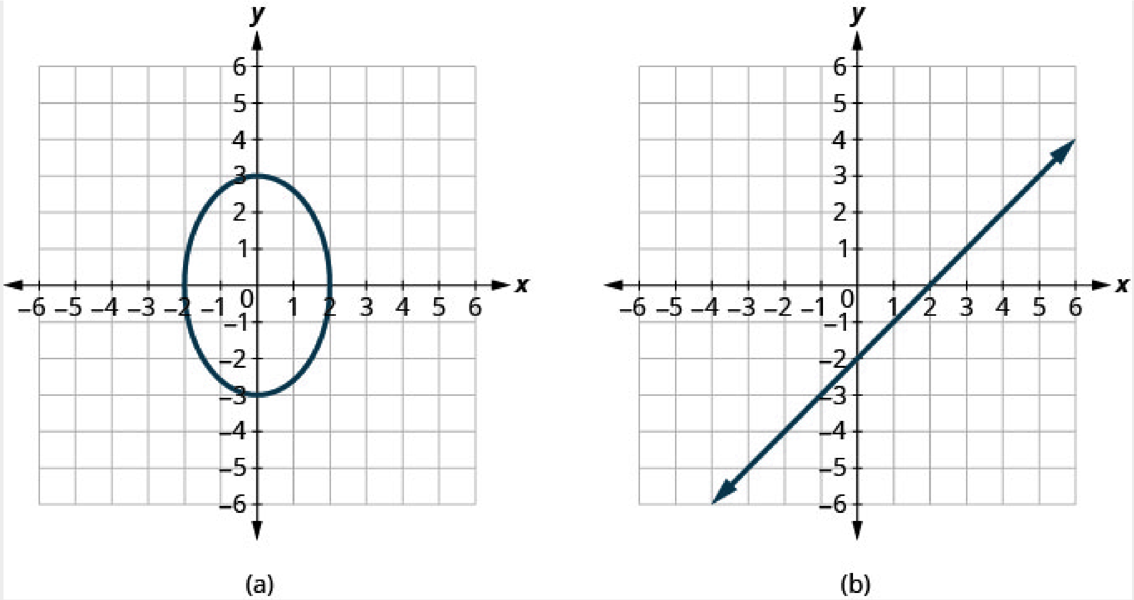


Solution:

ⓐ yes ⓑ no

Note:
Exercise:

Problem: Determine whether each graph is the graph of a function.



Solution:

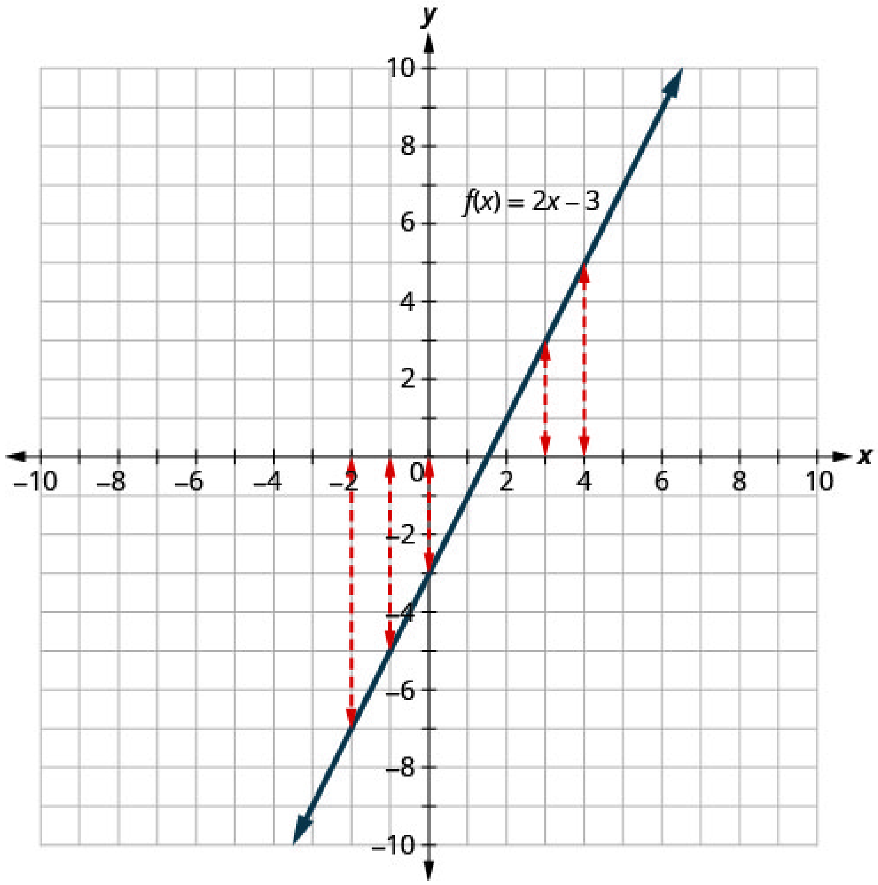
ⓐ no ⓑ yes

Identify Graphs of Basic Functions

We used the equation $y=2x-3$ and its graph as we developed the vertical line test. We said that the relation defined by the equation $y=2x-3$ is a function.

We can write this as in function notation as $f\left(x\right)=2x-3$. It still means the same thing. The graph of the function is the graph of all ordered pairs $\left(x,y\right)$ where $y=f\left(x\right)$. So we can write the ordered pairs as $\left(x,f\left(x\right)\right)$. It looks different but the graph will be the same.

Compare the graph of $y=2x-3$ previously shown in [link] with the graph of $f\left(x\right)=2x-3$ shown in [link]. Nothing has changed but the notation.



$f(x) = 2x - 3$		
x	$f(x)$	$(x, f(x))$
–2	–7	(–2, –7)
–1	–5	(–1, –5)
0	–3	(0, –3)
3	3	(3, 3)
4	5	(4, 5)

Note:
Graph of a Function
The graph of a function is the graph of all its ordered pairs, (x, y) or using function notation, $(x, f(x))$ where $y = f(x)$.
Equation:

$$\begin{array}{rl} f & \text{name of function} \\ x & x\text{-coordinate of the ordered pair} \\ f(x) & y\text{-coordinate of the ordered pair} \end{array}$$

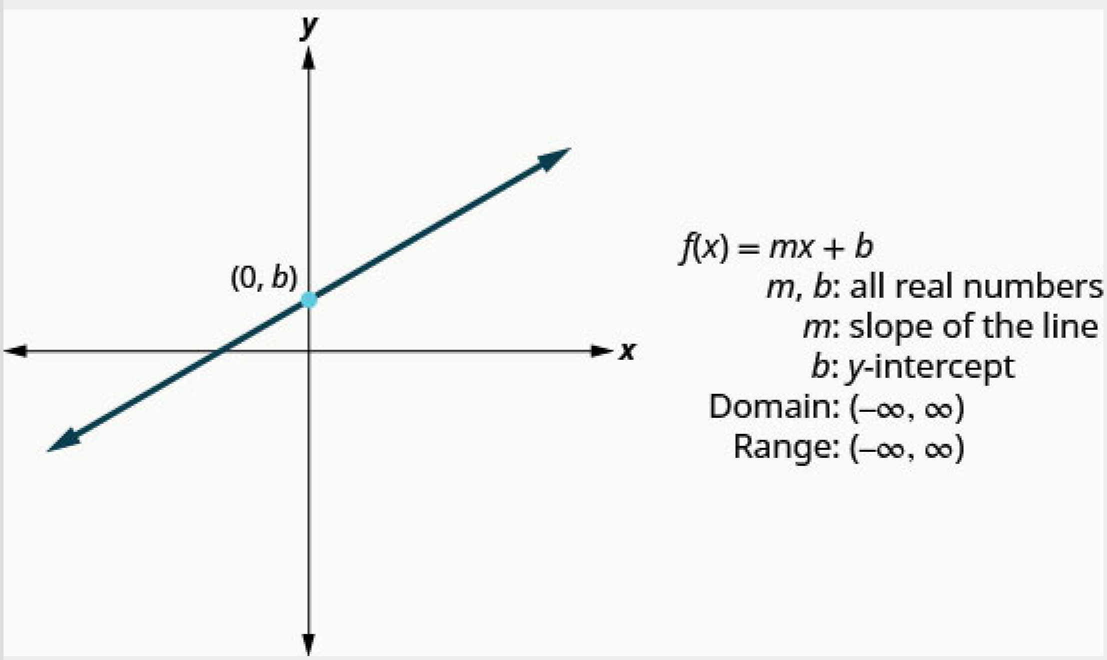
As we move forward in our study, it is helpful to be familiar with the graphs of several basic functions and be able to identify them.

Through our earlier work, we are familiar with the graphs of linear equations. The process we used to decide if $y = 2x - 3$ is a function would apply to all linear equations. All non-vertical linear equations are functions. Vertical lines are not functions as the *x*-value has infinitely many *y*-values.

We wrote linear equations in several forms, but it will be most helpful for us here to use the slope-intercept form of the linear equation. The slope-intercept form of a linear equation is $y = mx + b$. In function notation, this linear function becomes $f(x) = mx + b$ where *m* is the slope of the line and *b* is the *y*-intercept.

The domain is the set of all real numbers, and the range is also the set of all real numbers.

Note:
Linear Function



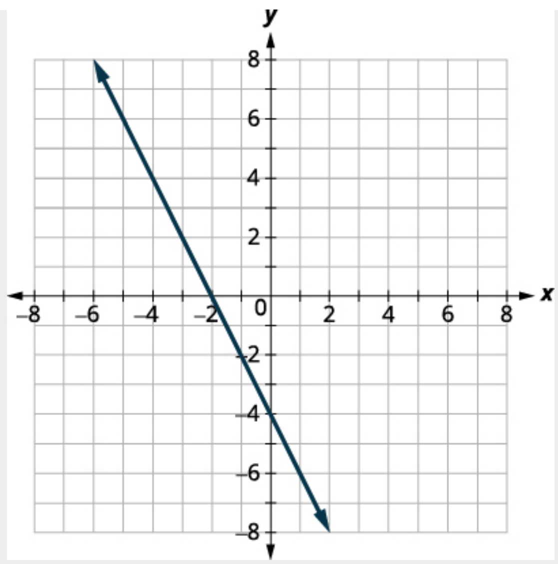
We will use the graphing techniques we used earlier, to graph the basic functions.

Example:
Exercise:

Problem: Graph: $f(x) = -2x - 4$.

Solution:

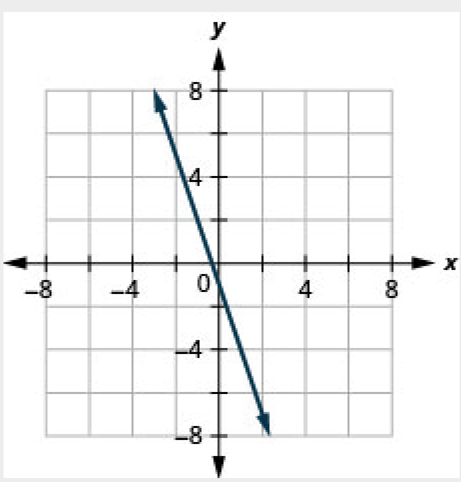
	$f(x) = -2x - 4$
We recognize this as a linear function.	
Find the slope and *y*-intercept.	$m = -2$ $b = -4$
Graph using the slope intercept.	



Note:
Exercise:

Problem: Graph: $f(x) = -3x - 1$

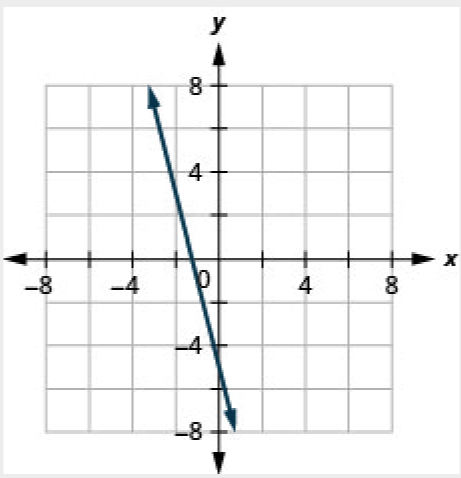
Solution:



Note:
Exercise:

Problem: Graph: $f\left(x\right)=-4x-5$

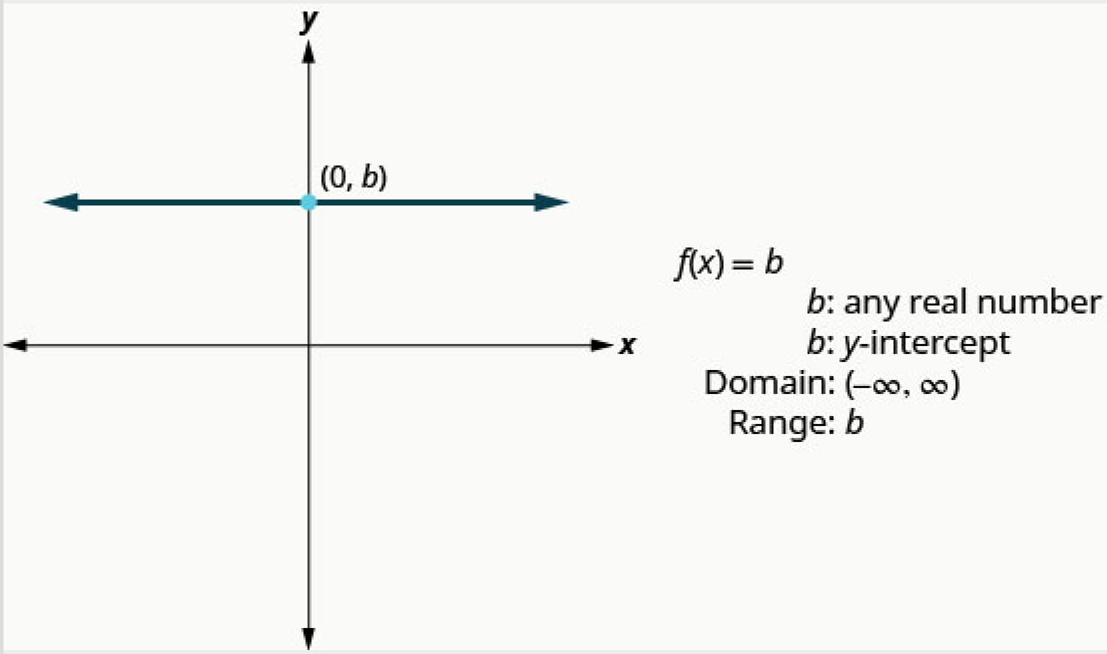
Solution:



The next function whose graph we will look at is called the constant function and its equation is of the form $f\left(x\right)=b$, where b is any real number. If we replace the $f\left(x\right)$ with y, we get $y=b$. We recognize this as the horizontal line whose *y*-intercept is b. The graph of the function $f\left(x\right)=b$, is also the horizontal line whose *y*-intercept is b.

Notice that for any real number we put in the function, the function value will be b. This tells us the range has only one value, b.

Note:
Constant Function



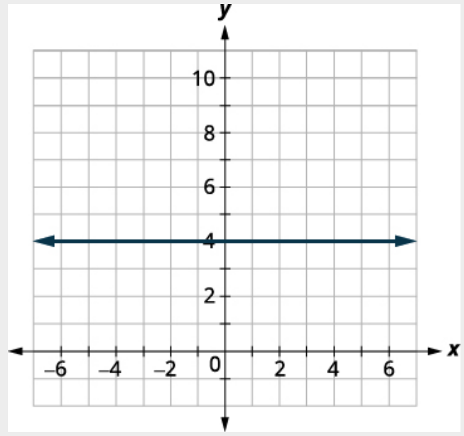
Example:
Exercise:

Problem: Graph: $f\left(x\right)=4.$

Solution:

	$f\left(x\right)=4$
We recognize this as a constant function.	

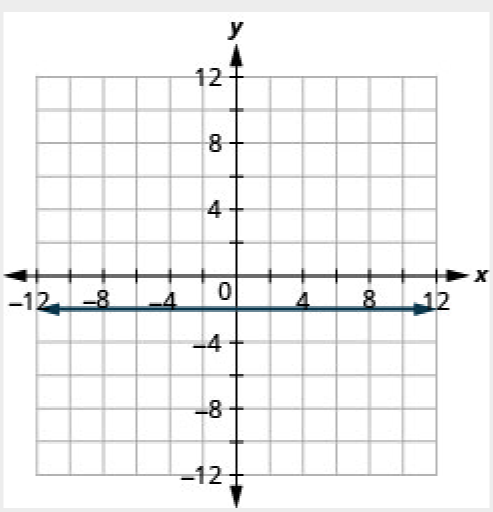
The graph will be a horizontal line through $(0, 4)$.



Note:
Exercise:

Problem: Graph: $f\left(x\right) = -2$.

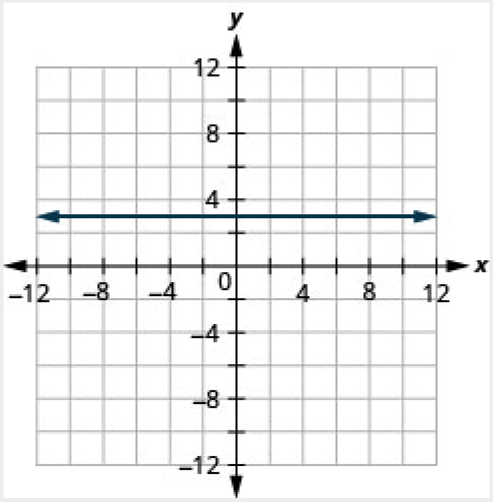
Solution:



Note:
Exercise:

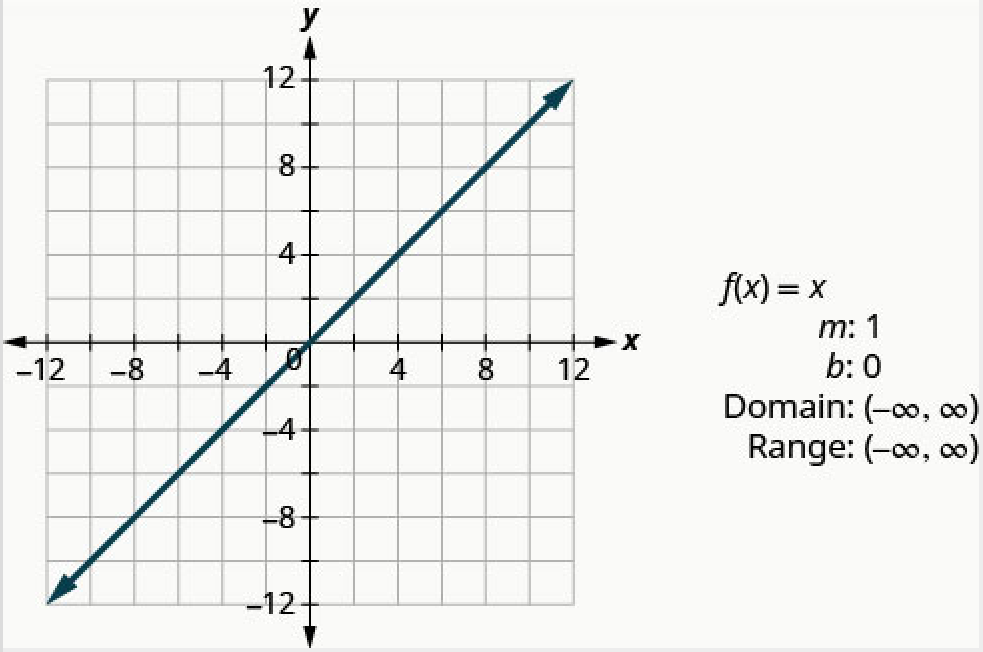
Problem: Graph: $f\left(x\right)=3.$

Solution:



The identity function, $f\left(x\right)=x$ is a special case of the linear function. If we write it in linear function form, $f\left(x\right)=1x+0$, we see the slope is 1 and the *y*-intercept is 0.

Note:
Identity Function



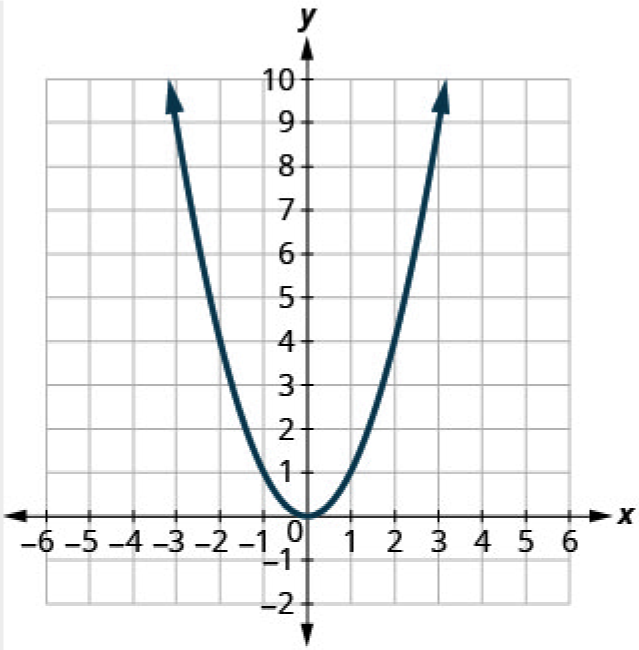
The next function we will look at is not a linear function. So the graph will not be a line. The only method we have to graph this function is point plotting. Because this is an unfamiliar function, we make sure to choose several positive and negative values as well as 0 for our x-values.

Example:
Exercise:

Problem: Graph: $f\left(x\right)={x}^{2}$.

Solution:

We choose *x*-values. We substitute them in and then create a chart as shown.

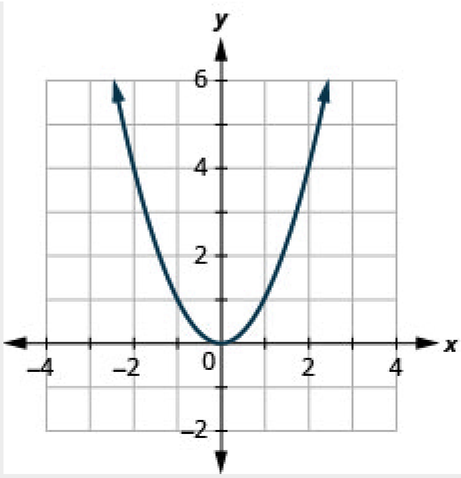


x	$f(x) = x^2$	$(x, f(x))$
–3	9	(–3, 9)
–2	4	(–2, 4)
–1	1	(–1, 1)
0	0	(0, 0)
1	1	(1, 1)
2	4	(2, 4)
3	9	(3, 9)

Note:
Exercise:

Problem: Graph: $f(x) = x^2$.

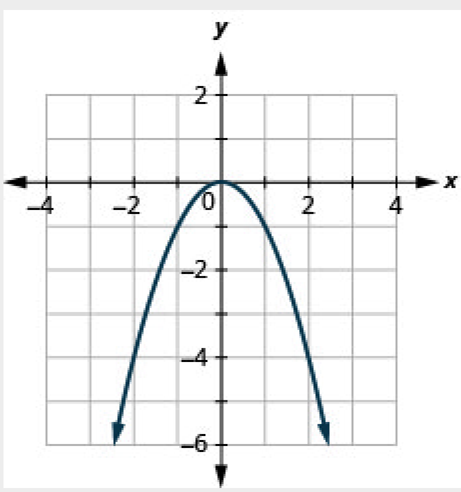
Solution:



Note:
Exercise:

Problem: $f(x) = -x^2$

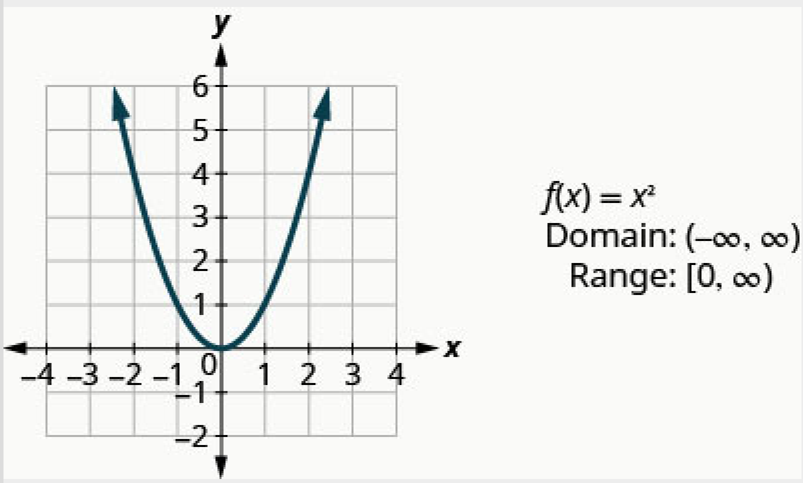
Solution:



Looking at the result in [link], we can summarize the features of the square function. We call this graph a parabola. As we consider the domain, notice any real number can be used as an *x*-value. The domain is all real numbers.

The range is not all real numbers. Notice the graph consists of values of *y* never go below zero. This makes sense as the square of any number cannot be negative. So, the range of the square function is all non-negative real numbers.

Note:
Square Function



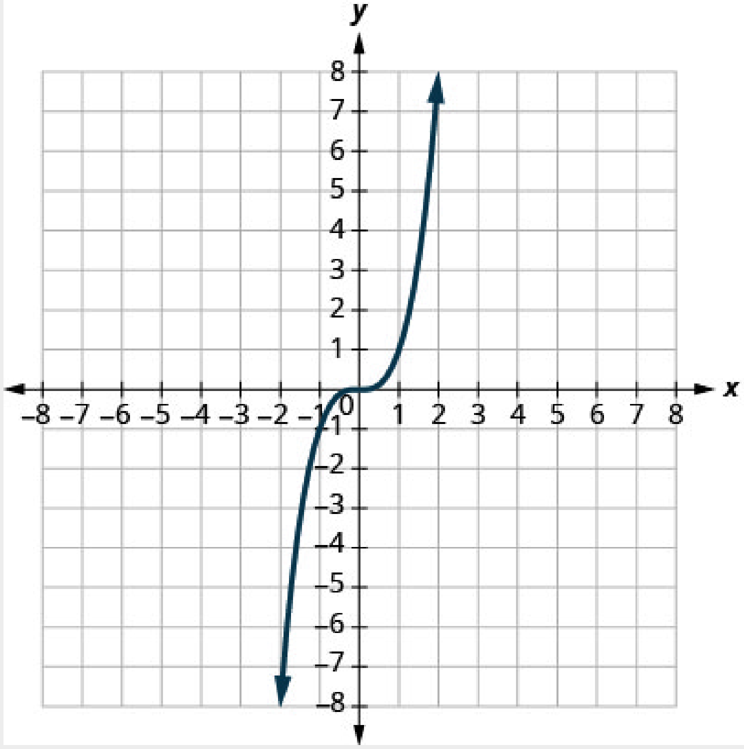
The next function we will look at is also not a linear function so the graph will not be a line. Again we will use point plotting, and make sure to choose several positive and negative values as well as 0 for our *x*-values.

Example:
Exercise:

Problem: Graph: $f\left(x\right)=x^3$.

Solution:

We choose *x*-values. We substitute them in and then create a chart.

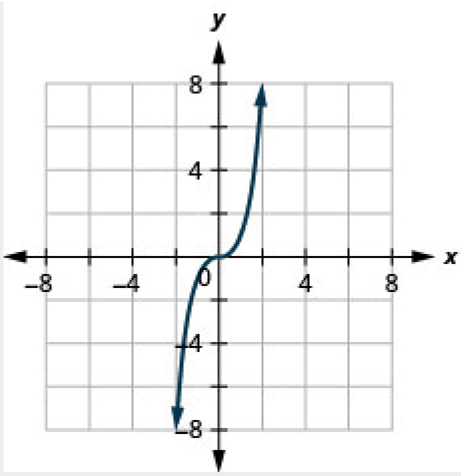


x	$f(x) = x^3$	$(x, f(x))$
–2	–8	(–2, –8)
–1	–1	(–1, –1)
0	0	(0, 0)
1	1	(1, 1)
2	8	(2, 8)

Note:
Exercise:

Problem: Graph: $f\left(x\right)=x^3$.

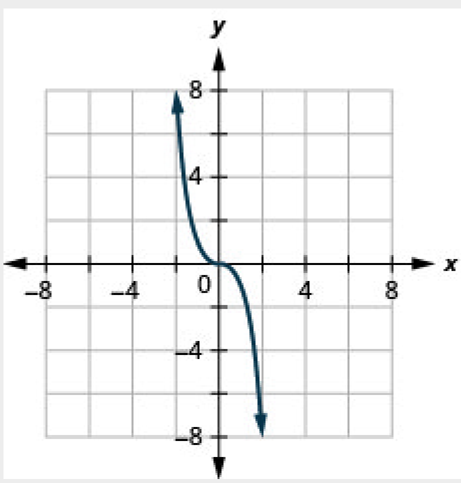
Solution:



Note:
Exercise:

Problem: Graph: $f(x) = -x^3$.

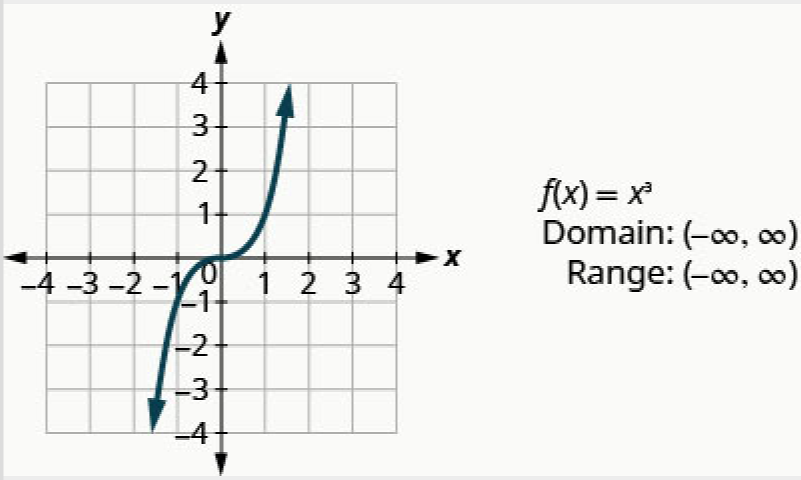
Solution:



Looking at the result in [link], we can summarize the features of the cube function. As we consider the domain, notice any real number can be used as an *x*-value. The domain is all real numbers.

The range is all real numbers. This makes sense as the cube of any non-zero number can be positive or negative. So, the range of the cube function is all real numbers.

Note:
Cube Function



The next function we will look at does not square or cube the input values, but rather takes the square root of those values.

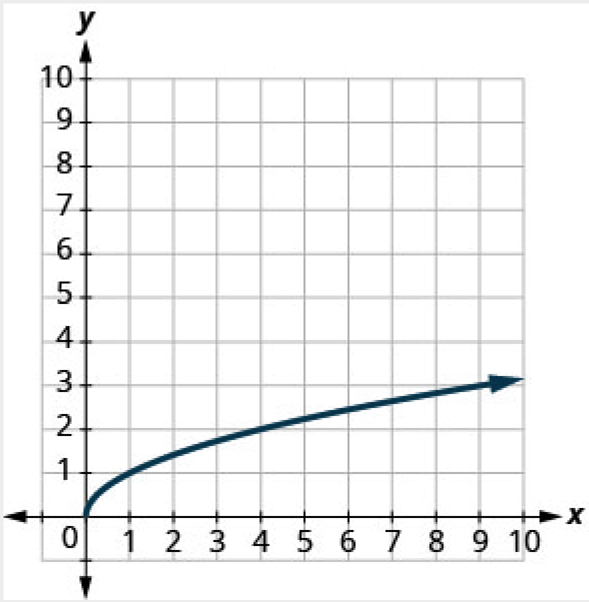
Let's graph the function $f\left(x\right)=\sqrt{x}$ and then summarize the features of the function. Remember, we can only take the square root of non-negative real numbers, so our domain will be the non-negative real numbers.

Example:
Exercise:

Problem: $f\left(x\right)=\sqrt{x}$

Solution:

We choose *x*-values. Since we will be taking the square root, we choose numbers that are perfect squares, to make our work easier. We substitute them in and then create a chart.

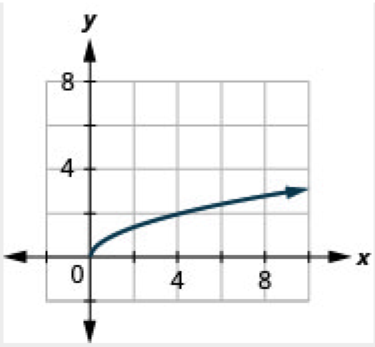


x	$f(x)=\sqrt{x}$	$(x, f(x))$
0	0	(0, 0)
1	1	(1, 1)
4	2	(4, 2)
9	3	(9, 3)

Note:
Exercise:

Problem: Graph: $f\left(x\right)=\sqrt{x}$.

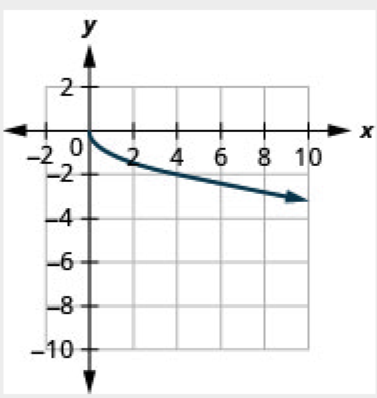
Solution:



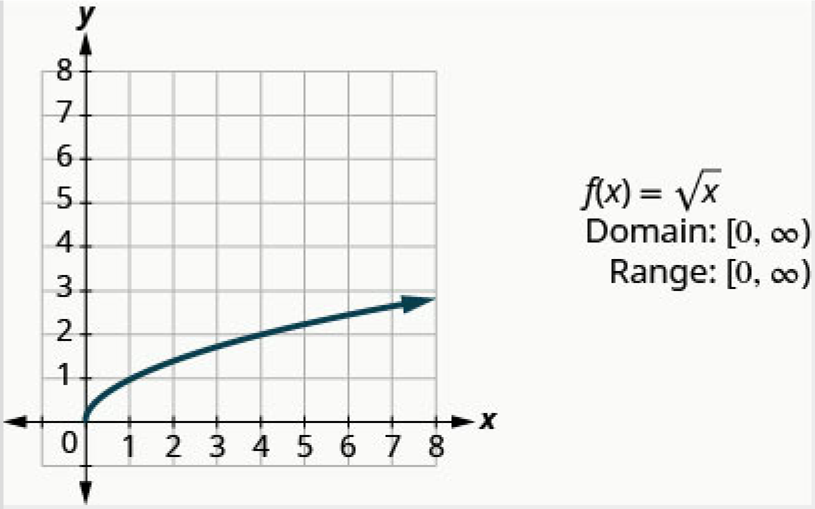
Note:
Exercise:

Problem: Graph: $f\left(x\right)=-\sqrt{x}.$

Solution:



Note:
Square Root Function



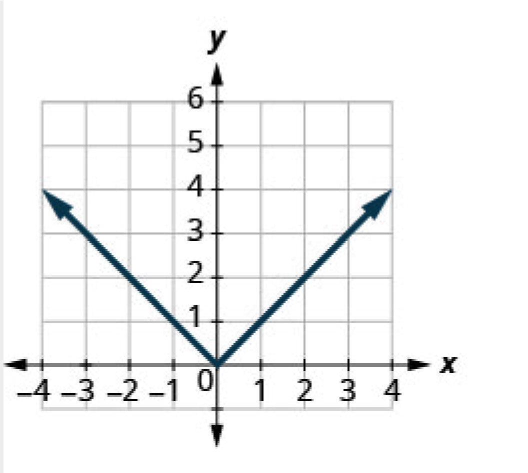
Our last basic function is the absolute value function, $f\left(x\right)=|x|$. Keep in mind that the absolute value of a number is its distance from zero. Since we never measure distance as a negative number, we will never get a negative number in the range.

Example:
Exercise:

Problem: Graph: $f\left(x\right)=|x|$.

Solution:

We choose *x*-values. We substitute them in and then create a chart.

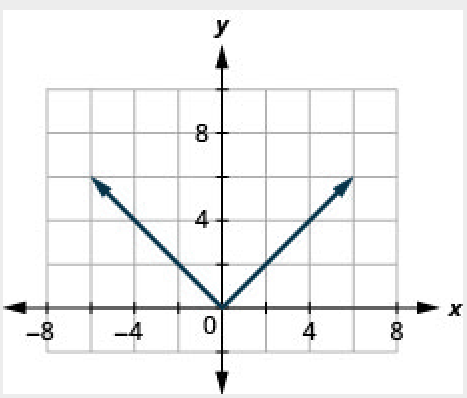


x	$f(x) = \|x\|$	$(x, f(x))$
–3	3	(–3, 3)
–2	2	(–2, 2)
–1	1	(–1, 1)
0	0	(0, 0)
1	1	(1, 1)
2	2	(2, 2)
3	3	(3, 3)

Note:
Exercise:

Problem: Graph: $f(x) = |x|$.

Solution:

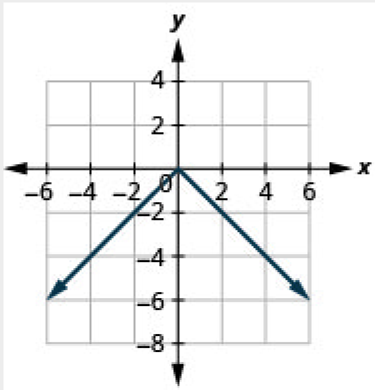


Note:

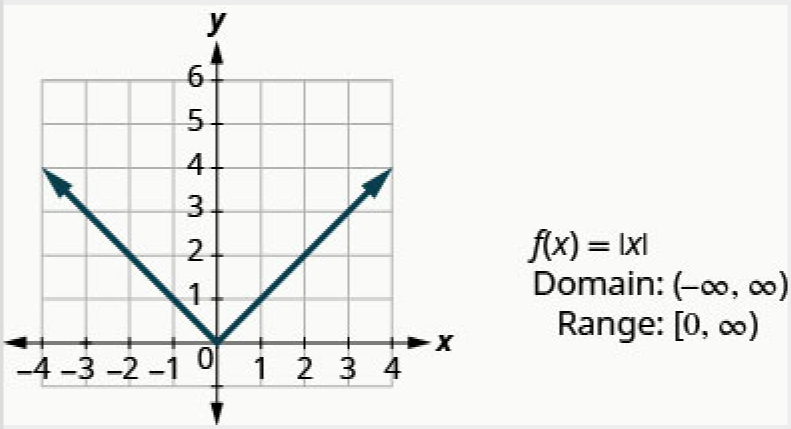
Exercise:

Problem: Graph: $f\left(x\right)=-\left|x\right|.$

Solution:



Note:
Absolute Value Function



Read Information from a Graph of a Function

In the sciences and business, data is often collected and then graphed. The graph is analyzed, information is obtained from the graph and then often predictions are made from the data.

We will start by reading the domain and range of a function from its graph.

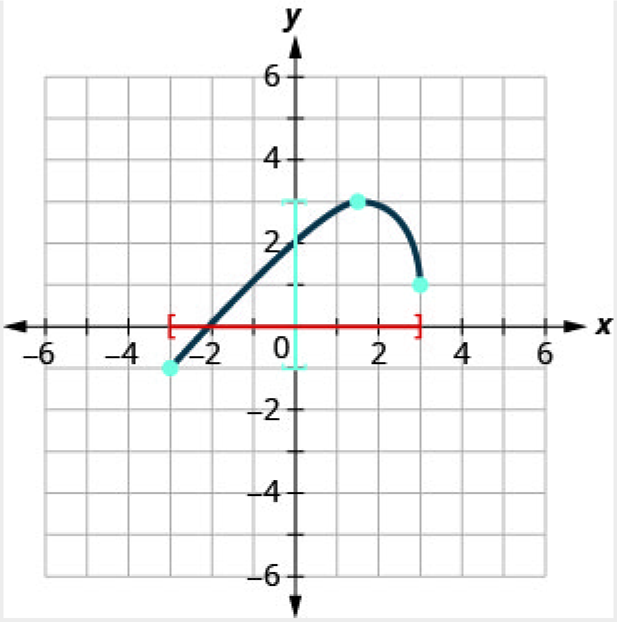
Remember the domain is the set of all the x-values in the ordered pairs in the function. To find the domain we look at the graph and find all the values of x that have a corresponding value on the graph. Follow the value x up or down vertically. If you hit the graph of the function then x is in the domain.

Remember the range is the set of all the y-values in the ordered pairs in the function. To find the range we look at the graph and find all the values of y that have a corresponding value on the graph. Follow the value y left or right horizontally. If you hit the graph of the function then y is in the range.

Example:
Exercise:

Problem:

Use the graph of the function to find its domain and range. Write the domain and range in interval notation.



Solution:

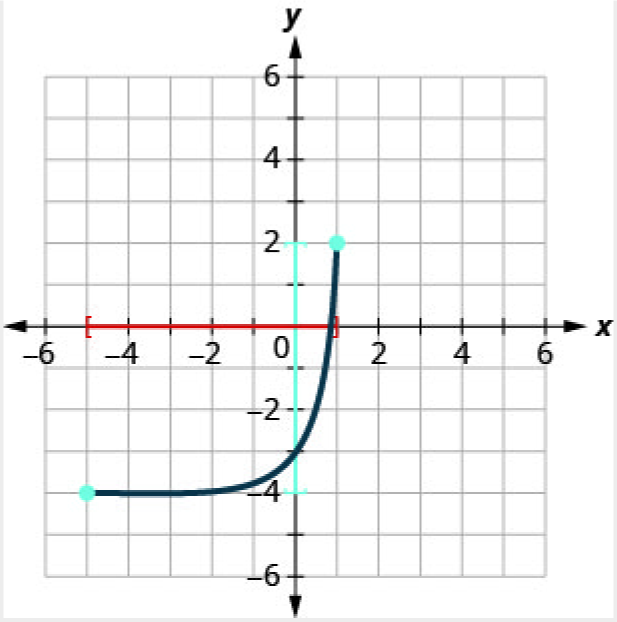
To find the domain we look at the graph and find all the values of x that correspond to a point on the graph. The domain is highlighted in red on the graph. The domain is $[-3, 3]$.

To find the range we look at the graph and find all the values of y that correspond to a point on the graph. The range is highlighted in blue on the graph. The range is $[-1, 3]$.

Note:
Exercise:

Problem:

Use the graph of the function to find its domain and range. Write the domain and range in interval notation.



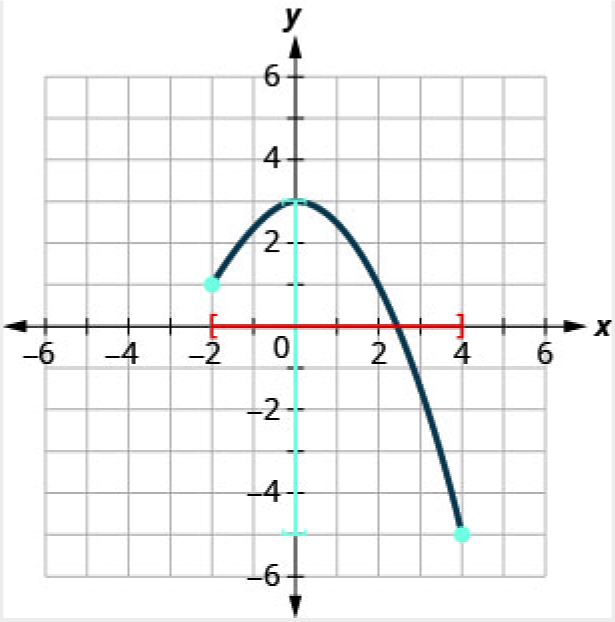
Solution:

The domain is $[-5, 1]$. The range is $[-4, 2]$.

Note:
Exercise:

Problem:

Use the graph of the function to find its domain and range. Write the domain and range in interval notation.



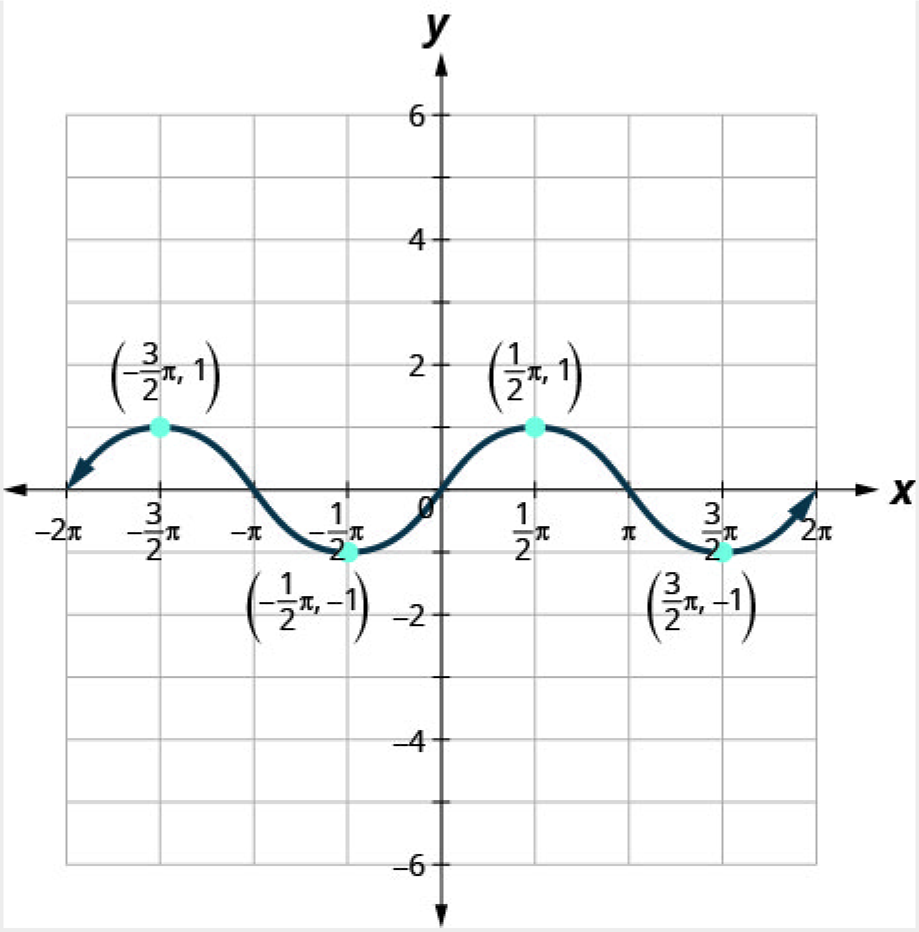
Solution:

The domain is $[-2, 4]$. The range is $[-5, 3]$.

We are now going to read information from the graph that you may see in future math classes.

Example:
Exercise:

Problem: Use the graph of the function to find the indicated values.



ⓐ Find: $f(0)$.
ⓑ Find: $f\left(\frac{3}{2}\pi\right)$.
ⓒ Find: $f\left(-\frac{1}{2}\pi\right)$.
ⓓ Find the values for *x* when $f(x) = 0$.
ⓔ Find the *x*-intercepts.
ⓕ Find the *y*-intercepts.
ⓖ Find the domain. Write it in interval notation.
ⓗ Find the range. Write it in interval notation.

Solution:

ⓐ When $x = 0$, the function crosses the *y*-axis at 0. So, $f(0) = 0$.
ⓑ When $x = \frac{3}{2}\pi$, the *y*-value of the function is -1. So, $f\left(\frac{3}{2}\pi\right) = -1$.
ⓒ When $x = -\frac{1}{2}\pi$, the *y*-value of the function is -1. So,

$f\left(-\frac{1}{2}\pi\right) = -1$.

ⓓ The function is 0 at the points, $(-2\pi, 0), (-\pi, 0), (0, 0), (\pi, 0), (2\pi, 0)$. The *x*-values when $f(x) = 0$ are $-2\pi, -\pi, 0, \pi, 2\pi$.

ⓔ The *x*-intercepts occur when $y = 0$. So the *x*-intercepts occur when $f(x) = 0$. The *x*-intercepts are $(-2\pi, 0), (-\pi, 0), (0, 0), (\pi, 0), (2\pi, 0)$.

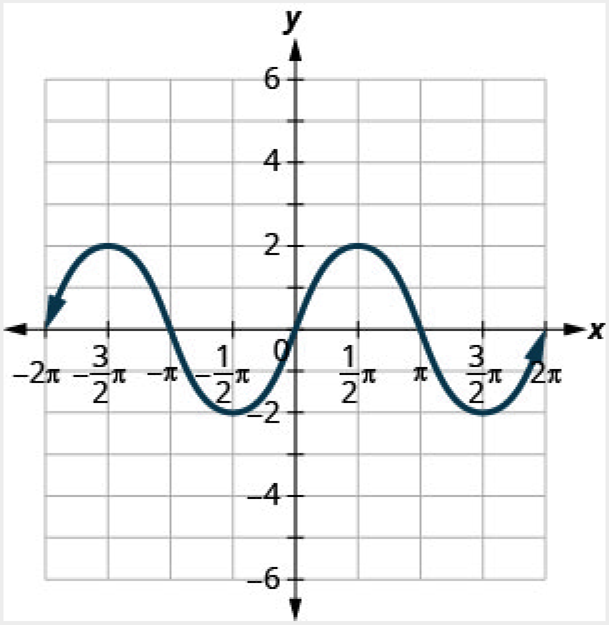
ⓕ The *y*-intercepts occur when $x = 0$. So the *y*-intercepts occur at $f(0)$. The *y*-intercept is $(0, 0)$.

ⓖ This function has a value when *x* is from -2π to 2π. Therefore, the domain in interval notation is $[-2\pi, 2\pi]$.

ⓗ This function values, or *y*-values go from -1 to 1. Therefore, the range, in interval notation, is $[-1, 1]$.

Note:
Exercise:

Problem: Use the graph of the function to find the indicated values.



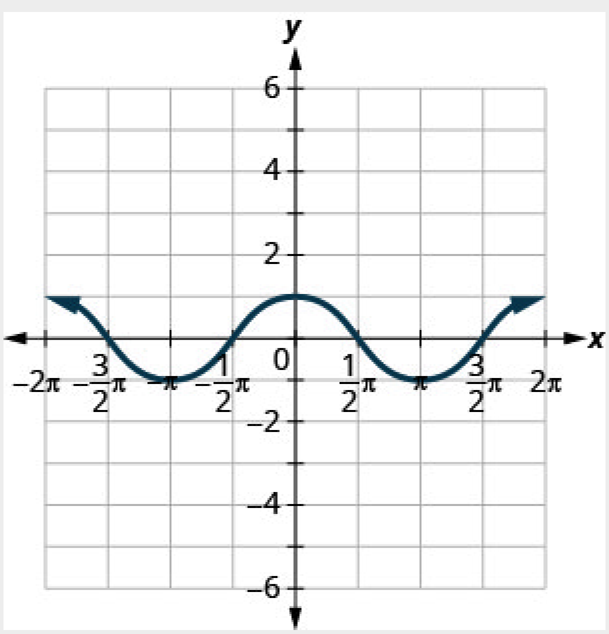
ⓐ Find: $f(0)$.
ⓑ Find: $f\left(\frac{1}{2}\pi\right)$.
ⓒ Find: $f\left(-\frac{3}{2}\pi\right)$.
ⓓ Find the values for *x* when $f(x) = 0$.
ⓔ Find the *x*-intercepts.
ⓕ Find the *y*-intercepts.
ⓖ Find the domain. Write it in interval notation.
ⓗ Find the range. Write it in interval notation.

Solution:

ⓐ $f(0) = 0$ ⓑ $f = \left(\frac{\pi}{2}\right) = 2$ ⓒ $f = \left(\frac{-3\pi}{2}\right) = 2$ ⓓ $f(x) = 0$ for $x = -2\pi, -\pi, 0, \pi, 2\pi$ ⓔ $(-2\pi, 0), (-\pi, 0), (0, 0), (\pi, 0), (2\pi, 0)$ ⓕ $(0, 0)$ ⓖ $[-2\pi, 2\pi]$ ⓗ $[-2, 2]$

Note:
Exercise:

Problem: Use the graph of the function to find the indicated values.



ⓐ Find: $f(0)$.
ⓑ Find: $f(\pi)$.
ⓒ Find: $f(-\pi)$.
ⓓ Find the values for *x* when $f(x) = 0$.
ⓔ Find the *x*-intercepts.
ⓕ Find the *y*-intercepts.
ⓖ Find the domain. Write it in interval notation.
ⓗ Find the range. Write it in interval notation.

Solution:

ⓐ $f(0) = 1$ ⓑ $f(\pi) = -1$ ⓒ $f(-\pi) = -1$ ⓓ $f(x) = 0$ for $x = -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$ ⓔ $(-2\text{pi}, 0), (-\text{pi}, 0), (0, 0), (\text{pi}, 0), (2\text{pi}, 0)$ ⓕ $(0, 1)$ ⓖ $[-2\text{pi}, 2\text{pi}]$ ⓗ $[-1, 1]$

Note:
Access this online resource for additional instruction and practice with graphs of functions.

- Find Domain and Range

Key Concepts

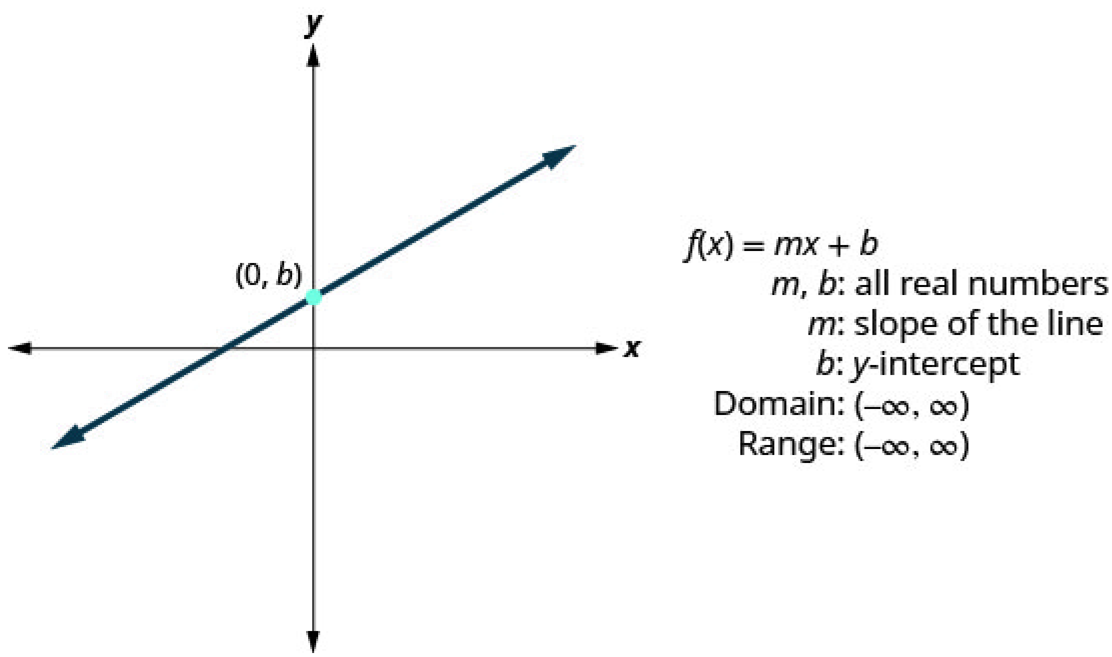
- **Vertical Line Test**
  - A set of points in a rectangular coordinate system is the graph of a function if every vertical line intersects the graph in at most one point.
  - If any vertical line intersects the graph in more than one point, the graph does not represent a function.
- **Graph of a Function**

- The graph of a function is the graph of all its ordered pairs, (x, y) or using function notation, $(x, f(x))$ where $y = f(x)$.

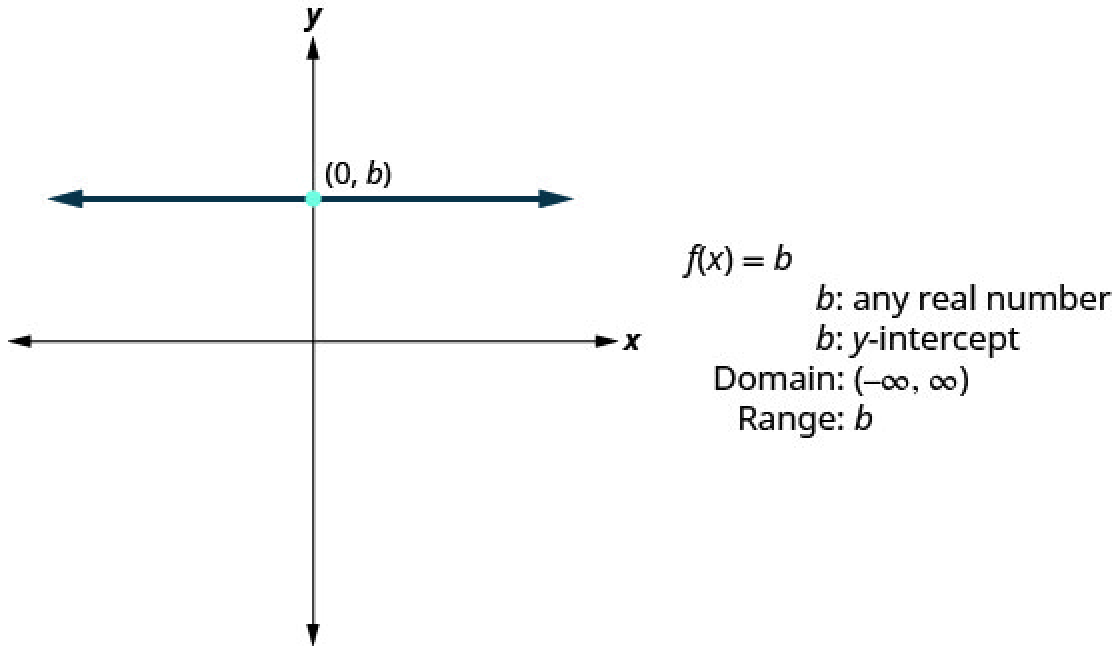
  Equation:

f	name of function
x	x-coordinate of the ordered pair
$f(x)$	y-coordinate of the ordered pair

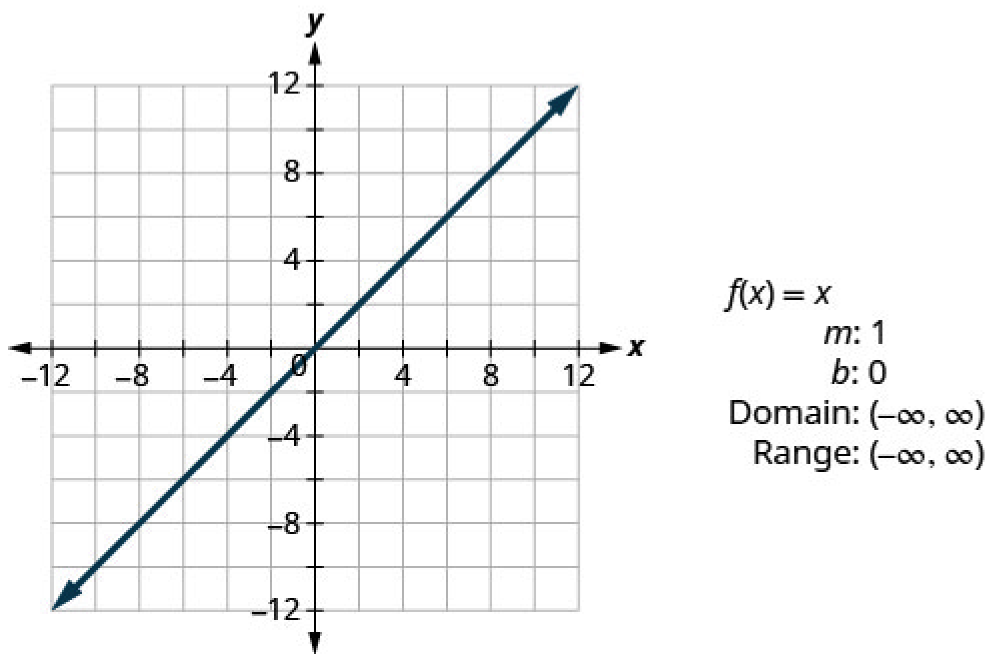
- **Linear Function**



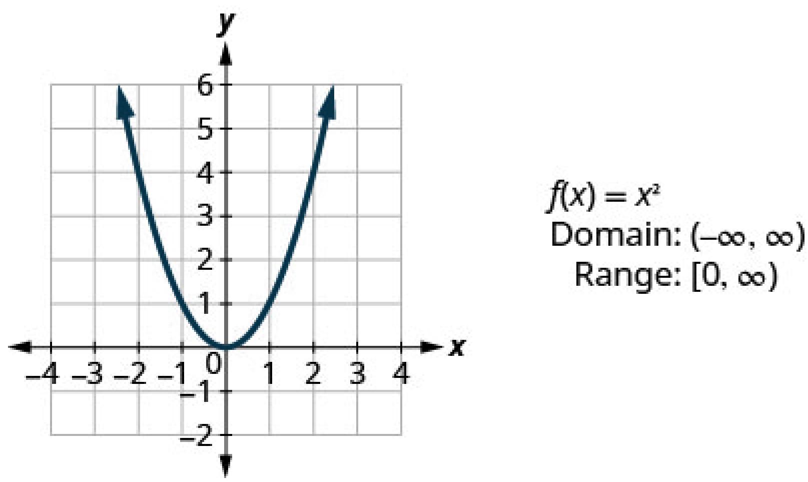
- **Constant Function**



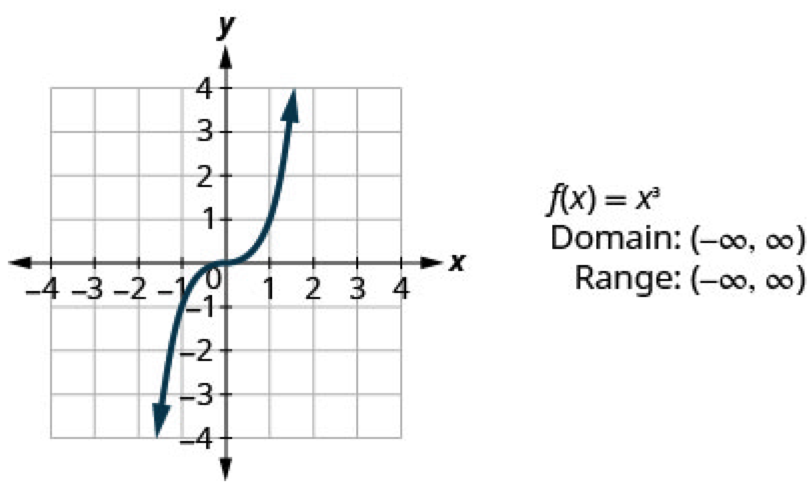
- **Identity Function**



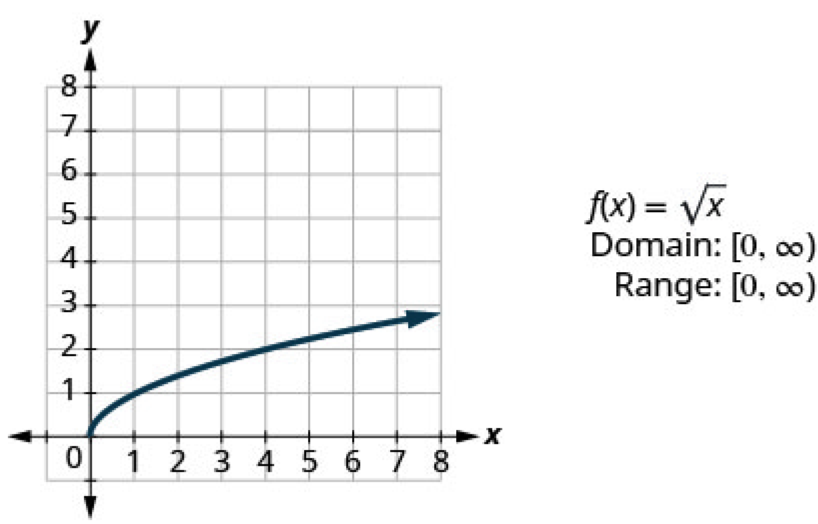
- **Square Function**



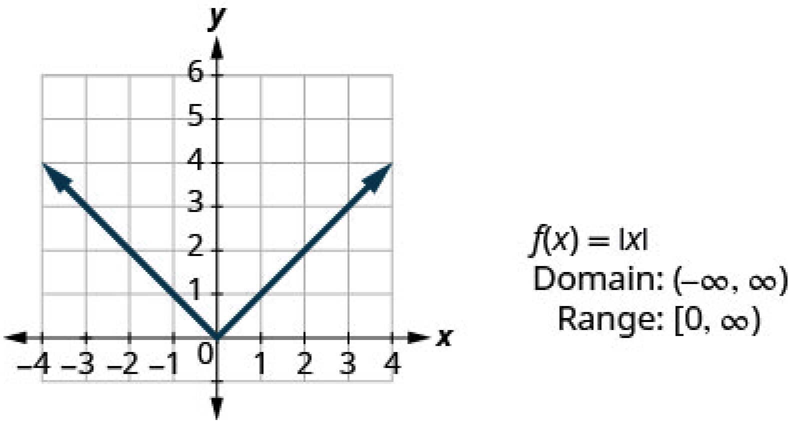
- **Cube Function**



- **Square Root Function**



- **Absolute Value Function**



Section Exercises

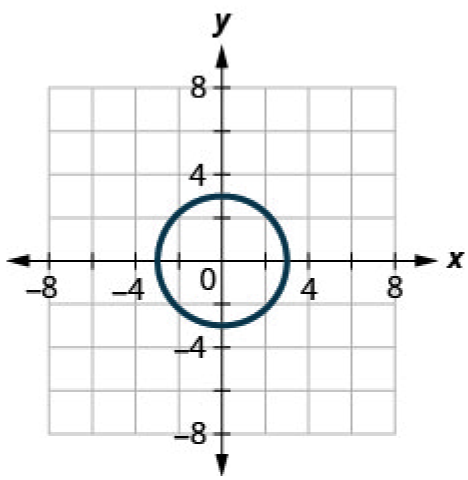
Practice Makes Perfect

Use the Vertical Line Test

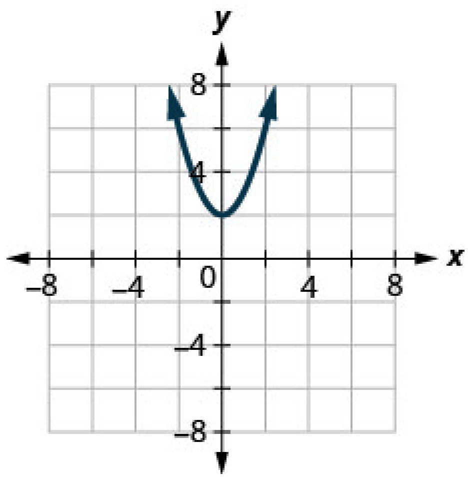
In the following exercises, determine whether each graph is the graph of a function.

Exercise:

Problem: ⓐ



ⓑ

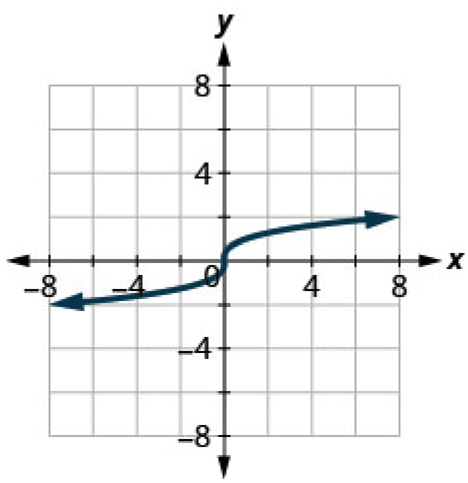


Solution:

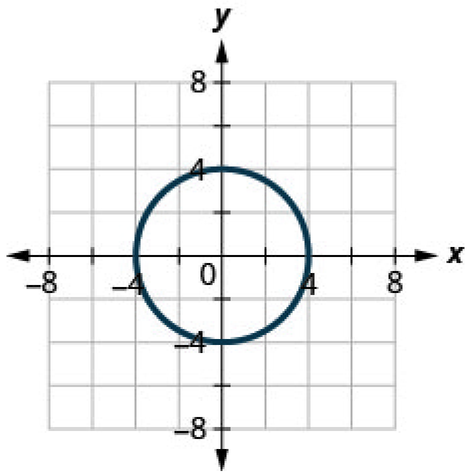
ⓐ no ⓑ yes

Exercise:

Problem: ⓐ

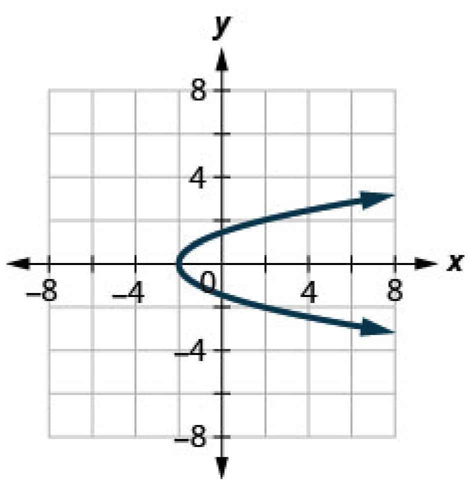


ⓑ

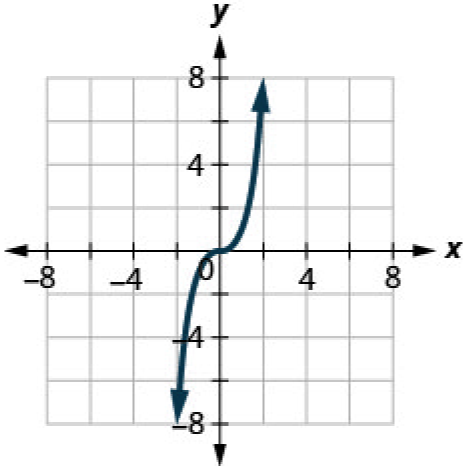


Exercise:

Problem: ⓐ



ⓑ

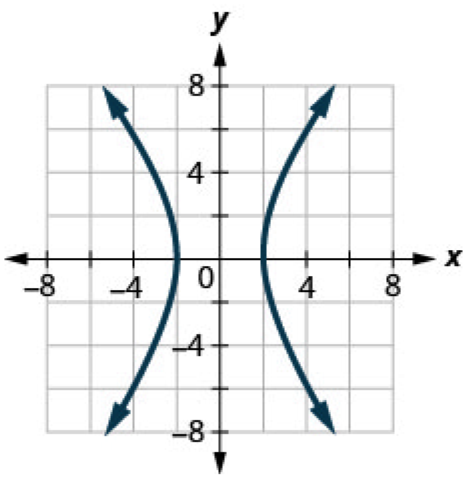


Solution:

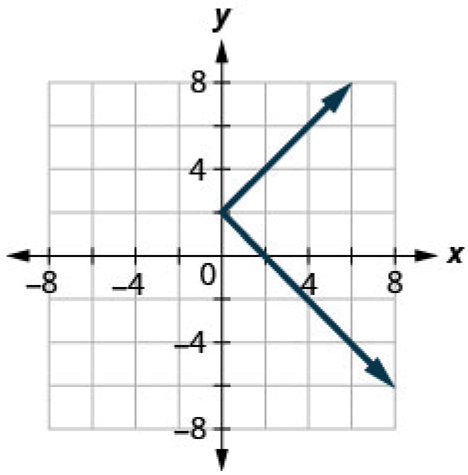
ⓐ no ⓑ yes

Exercise:

Problem: ⓐ



ⓑ



Identify Graphs of Basic Functions

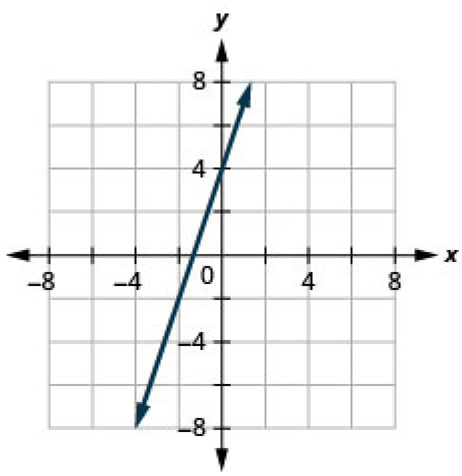
In the following exercises, ⓐ graph each function ⓑ state its domain and range. Write the domain and range in interval notation.

Exercise:

Problem: $f(x) = 3x + 4$

Solution:

ⓐ



ⓑ D:(-∞,∞), R:(-∞,∞)

Exercise:

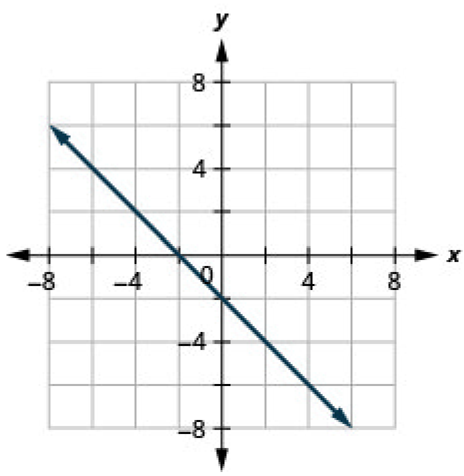
Problem: $f(x) = 2x + 5$

Exercise:

Problem: $f(x) = -x - 2$

Solution:

ⓐ



ⓑ D:(-∞,∞), R:(-∞,∞)

Exercise:

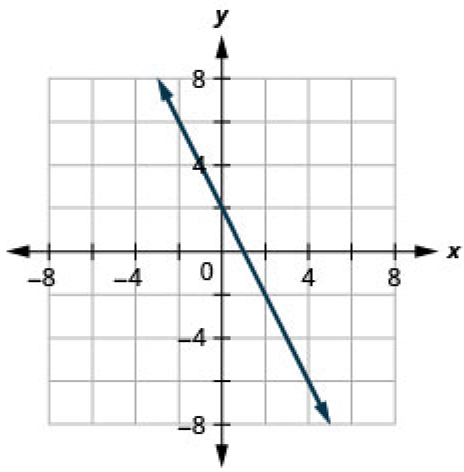
Problem: $f(x) = -4x - 3$

Exercise:

Problem: $f(x) = -2x + 2$

Solution:

ⓐ



ⓑ D:(-∞,∞), R:(-∞,∞)

Exercise:

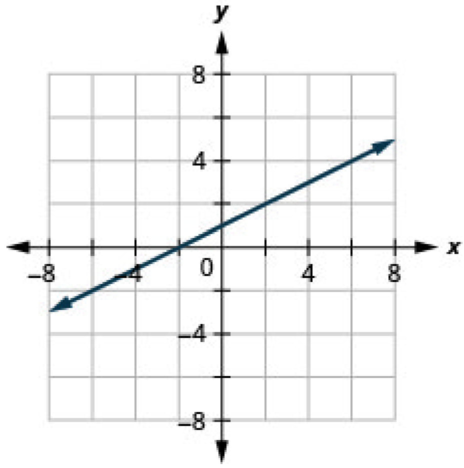
Problem: $f\left(x\right)=-3x+3$

Exercise:

Problem: $f\left(x\right)=\frac{1}{2}x+1$

Solution:

ⓐ



ⓑ D:(-∞,∞), R:(-∞,∞)

Exercise:

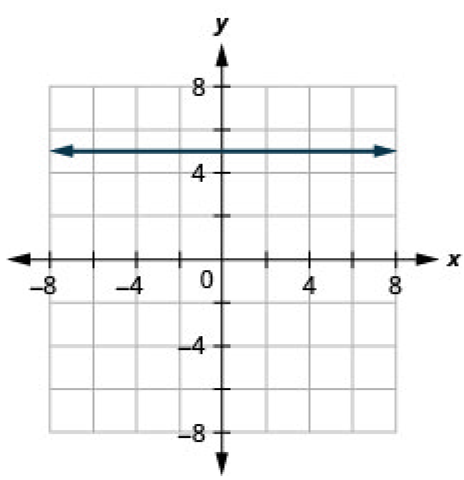
Problem: $f\left(x\right)=\frac{2}{3}x-2$

Exercise:

Problem: $f\left(x\right)=5$

Solution:

ⓐ



ⓑ D:(-∞,∞), R:{5}

Exercise:

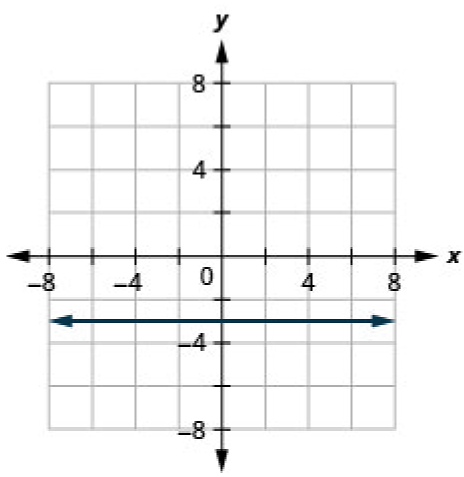
Problem: $f(x) = 2$

Exercise:

Problem: $f(x) = -3$

Solution:

ⓐ



ⓑ D:(-∞,∞), R: $\{-3\}$

Exercise:

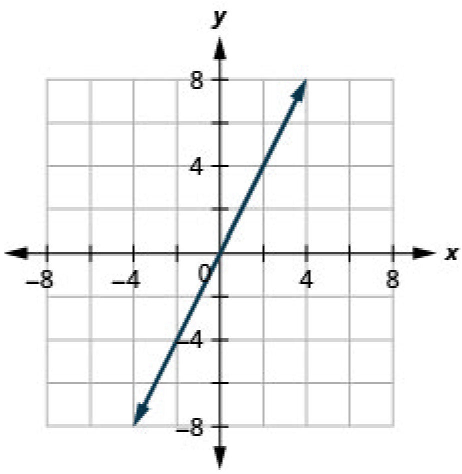
Problem: $f(x) = -1$

Exercise:

Problem: $f(x) = 2x$

Solution:

ⓐ



ⓑ D:(-∞,∞), R:(-∞,∞)

Exercise:

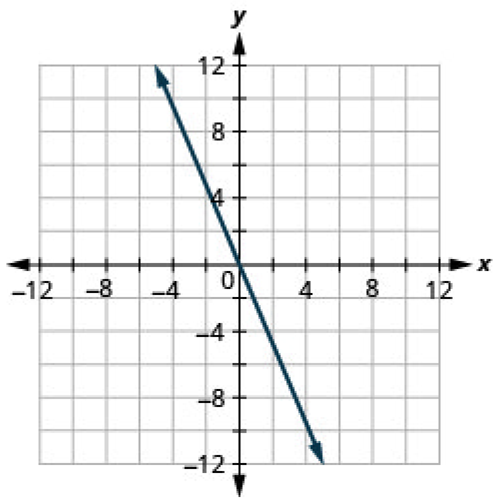
Problem: $f\left(x\right)=3x$

Exercise:

Problem: $f\left(x\right)=-2x$

Solution:

ⓐ



ⓑ D:(-∞,∞), R:(-∞,∞)

Exercise:

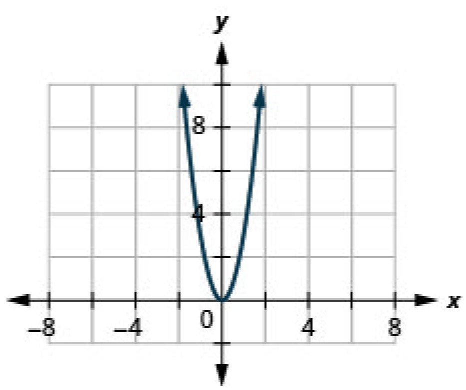
Problem: $f(x) = -3x$

Exercise:

Problem: $f(x) = 3x^2$

Solution:

ⓐ



ⓑ D:(-∞,∞), R:[0,∞)

Exercise:

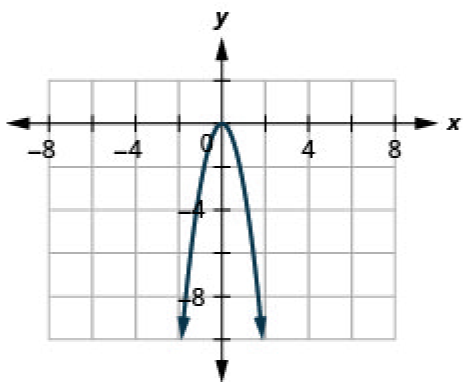
Problem: $f\left(x\right)=2x^2$

Exercise:

Problem: $f\left(x\right)=-3x^2$

Solution:

ⓐ



ⓑ (-∞,∞), R:(-∞,0]

Exercise:

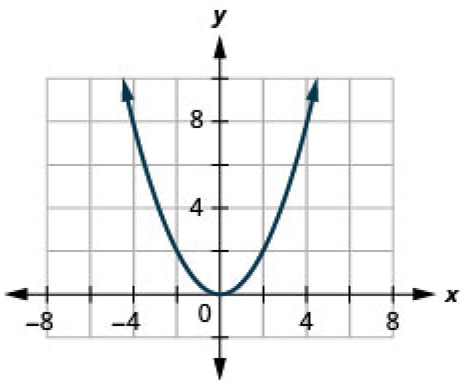
Problem: $f\left(x\right)=-2x^2$

Exercise:

Problem: $f\left(x\right)=\frac{1}{2}x^2$

Solution:

ⓐ



ⓑ (-∞,∞), R:[-∞,0)

Exercise:

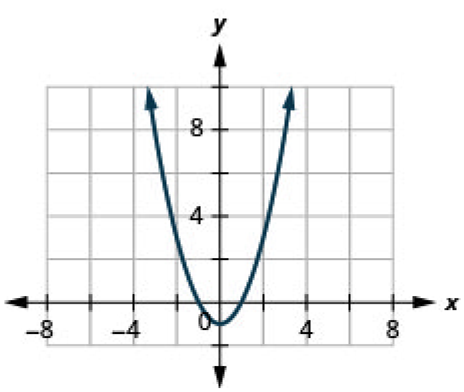
Problem: $f\left(x\right)=\frac{1}{3}x^2$

Exercise:

Problem: $f\left(x\right)=x^2-1$

Solution:

ⓐ



ⓑ (-∞,∞), R:$[-1, \infty)$

Exercise:

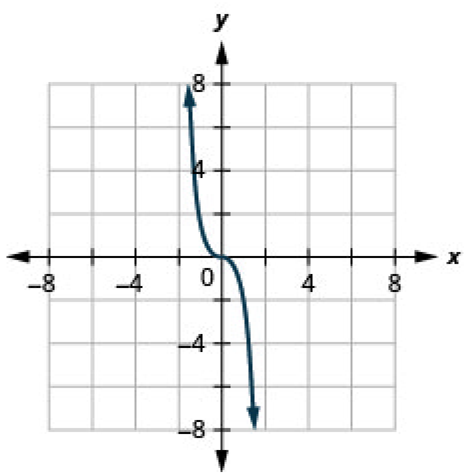
Problem: $f\left(x\right)=x^2+1$

Exercise:

Problem: $f\left(x\right)=-2x^3$

Solution:

ⓐ



ⓑ D:(-∞,∞), R:(-∞,∞)

Exercise:

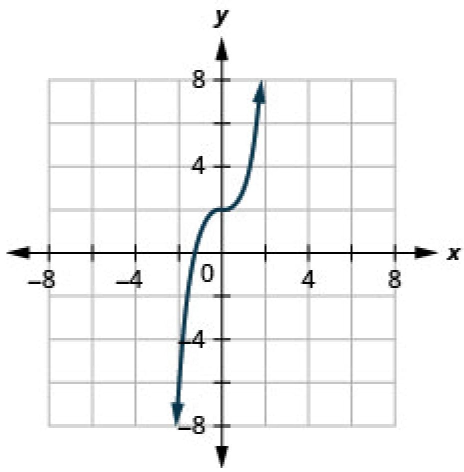
Problem: $f\left(x\right)=2x^3$

Exercise:

Problem: $f\left(x\right)=x^3+2$

Solution:

ⓐ



ⓑ D:(-∞,∞), R:(-∞,∞)

Exercise:

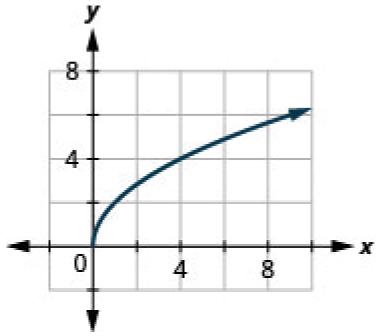
Problem: $f\left(x\right)={x}^{3}-2$

Exercise:

Problem: $f\left(x\right)=2\sqrt{x}$

Solution:

ⓐ



ⓑ D:[0,∞), R:[0,∞)

Exercise:

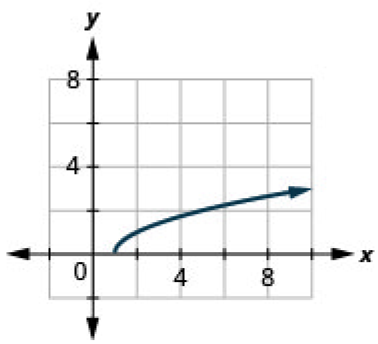
Problem: $f\left(x\right)=-2\sqrt{x}$

Exercise:

Problem: $f\left(x\right)=\sqrt{x-1}$

Solution:

ⓐ



ⓑ D:[1,∞), R:[0,∞)

Exercise:

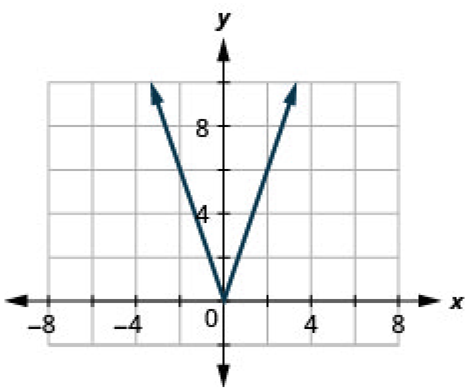
Problem: $f(x) = \sqrt{x+1}$

Exercise:

Problem: $f(x) = 3|x|$

Solution:

ⓐ



ⓑ D:[-1, ∞), R:[−∞,∞)

Exercise:

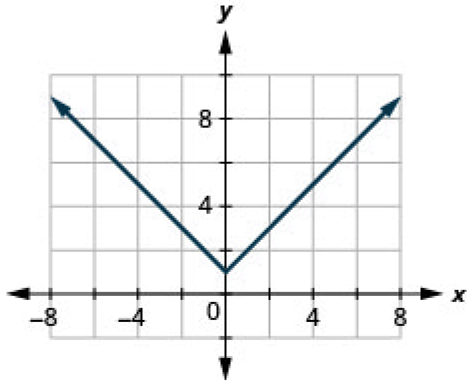
Problem: $f(x) = -2|x|$

Exercise:

Problem: $f(x) = |x| + 1$

Solution:

ⓐ



ⓑ D:(-∞,∞), R:[1,∞)

Exercise:

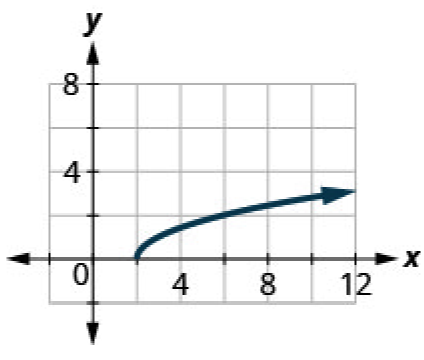
Problem: $f(x) = |x| - 1$

Read Information from a Graph of a Function

In the following exercises, use the graph of the function to find its domain and range. Write the domain and range in interval notation.

Exercise:

Problem:

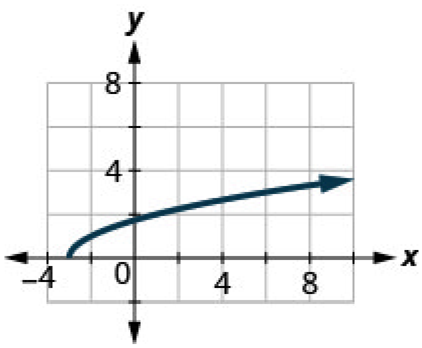


Solution:

D: [2,∞), R: [0,∞)

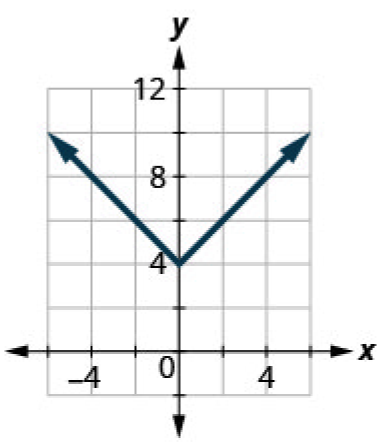
Exercise:

Problem:



Exercise:

Problem:

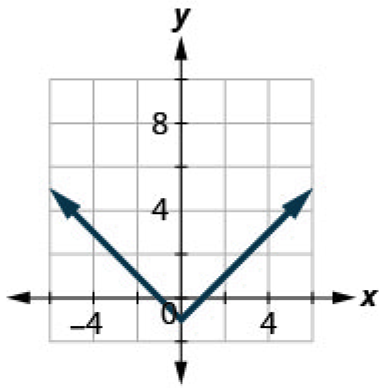


Solution:

D: (-∞,∞), R: [4,∞)

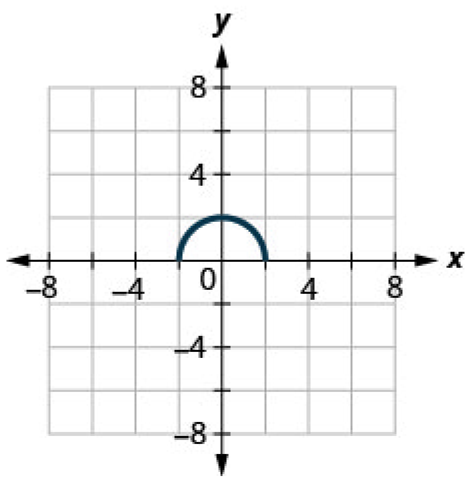
Exercise:

Problem:



Exercise:

Problem:

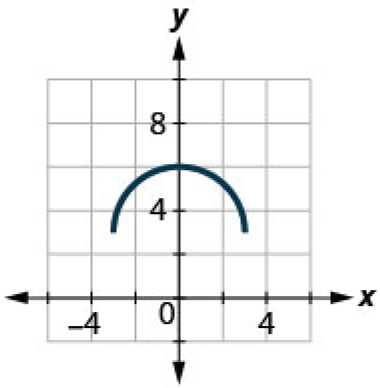


Solution:

D: $[-2, 2]$, R: [0, 2]

Exercise:

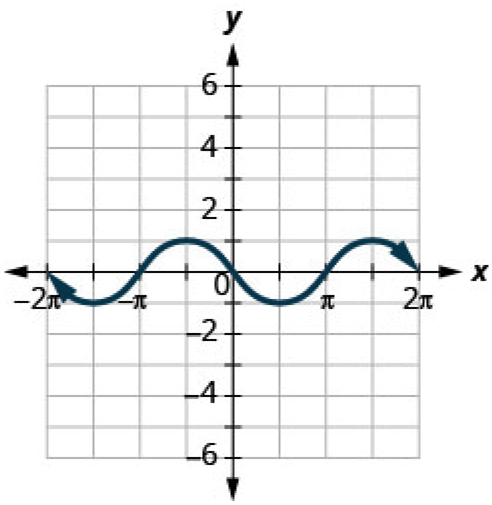
Problem:



In the following exercises, use the graph of the function to find the indicated values.

Exercise:

Problem:



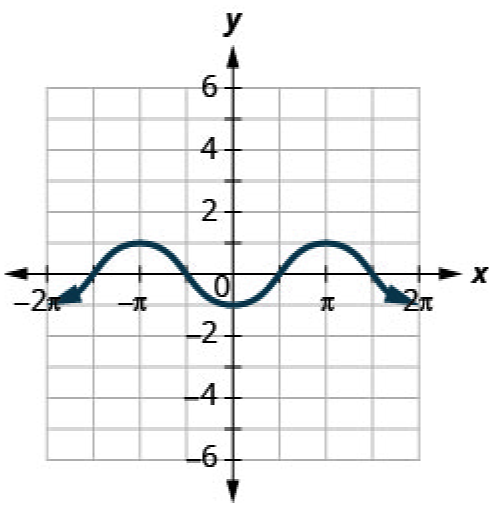
ⓐ Find: $f(0)$.
ⓑ Find: $f\left(\frac{1}{2}\pi\right)$.
ⓒ Find: $f\left(-\frac{3}{2}\pi\right)$.
ⓓ Find the values for *x* when $f(x) = 0$.
ⓔ Find the *x*-intercepts.
ⓕ Find the *y*-intercepts.
ⓖ Find the domain. Write it in interval notation.
ⓗ Find the range. Write it in interval notation.

Solution:

ⓐ $f(0) = 0$ ⓑ $(\text{pi}/2) = -1$
ⓒ $f(-3\text{pi}/2) = -1$ ⓓ $f(x) = 0$ for $x = -2\text{pi}, -\text{pi}, 0, \text{pi}, 2\text{pi}$
ⓔ $(-2\text{pi}, 0), (-\text{pi}, 0), (0, 0), (\text{pi}, 0), (2\text{pi}, 0)$ (f) $(0, 0)$
ⓖ $[-2\text{pi}, 2\text{pi}]$ ⓗ $[-1, 1]$

Exercise:

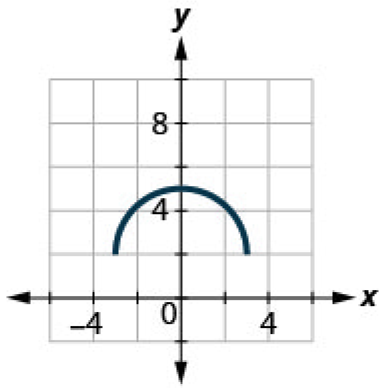
Problem:



ⓐ Find: $f(0)$.
ⓑ Find: $f(\pi)$.
ⓒ Find: $f(-\pi)$.
ⓓ Find the values for *x* when $f(x) = 0$.
ⓔ Find the *x*-intercepts.
ⓕ Find the *y*-intercepts.
ⓖ Find the domain. Write it in interval notation.
ⓗ Find the range. Write it in interval notation

Exercise:

Problem:



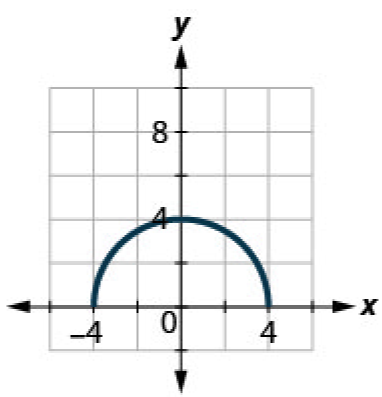
ⓐ Find: $f(0)$.
ⓑ Find: $f(-3)$.
ⓒ Find: $f(3)$.
ⓓ Find the values for *x* when $f(x) = 0$.
ⓔ Find the *x*-intercepts.
ⓕ Find the *y*-intercepts.
ⓖ Find the domain. Write it in interval notation.
ⓗ Find the range. Write it in interval notation.

Solution:

ⓐ $f(0) = -6$ ⓑ $f(-3) = 3$ ⓒ $f(3) = 3$ ⓓ $f(x) = 0$ for no *x* ⓔ none ⓕ $y = 6$ ⓖ $[-3, 3]$
ⓗ $[-3, 6]$

Exercise:

Problem:



ⓐ Find: $f(0)$.
ⓑ Find the values for *x* when $f(x) = 0$.
ⓒ Find the *x*-intercepts.
ⓓ Find the *y*-intercepts.
ⓔ Find the domain. Write it in interval notation.
ⓕ Find the range. Write it in interval notation

Writing Exercises

Exercise:

Problem:

Explain in your own words how to find the domain from a graph.

Exercise:

Problem:

Explain in your own words how to find the range from a graph.

Exercise:

Problem: Explain in your own words how to use the vertical line test.

Exercise:

Problem:

Draw a sketch of the square and cube functions. What are the similarities and differences in the graphs?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
use the vertical line test.			
identify graphs of basic functions.			
read information from a graph.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

Chapter Review Exercises

Graph Linear Equations in Two Variables

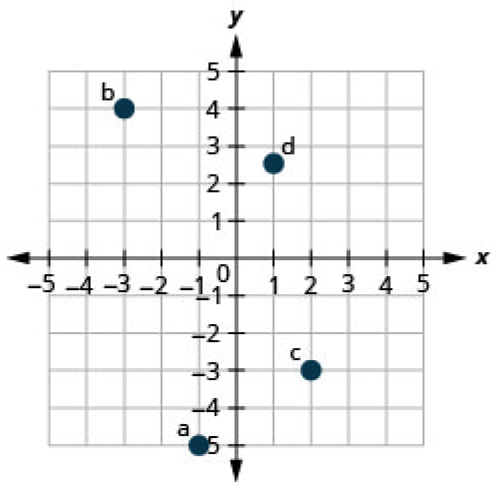
Plot Points in a Rectangular Coordinate System

In the following exercises, plot each point in a rectangular coordinate system.

Exercise:

Problem: ⓐ $(-1, -5)$ ⓑ $(-3, 4)$ ⓒ $(2, -3)$ ⓓ $\left(1, \frac{5}{2}\right)$

Solution:



Exercise:

Problem: ⓐ $(-2,0)$
ⓑ $(0,-4)$
ⓒ $(0,5)$
ⓓ $(3,0)$

In the following exercises, determine which ordered pairs are solutions to the given equations.

Exercise:

Problem: $5x+y=10;$
ⓐ $(5,1)$
ⓑ $(2,0)$
ⓒ $(4,-10)$

Solution:

ⓑ, ⓒ

Exercise:

Problem: $y = 6x - 2$;
ⓐ $(1, 4)$
ⓑ $\left(\frac{1}{3}, 0\right)$
ⓒ $(6, -2)$

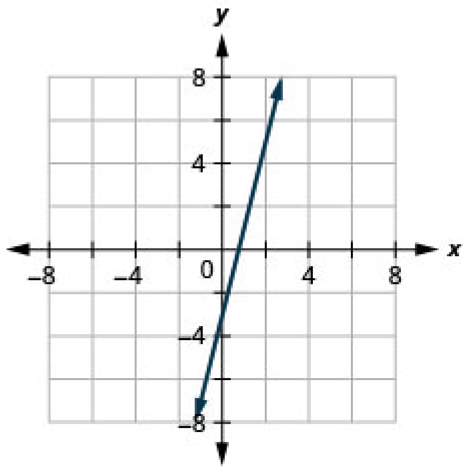
Graph a Linear Equation by Plotting Points

In the following exercises, graph by plotting points.

Exercise:

Problem: $y = 4x - 3$

Solution:



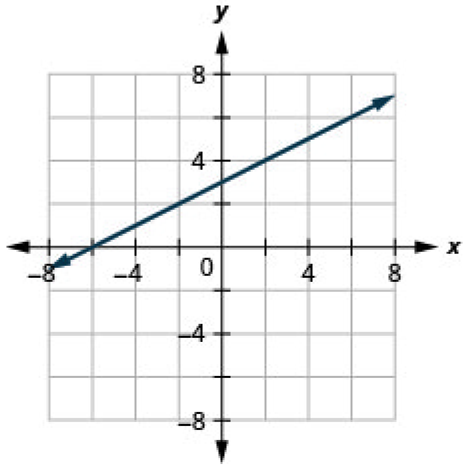
Exercise:

Problem: $y = -3x$

Exercise:

Problem: $y = \frac{1}{2}x + 3$

Solution:



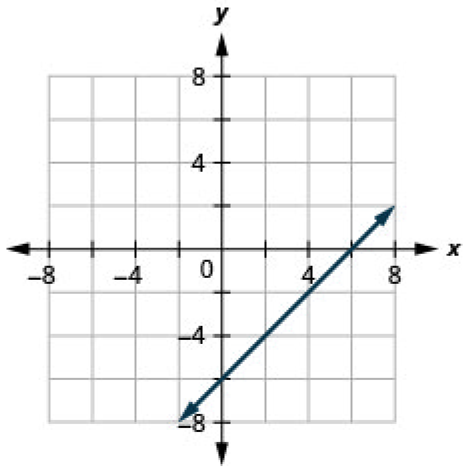
Exercise:

Problem: $y=-\frac{4}{5}x-1$

Exercise:

Problem: $x-y=6$

Solution:



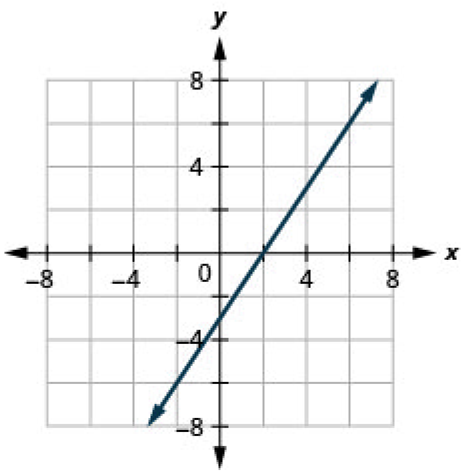
Exercise:

Problem: $2x+y=7$

Exercise:

Problem: $3x - 2y = 6$

Solution:



Graph Vertical and Horizontal lines

In the following exercises, graph each equation.

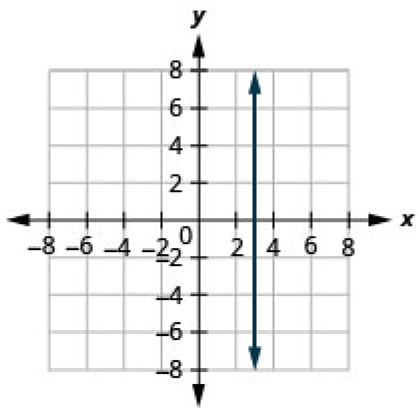
Exercise:

Problem: $y = -2$

Exercise:

Problem: $x = 3$

Solution:



In the following exercises, graph each pair of equations in the same rectangular coordinate system.

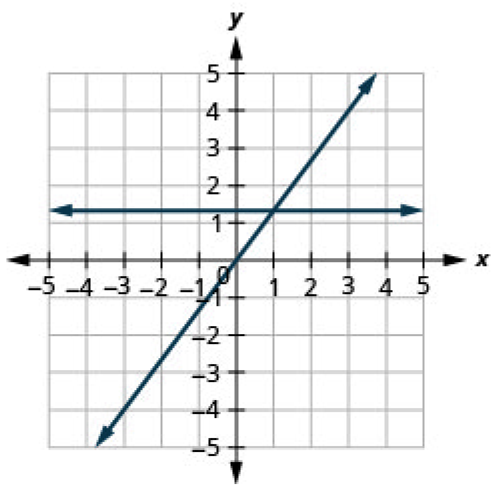
Exercise:

Problem: $y = -2x$ and $y = -2$

Exercise:

Problem: $y = \frac{4}{3}x$ and $y = \frac{4}{3}$

Solution:

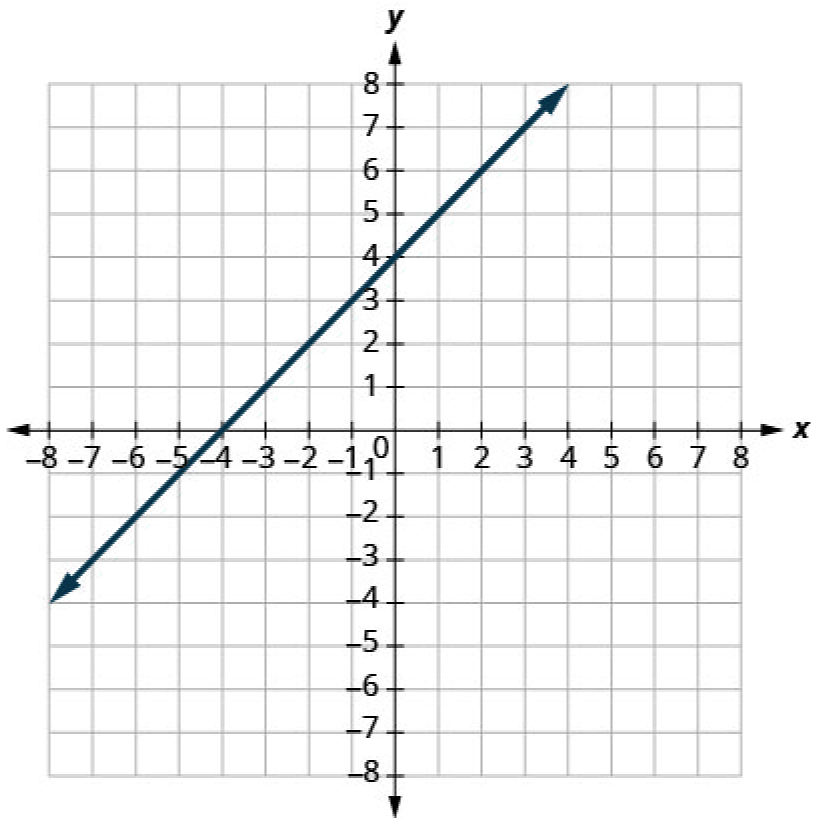


Find *x*- and *y*-Intercepts

In the following exercises, find the *x*- and *y*-intercepts.

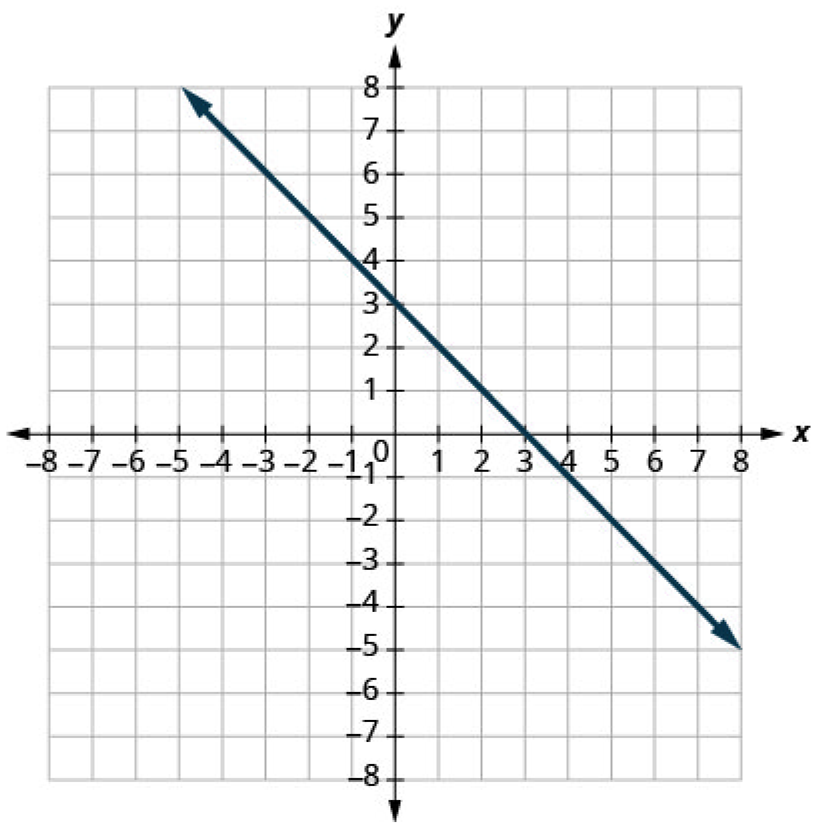
Exercise:

Problem:



Exercise:

Problem:



Solution:

$(0,3)(3,0)$

In the following exercises, find the intercepts of each equation.

Exercise:

Problem: $x - y = -1$

Exercise:

Problem: $x + 2y = 6$

Solution:

$(6,0),(0,3)$

Exercise:

Problem: $2x + 3y = 12$

Exercise:

Problem: $y = \frac{3}{4}x - 12$

Solution:

$(16, 0), (0, -12)$

Exercise:

Problem: $y = 3x$

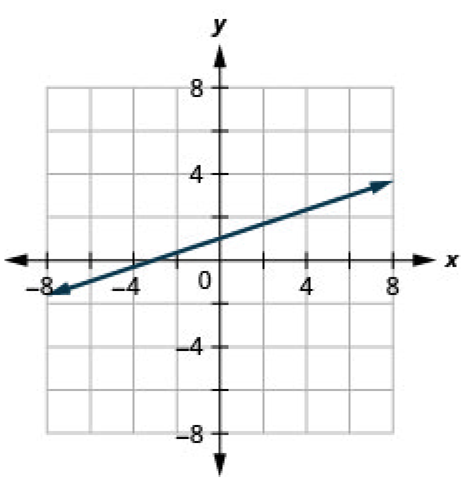
Graph a Line Using the Intercepts

In the following exercises, graph using the intercepts.

Exercise:

Problem: $-x + 3y = 3$

Solution:



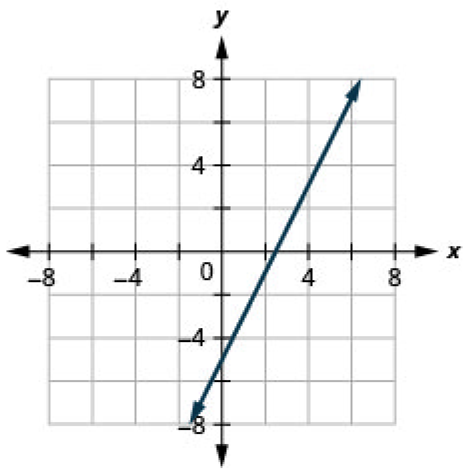
Exercise:

Problem: $x - y = 4$

Exercise:

Problem: $2x - y = 5$

Solution:



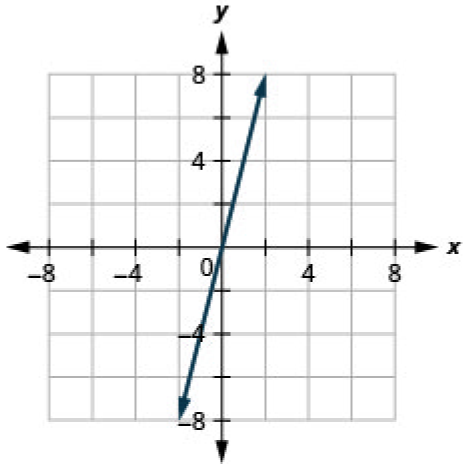
Exercise:

Problem: $2x - 4y = 8$

Exercise:

Problem: $y = 4x$

Solution:



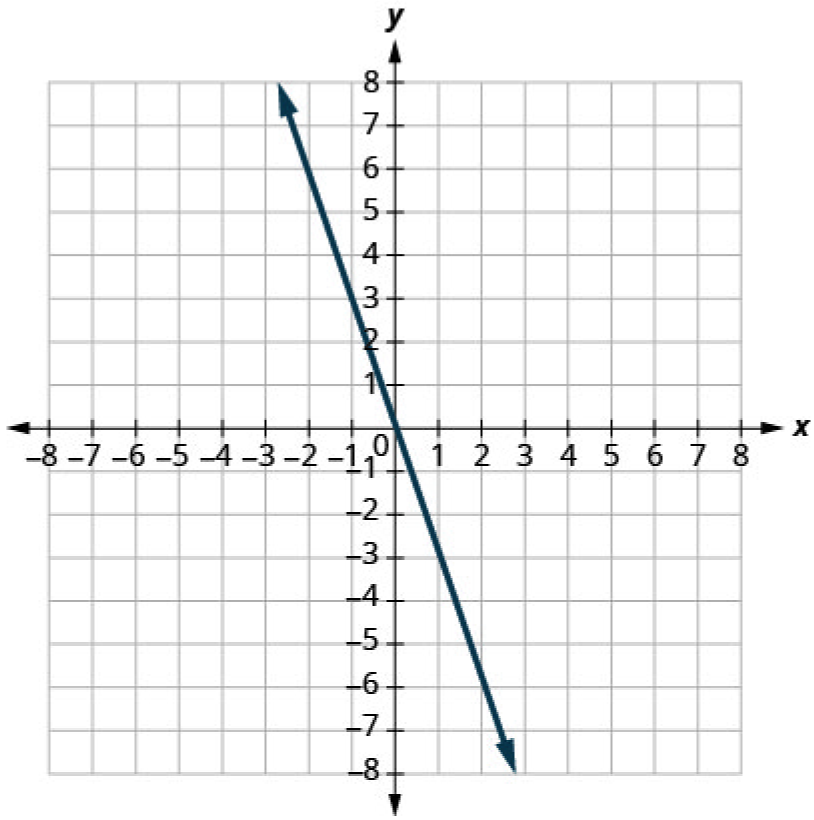
[Slope of a Line](#)

Find the Slope of a Line

In the following exercises, find the slope of each line shown.

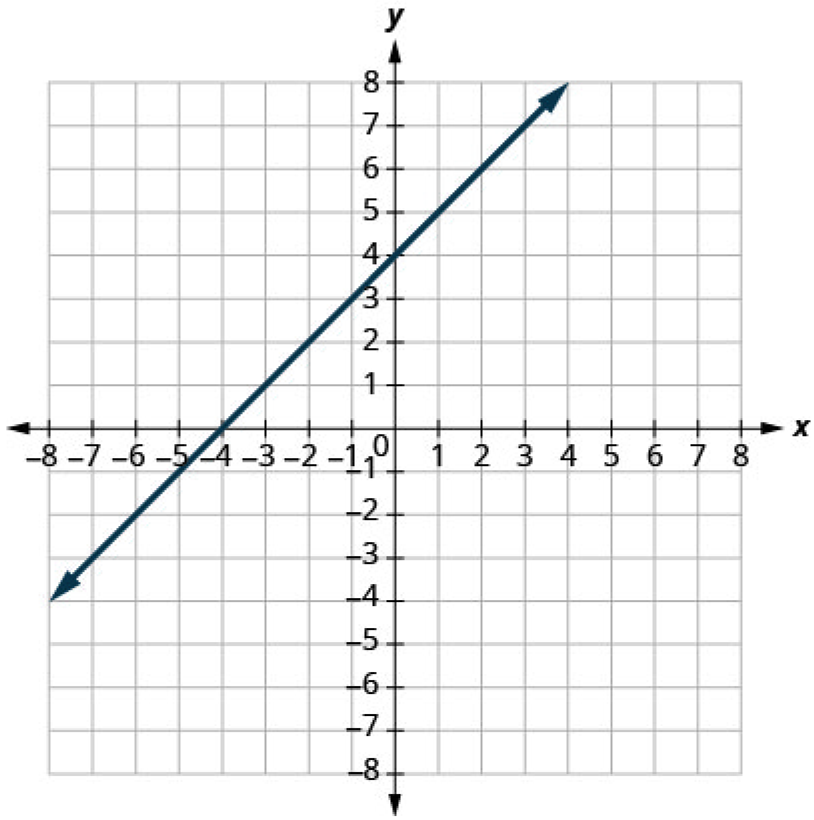
Exercise:

Problem:



Exercise:

Problem:

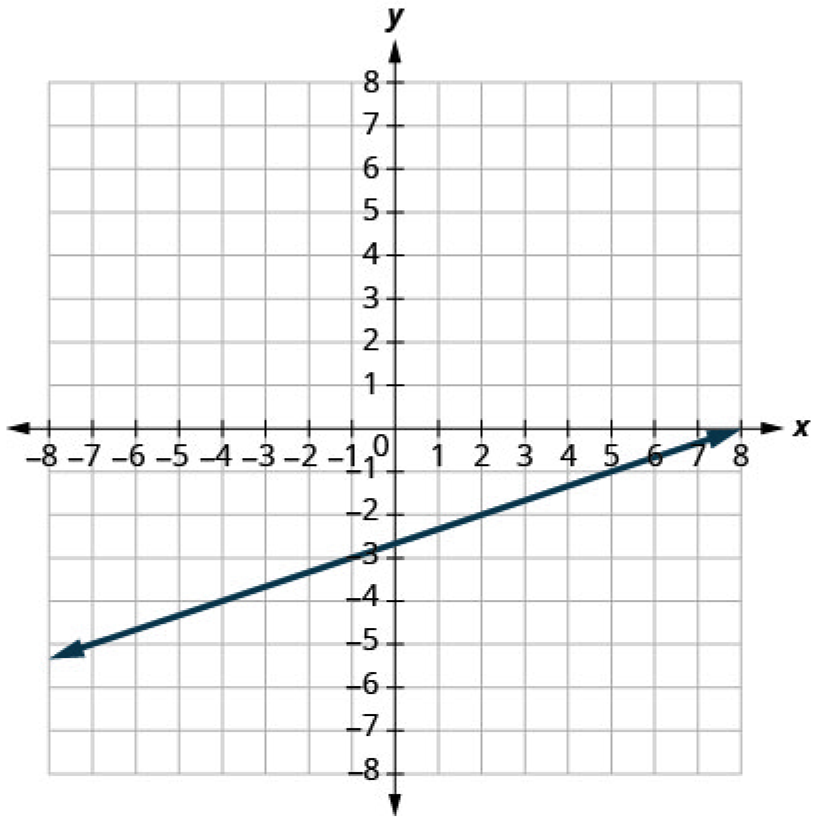


Solution:

1

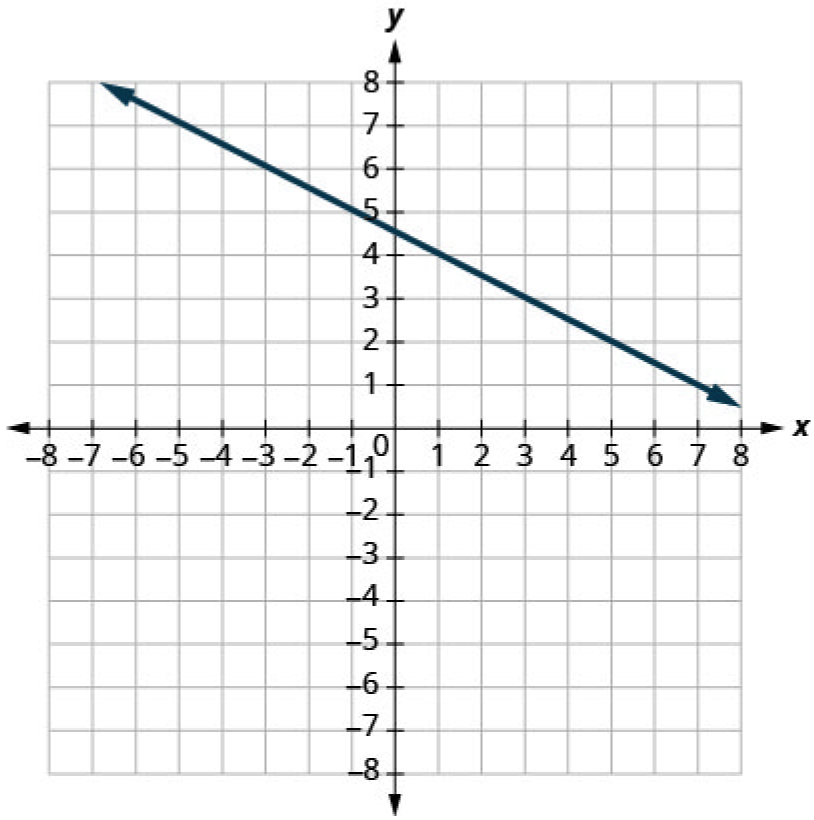
Exercise:

Problem:



Exercise:

Problem:



Solution:

$-\frac{1}{2}$

In the following exercises, find the slope of each line.

Exercise:

Problem: $y = 2$

Exercise:

Problem: $x = 5$

Solution:

undefined

Exercise:

Problem: $x = -3$

Exercise:

Problem: $y = -1$

Solution:

0

Use the Slope Formula to find the Slope of a Line between Two Points

In the following exercises, use the slope formula to find the slope of the line between each pair of points.

Exercise:

Problem: $(-1, -1), (0, 5)$

Exercise:

Problem: $(3.5), (4, -1)$

Solution:

-6

Exercise:

Problem: $(-5, -2), (3, 2)$

Exercise:

Problem: $(2, 1), (4, 6)$

Solution:

$\frac{5}{2}$

Graph a Line Given a Point and the Slope

In the following exercises, graph each line with the given point and slope.

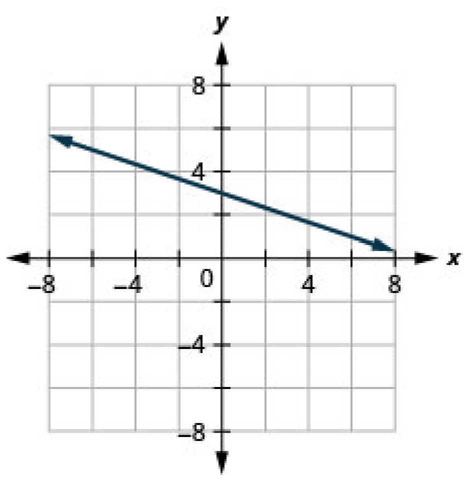
Exercise:

Problem: $(2, -2)$; $m = \frac{5}{2}$

Exercise:

Problem: $(-3, 4)$; $m = -\frac{1}{3}$

Solution:



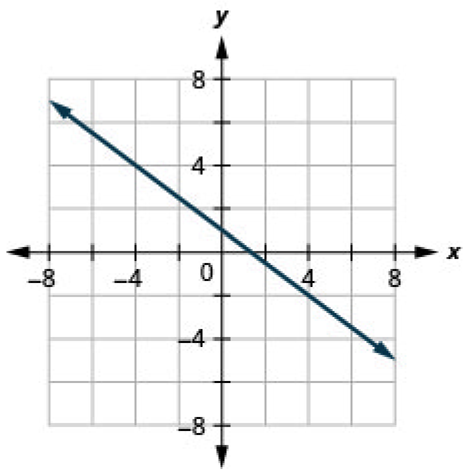
Exercise:

Problem: *x*-intercept -4; $m = 3$

Exercise:

Problem: *y*-intercept 1; $m = -\frac{3}{4}$

Solution:



Graph a Line Using Its Slope and Intercept

In the following exercises, identify the slope and *y*-intercept of each line.

Exercise:

Problem: $y = -4x + 9$

Exercise:

Problem: $y = \frac{5}{3}x - 6$

Solution:

$m = \frac{5}{3}; (0, -6)$

Exercise:

Problem: $5x + y = 10$

Exercise:

Problem: $4x - 5y = 8$

Solution:

$m = \frac{4}{5}$; $\left(0, -\frac{8}{5}\right)$

In the following exercises, graph the line of each equation using its slope and *y*-intercept.

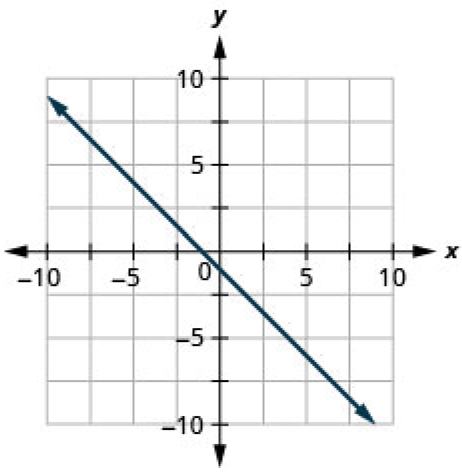
Exercise:

Problem: $y = 2x + 3$

Exercise:

Problem: $y = -x - 1$

Solution:



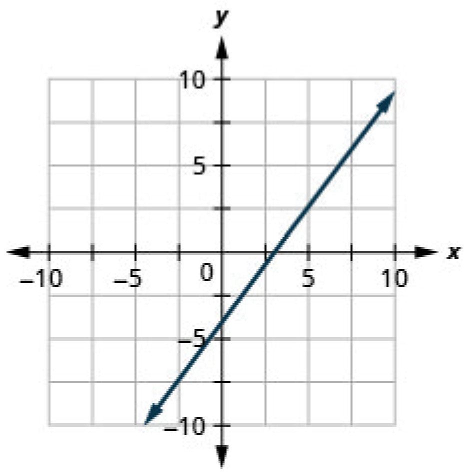
Exercise:

Problem: $y = -\frac{2}{5}x + 3$

Exercise:

Problem: $4x - 3y = 12$

Solution:



In the following exercises, determine the most convenient method to graph each line.

Exercise:

Problem: $x = 5$

Exercise:

Problem: $y = -3$

Solution:

horizontal line

Exercise:

Problem: $2x + y = 5$

Exercise:

Problem: $x - y = 2$

Solution:

intercepts

Exercise:

Problem: $y = \frac{2}{2}x + 2$

Exercise:

Problem: $y = \frac{3}{4}x - 1$

Solution:

plotting points

Graph and Interpret Applications of Slope-Intercept

Exercise:

Problem:

Katherine is a private chef. The equation $C = 6.5m + 42$ models the relation between her weekly cost, *C*, in dollars and the number of meals, *m*, that she serves.

ⓐ Find Katherine’s cost for a week when she serves no meals.
ⓑ Find the cost for a week when she serves 14 meals.
ⓒ Interpret the slope and *C*-intercept of the equation.
ⓓ Graph the equation.

Exercise:

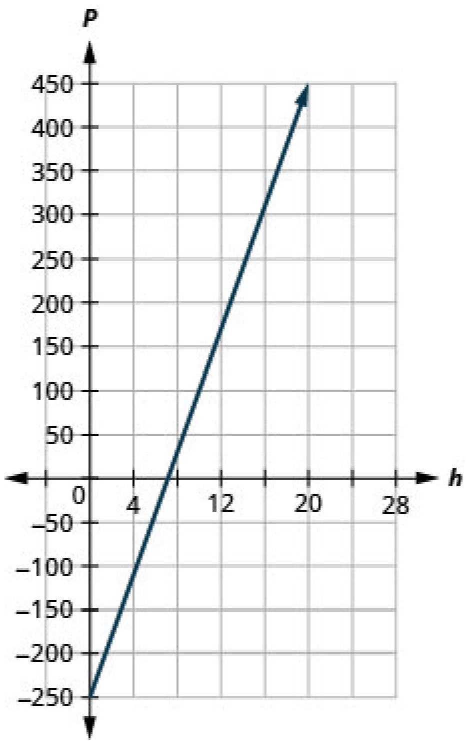
Problem:

Marjorie teaches piano. The equation $P = 35h - 250$ models the relation between her weekly profit, *P*, in dollars and the number of student lessons, *s*, that she teaches.

ⓐ Find Marjorie’s profit for a week when she teaches no student lessons.
ⓑ Find the profit for a week when she teaches 20 student lessons.
ⓒ Interpret the slope and *P*-intercept of the equation.
ⓓ Graph the equation.

Solution:

ⓐ $-\$250$
ⓑ $\$450$
ⓒ The slope, 35, means that Marjorie's weekly profit, *P*, increases by $35 for each additional student lesson she teaches.
The *P*-intercept means that when the number of lessons is 0, Marjorie loses $250.
ⓓ



Use Slopes to Identify Parallel and Perpendicular Lines

In the following exercises, use slopes and y-intercepts to determine if the lines are parallel, perpendicular, or neither.

Exercise:

Problem: $4x - 3y = -1; y = \frac{4}{3}x - 3$

Exercise:

Problem: $y = 5x - 1; 10x + 2y = 0$

Solution:

neither

Exercise:

Problem: $3x - 2y = 5; 2x + 3y = 6$

Exercise:

Problem: $2x - y = 8; x - 2y = 4$

Solution:

not parallel

Find the Equation of a Line

Find an Equation of the Line Given the Slope and *y*-Intercept

In the following exercises, find the equation of a line with given slope and y-intercept. Write the equation in slope–intercept form.

Exercise:

Problem: slope $\frac{1}{3}$ and y-intercept $(0, -6)$

Exercise:

Problem: slope -5 and y-intercept $(0, -3)$

Solution:

$y = -5x - 3$

Exercise:

Problem: slope 0 and y-intercept $(0, 4)$

Exercise:

Problem: slope -2 and y-intercept $(0, 0)$

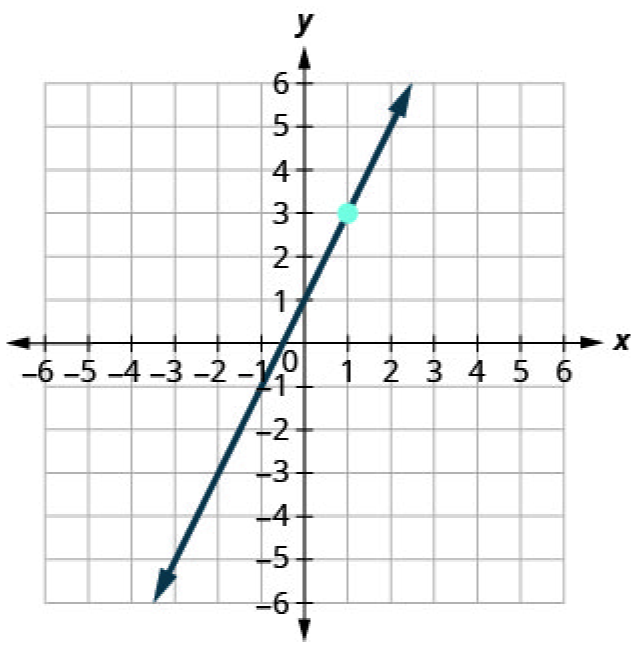
Solution:

$y = -2x$

In the following exercises, find the equation of the line shown in each graph. Write the equation in slope–intercept form.

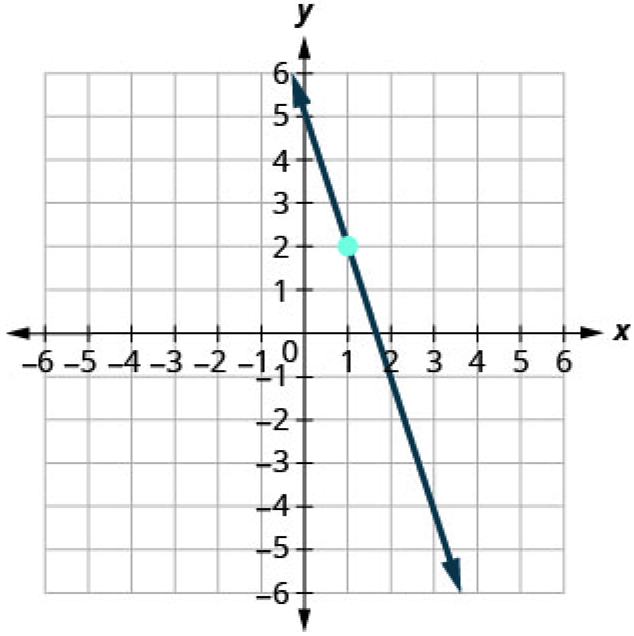
Exercise:

Problem:



Exercise:

Problem:

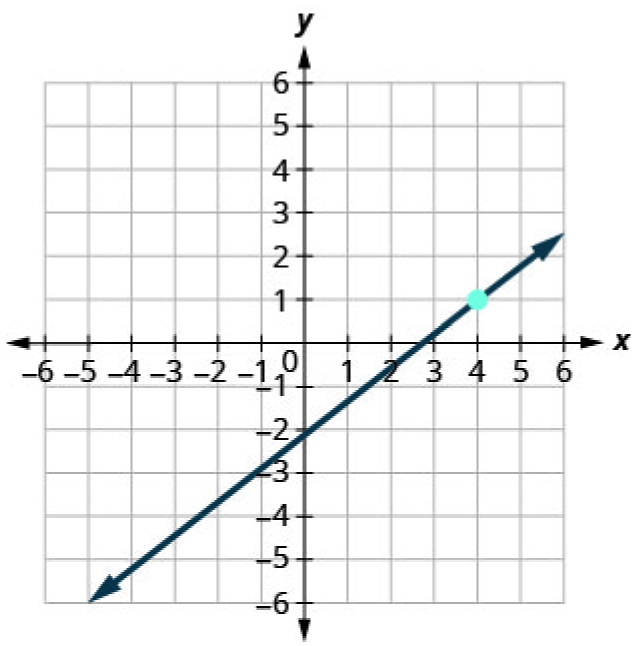


Solution:

$y = -3x + 5$

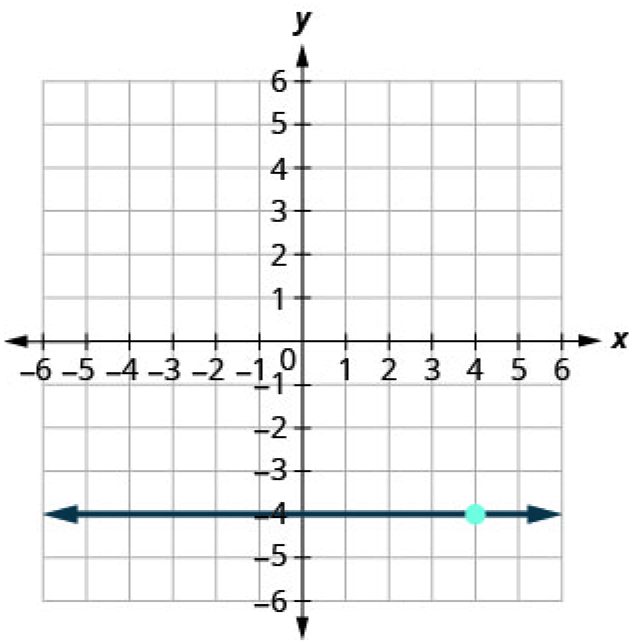
Exercise:

Problem:



Exercise:

Problem:



Solution:

$y = -4$

Find an Equation of the Line Given the Slope and a Point

In the following exercises, find the equation of a line with given slope and containing the given point. Write the equation in slope–intercept form.

Exercise:

Problem: $m = -\frac{1}{4}$, point $(-8, 3)$

Exercise:

Problem: $m = \frac{3}{5}$, point $(10, 6)$

Solution:

$y = \frac{3}{5}x$

Exercise:

Problem: Horizontal line containing $(-2, 7)$

Exercise:

Problem: $m = -2$, point $(-1, -3)$

Solution:

$y = -2x - 5$

Find an Equation of the Line Given Two Points

In the following exercises, find the equation of a line containing the given points. Write the equation in slope–intercept form.

Exercise:

Problem: $(2, 10)$ and $(-2, -2)$

Exercise:

Problem: $(7, 1)$ and $(5, 0)$

Solution:

$y = \frac{1}{2}x - \frac{5}{2}$

Exercise:

Problem: $(3, 8)$ and $(3, -4)$

Exercise:

Problem: $(5, 2)$ and $(-1, 2)$

Solution:

$y = 2$

Find an Equation of a Line Parallel to a Given Line

In the following exercises, find an equation of a line parallel to the given line and contains the given point. Write the equation in slope–intercept form.

Exercise:

Problem: line $y = -3x + 6$, point $(1, -5)$

Exercise:

Problem: line $2x + 5y = -10$, point $(10, 4)$

Solution:

$y = -\frac{2}{5}x + 8$

Exercise:

Problem: line $x = 4$, point $(-2, -1)$

Exercise:

Problem: line $y = -5$, point $(-4, 3)$

Solution:

$y = 3$

Find an Equation of a Line Perpendicular to a Given Line

In the following exercises, find an equation of a line perpendicular to the given line and contains the given point. Write the equation in slope–

intercept form.
Exercise:

Problem: line $y=-\frac{4}{5}x+2$, point $(8,9)$

Exercise:

Problem: line $2x-3y=9$, point $(-4,0)$

Solution:

$y=-\frac{3}{2}x-6$

Exercise:

Problem: line $y=3$, point $(-1,-3)$

Exercise:

Problem: line $x=-5$ point $(2,1)$

Solution:

$y=1$

Graph Linear Inequalities in Two Variables

Verify Solutions to an Inequality in Two Variables

In the following exercises, determine whether each ordered pair is a solution to the given inequality.
Exercise:

Problem:

Determine whether each ordered pair is a solution to the inequality $y < x - 3$:

ⓐ $(0, 1)$ ⓑ $(-2, -4)$ ⓒ $(5, 2)$ ⓓ $(3, -1)$
ⓔ $(-1, -5)$

Exercise:

Problem:

Determine whether each ordered pair is a solution to the inequality $x + y > 4$:

ⓐ $(6, 1)$ ⓑ $(-3, 6)$ ⓒ $(3, 2)$ ⓓ $(-5, 10)$ ⓔ $(0, 0)$

Solution:

ⓐ yes ⓑ no ⓒ yes ⓓ yes; ⓔ no

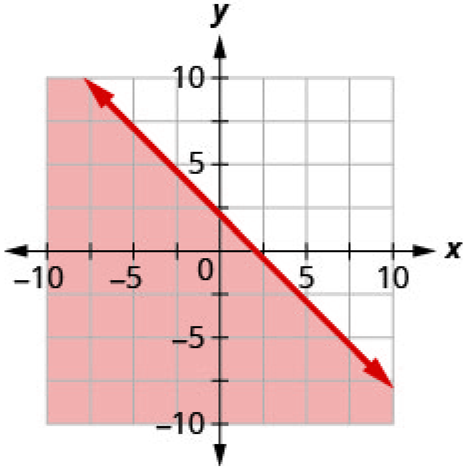
Recognize the Relation Between the Solutions of an Inequality and its Graph

In the following exercises, write the inequality shown by the shaded region.

Exercise:

Problem:

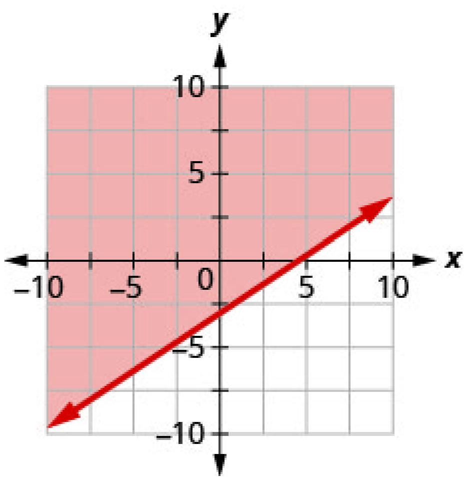
Write the inequality shown by the graph with the boundary line $y = -x + 2$.



Exercise:

Problem:

Write the inequality shown by the graph with the boundary line $y = \frac{2}{3}x - 3.$



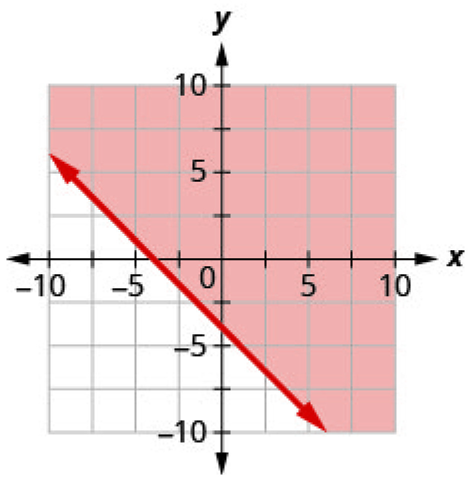
Solution:

$y > \frac{2}{3}x - 3$

Exercise:

Problem:

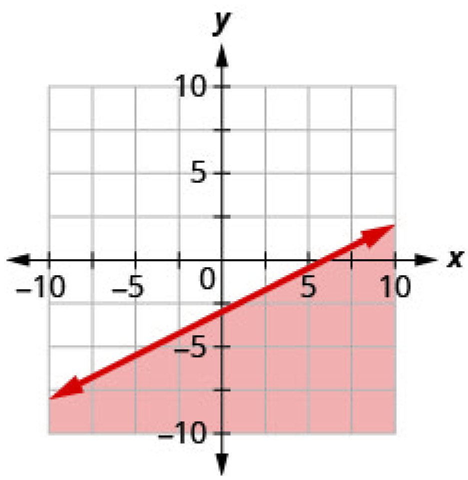
Write the inequality shown by the shaded region in the graph with the boundary line $x + y = -4$.



Exercise:

Problem:

Write the inequality shown by the shaded region in the graph with the boundary line $x - 2y = 6$.



Solution:

$x - 2y \geq 6$

Graph Linear Inequalities in Two Variables

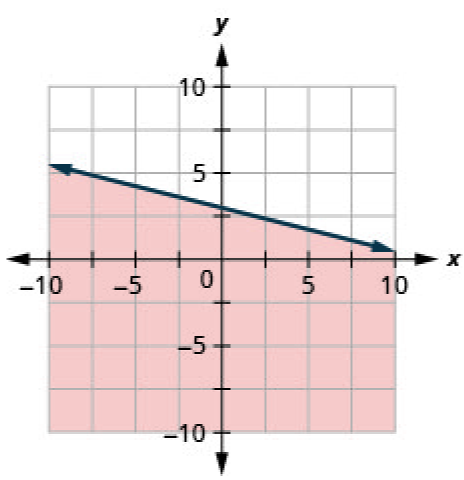
In the following exercises, graph each linear inequality.
Exercise:

Problem: Graph the linear inequality $y > \frac{2}{5}x - 4$.

Exercise:

Problem: Graph the linear inequality $y \leq -\frac{1}{4}x + 3$.

Solution:



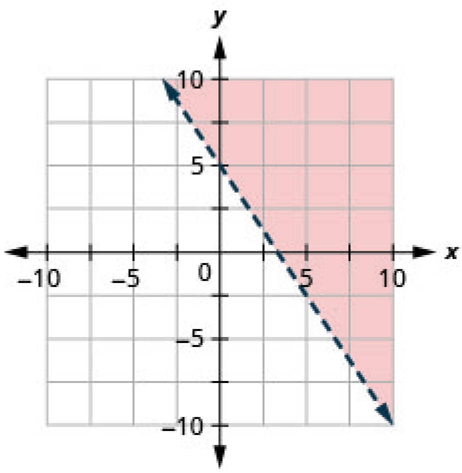
Exercise:

Problem: Graph the linear inequality $x - y \leq 5$.

Exercise:

Problem: Graph the linear inequality $3x + 2y > 10$.

Solution:



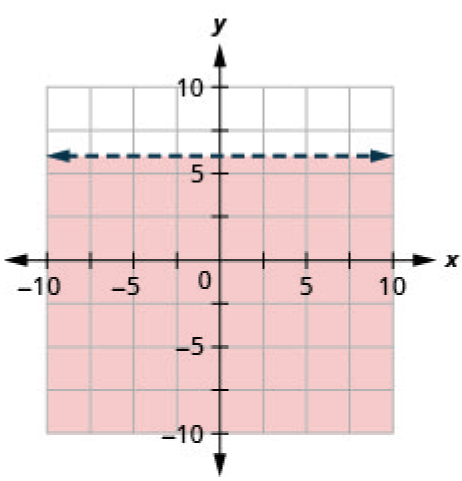
Exercise:

Problem: Graph the linear inequality $y \leq -3x$.

Exercise:

Problem: Graph the linear inequality $y < 6$.

Solution:



Solve Applications using Linear Inequalities in Two Variables

Exercise:

Problem:

Shanthie needs to earn at least $500 a week during her summer break to pay for college. She works two jobs. One as a swimming instructor that pays $10 an hour and the other as an intern in a law office for $25 hour. How many hours does Shanthie need to work at each job to earn at least $500 per week?

ⓐ Let *x* be the number of hours she works teaching swimming and let *y* be the number of hours she works as an intern. Write an inequality that would model this situation.
ⓑ Graph the inequality.
ⓒ Find three ordered pairs (x, y) that would be solutions to the inequality. Then, explain what that means for Shanthie.

Exercise:

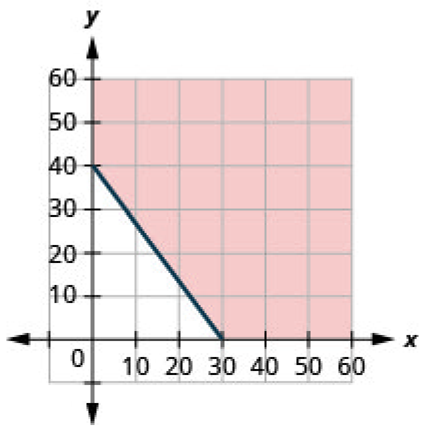
Problem:

Atsushi he needs to exercise enough to burn 600 calories each day. He prefers to either run or bike and burns 20 calories per minute while running and 15 calories a minute while biking.

ⓐ If *x* is the number of minutes that Atsushi runs and *y* is the number minutes he bikes, find the inequality that models the situation.
ⓑ Graph the inequality.
ⓒ List three solutions to the inequality. What options do the solutions provide Atsushi?

Solution:

ⓐ $20x + 15y \geq 600$
ⓑ



ⓒ Answers will vary.

Relations and Functions

Find the Domain and Range of a Relation

In the following exercises, for each relation, ⓐ find the domain of the relation ⓑ find the range of the relation.

Exercise:

Problem: $\{(5,-2),(5,-4),(7,-6),(8,-8),(9,-10)\}$

Exercise:

Problem: $\{(-3,7),(-2,3),(-1,9),(0,-3),(-1,8)\}$

Solution:

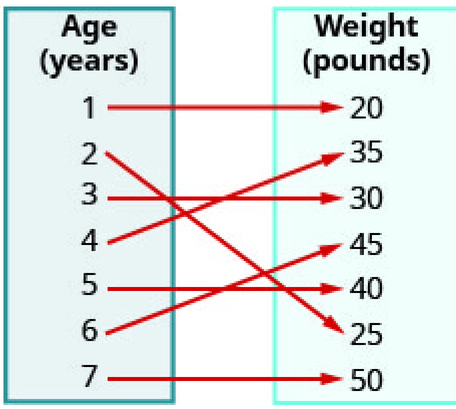
ⓐ D: {−3, −2, −1, 0}
ⓑ R: {7, 3, 9, −3, 8}

In the following exercise, use the mapping of the relation to ⓐ list the ordered pairs of the relation ⓑ find the domain of the relation ⓒ find the range of the relation.

Exercise:

Problem:

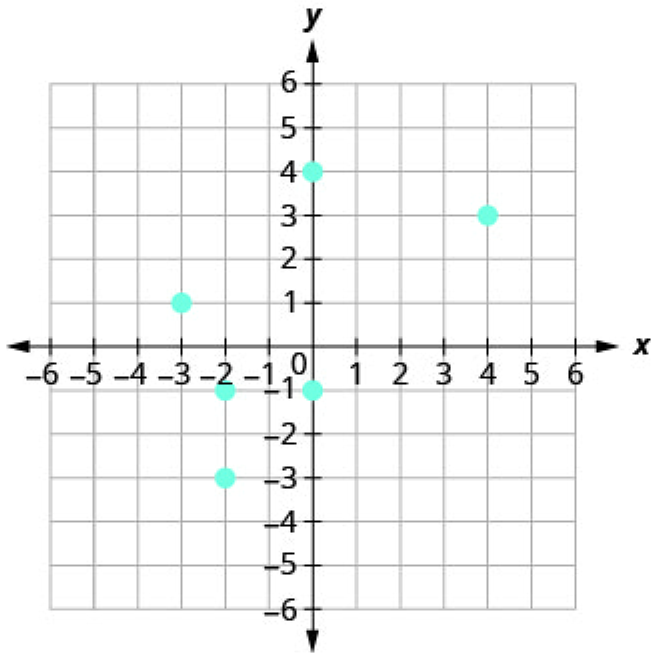
The mapping below shows the average weight of a child according to age.



In the following exercise, use the graph of the relation to ⓐ list the ordered pairs of the relation ⓑ find the domain of the relation ⓒ find the range of the relation.

Exercise:

Problem:



Solution:

ⓐ (4, 3), (−2, −3), (−2, −1), (−3, 1), (0, −1), (0, 4),
ⓑ D: {−3, −2, 0, 4}
ⓒ R: {−3, −1, 1, 3, 4}

Determine if a Relation is a Function

In the following exercises, use the set of ordered pairs to ⓐ determine whether the relation is a function ⓑ find the domain of the relation ⓒ find the range of the relation.

Exercise:

Problem: $\{(9,-5),(4,-3),(1,-1),(0,0),(1,1),(4,3),(9,5)\}$

Exercise:

Problem: $\{(-3,27),(-2,8),(-1,1),(0,0),(1,1),(2,8),(3,27)\}$

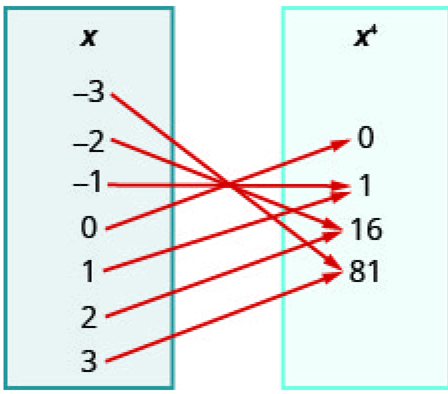
Solution:

ⓐ yes ⓑ {−3, −2, −1, 0, 1, 2, 3}
ⓒ {0, 1, 8, 27}

In the following exercises, use the mapping to ⓐ determine whether the relation is a function ⓑ find the domain of the function ⓒ find the range of the function.

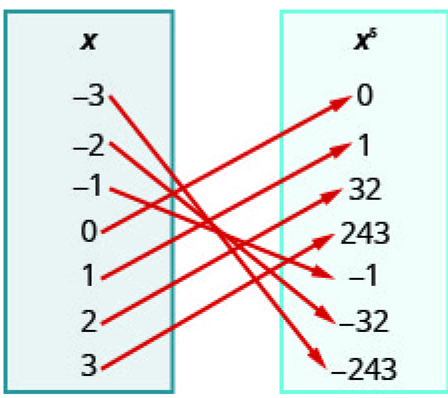
Exercise:

Problem:



Exercise:

Problem:



Solution:

ⓐ {−3, −2, −1, 0, 1, 2, 3}
ⓑ {−3, −2, −1, 0, 1, 2, 3}
ⓒ {−243, −32, −1, 0, 1, 32, 243}

In the following exercises, determine whether each equation is a function.

Exercise:

Problem: $2x + y = -3$

Exercise:

Problem: $y = x^2$

Solution:

yes

Exercise:

Problem: $y = 3x - 5$

Exercise:

Problem: $y = x^3$

Solution:

yes

Exercise:

Problem: $2x + y^2 = 4$

Find the Value of a Function

In the following exercises, evaluate the function:

ⓐ $f(-2)$ ⓑ $f(3)$ ⓒ $f(a)$.

Exercise:

Problem: $f(x) = 3x - 4$

Solution:

ⓐ $f\left(-2\right) = -10$ ⓑ $f\left(3\right) = 5$ ⓒ $f\left(a\right) = 3a - 4$

Exercise:

Problem: $f(x) = -2x + 5$

Exercise:

Problem: $f(x) = x^2 - 5x + 6$

Solution:

ⓐ $f\left(-2\right) = 20$ ⓑ $f\left(3\right) = 0$ ⓒ $f(a) = a^2 - 5a + 6$

Exercise:

Problem: $f(x) = 3x^2 - 2x + 1$

In the following exercises, evaluate the function.

Exercise:

Problem: $g(x) = 3x^2 - 5x;\, g(2)$

Solution:

2

Exercise:

Problem: $F(x) = 2x^2 - 3x + 1;$ $F(-1)$

Exercise:

Problem: $h(t) = 4\,|t-1| + 2;\; h(-3)$

Solution:

18

Exercise:

Problem: $f(x) = \frac{x+2}{x-1};\; f(3)$

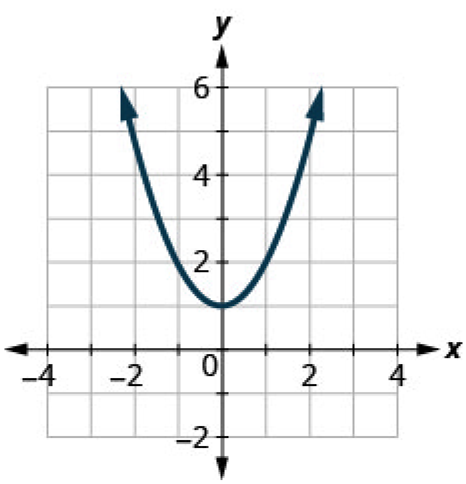
Graphs of Functions

Use the Vertical line Test

In the following exercises, determine whether each graph is the graph of a function.

Exercise:

Problem:

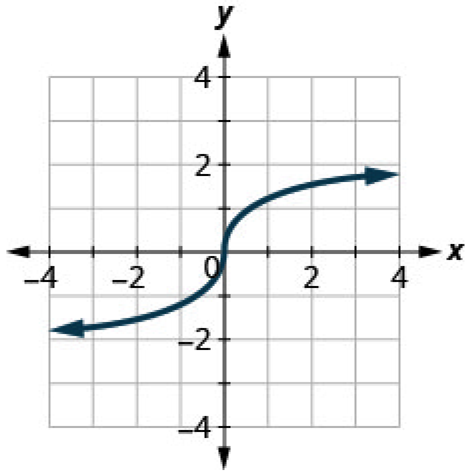


Solution:

yes

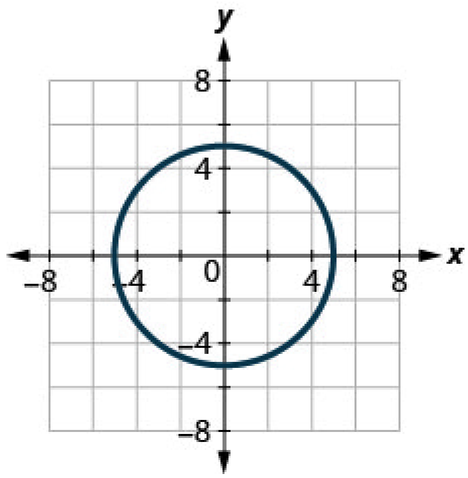
Exercise:

Problem:



Exercise:

Problem:

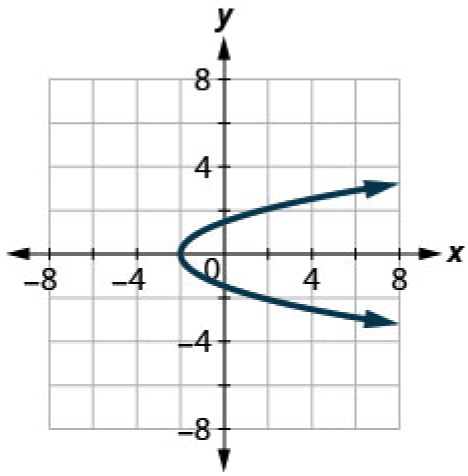


Solution:

no

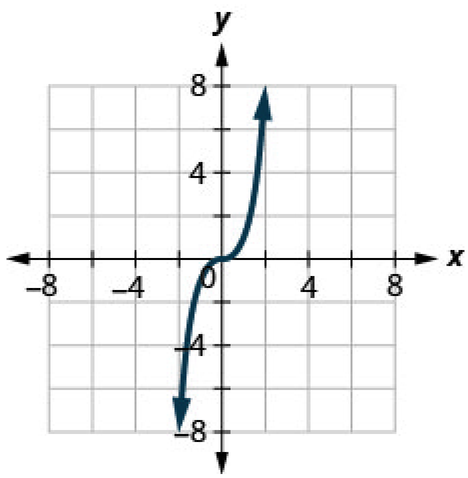
Exercise:

Problem:



Exercise:

Problem:

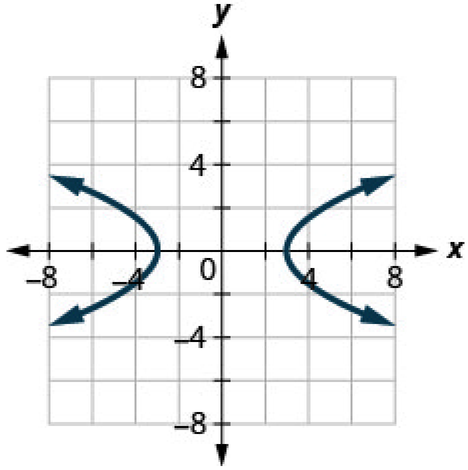


Solution:

yes

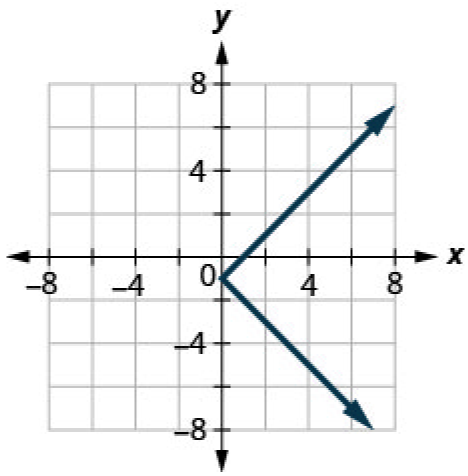
Exercise:

Problem:



Exercise:

Problem:



Solution:

no

Identify Graphs of Basic Functions

In the following exercises, ⓐ graph each function ⓑ state its domain and range. Write the domain and range in interval notation.

Exercise:

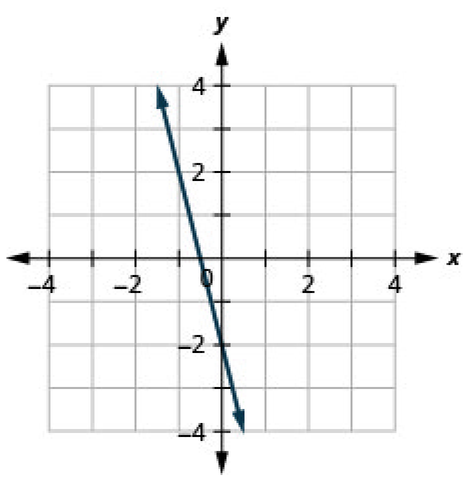
Problem: $f\left(x\right)=5x+1$

Exercise:

Problem: $f\left(x\right)=-4x-2$

Solution:

ⓐ



ⓑ D: (-∞,∞), R: (-∞,∞)

Exercise:

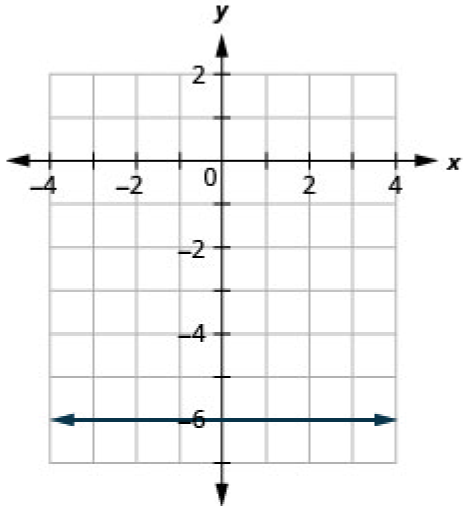
Problem: $f\left(x\right)=\frac{2}{3}x-1$

Exercise:

Problem: $f\left(x\right)=-6$

Solution:

ⓐ



ⓑ D: (-∞,∞), R: (-∞,∞)

Exercise:

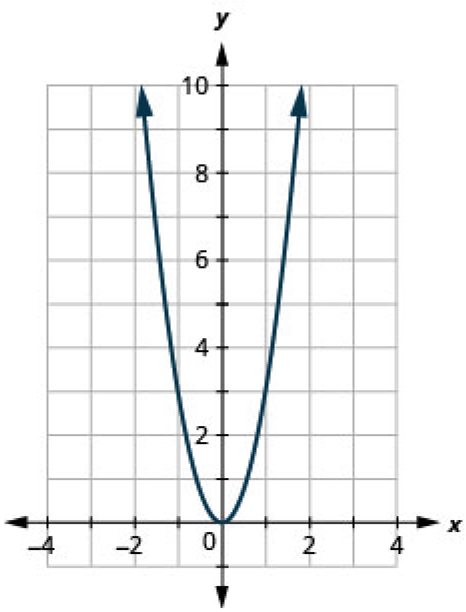
Problem: $f(x) = 2x$

Exercise:

Problem: $f(x) = 3x^2$

Solution:

ⓐ



ⓑ D: (-∞,∞), R: (-∞,0]

Exercise:

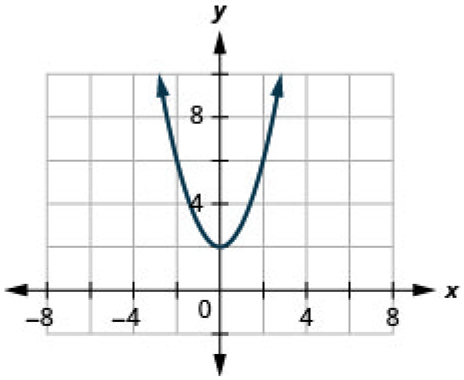
Problem: $f(x) = -\frac{1}{2}x^2$

Exercise:

Problem: $f(x) = x^2 + 2$

Solution:

ⓐ



ⓑ D: (-∞,∞), R: (-∞,∞)

Exercise:

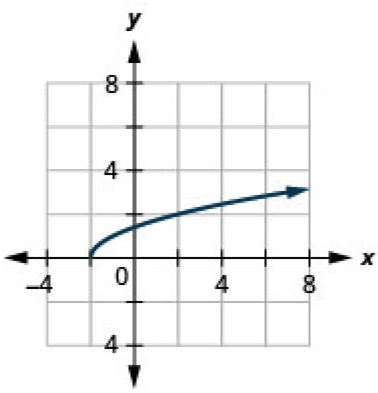
Problem: $f\left(x\right)={x}^{3}-2$

Exercise:

Problem: $f\left(x\right)=\sqrt{x+2}$

Solution:

ⓐ



ⓑ D: $[-2,\ \infty)$, R: [0,∞)

Exercise:

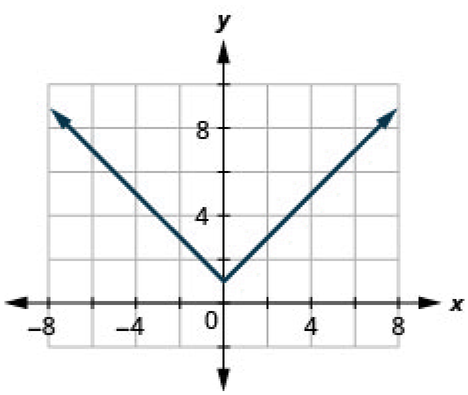
Problem: $f(x) = -|x|$

Exercise:

Problem: $f(x) = |x| + 1$

Solution:

ⓐ



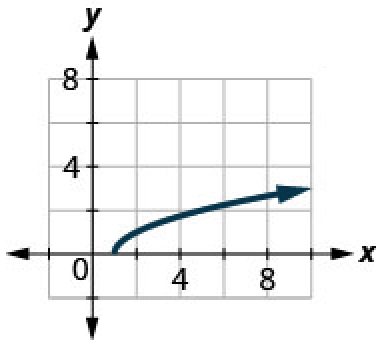
ⓑ D: (-∞,∞), R: [1,∞)

Read Information from a Graph of a Function

In the following exercises, use the graph of the function to find its domain and range. Write the domain and range in interval notation

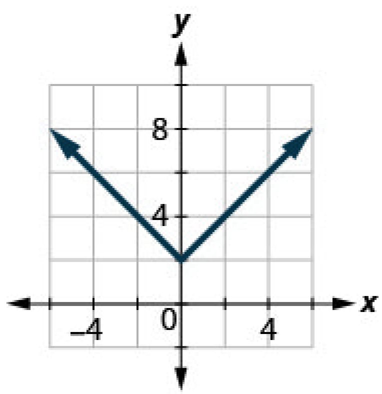
Exercise:

Problem:



Exercise:

Problem:

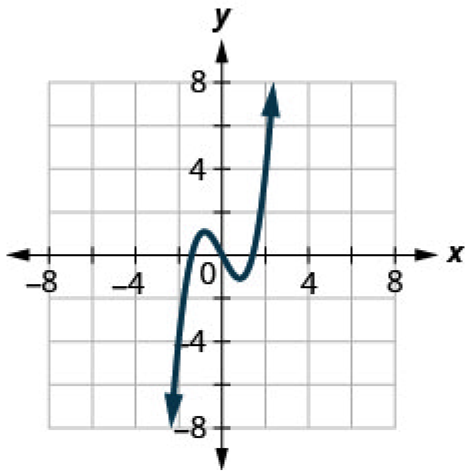


Solution:

D: (-∞,∞), R: [2,∞)

Exercise:

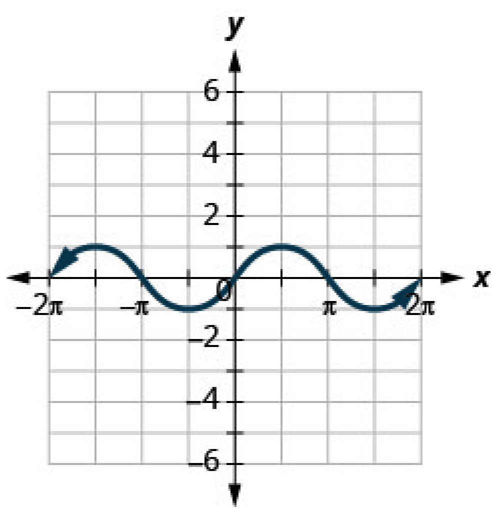
Problem:



In the following exercises, use the graph of the function to find the indicated values.

Exercise:

Problem:



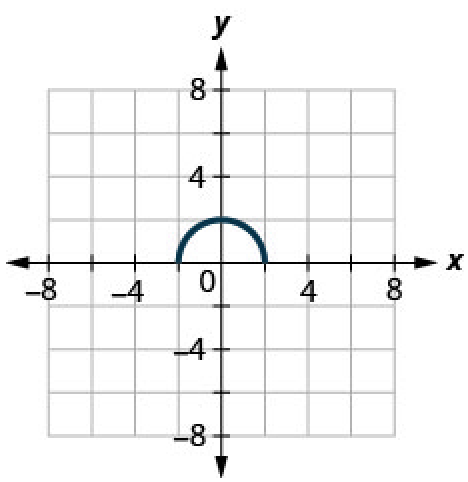
ⓐ Find $f(0)$.
ⓑ Find $f\left(\frac{1}{2}\pi\right)$.
ⓒ Find $f\left(-\frac{3}{2}\pi\right)$.
ⓓ Find the values for *x* when $f(x) = 0$.
ⓔ Find the x-intercepts.
ⓕ Find the y-intercepts.
ⓖ Find the domain. Write it in interval notation.
ⓗ Find the range. Write it in interval notation.

Solution:

ⓐ $f(x) = 0$ ⓑ $f(\pi/2) = 1$
ⓒ $f(-3\pi/2) = 1$ ⓓ $f(x) = 0$ for $x = -2\pi, -\pi, 0, \pi, 2\pi$
ⓔ $(-2\pi, 0)$, $(-\pi, 0)$, $(0, 0)$, $(\pi, 0)$, $(2\pi, 0)$ ⓕ $(0, 0)$
ⓖ $[-2\pi, 2\pi]$ ⓗ $[-1, 1]$

Exercise:

Problem:



ⓐ Find $f(0)$.
ⓑ Find the values for x when $f(x) = 0$.
ⓒ Find the x-intercepts.
ⓓ Find the y-intercepts.
ⓔ Find the domain. Write it in interval notation.
ⓕ Find the range. Write it in interval notation.

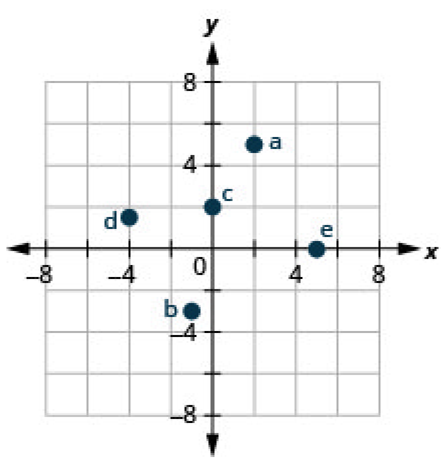
Practice Test

Exercise:

Problem: Plot each point in a rectangular coordinate system.

ⓐ $(2, 5)$
ⓑ $(-1, -3)$
ⓒ $(0, 2)$
ⓓ $\left(-4, \frac{3}{2}\right)$
ⓔ $(5, 0)$

Solution:



Exercise:

Problem:

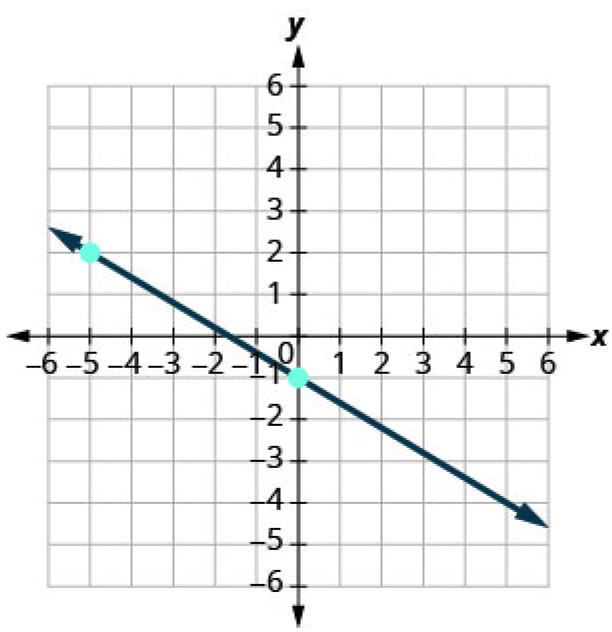
Which of the given ordered pairs are solutions to the equation $3x - y = 6$?

ⓐ $(3, 3)$ ⓑ $(2, 0)$ ⓒ $(4, -6)$

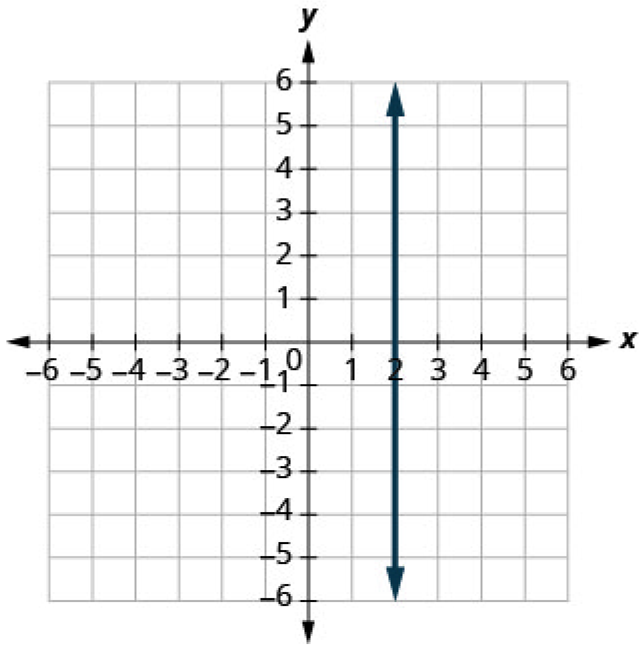
Find the slope of each line shown.

Exercise:

Problem: ⓐ



ⓑ



Solution:

ⓐ $-\frac{3}{5}$ ⓑ undefined

Exercise:

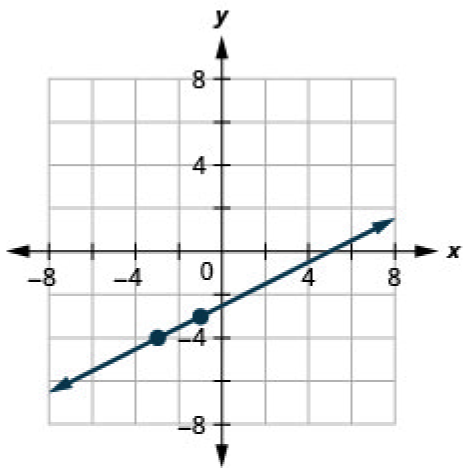
Problem:

Find the slope of the line between the points $(5, 2)$ and $(-1, -4)$.

Exercise:

Problem: Graph the line with slope $\frac{1}{2}$ containing the point $(-3, -4)$.

Solution:



Exercise:

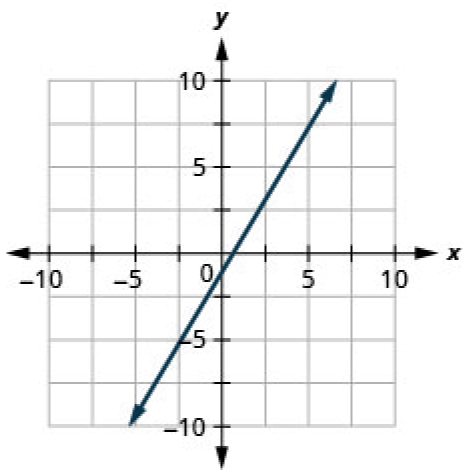
Problem: Find the intercepts of $4x + 2y = -8$ and graph.

Graph the line for each of the following equations.

Exercise:

Problem: $y = \frac{5}{3}x - 1$

Solution:



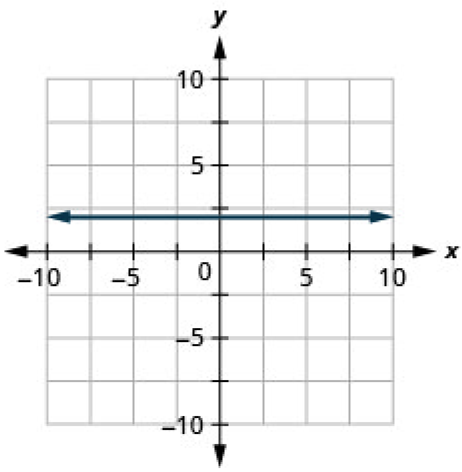
Exercise:

Problem: $y = -x$

Exercise:

Problem: $y = 2$

Solution:



Find the equation of each line. Write the equation in slope-intercept form.

Exercise:

Problem: slope $-\frac{3}{4}$ and y-intercept $(0,-2)$

Exercise:

Problem: $m=2$, point $(-3,-1)$

Solution:

$y=2x+5$

Exercise:

Problem: containing $(10,1)$ and $(6,-1)$

Exercise:

Problem:

perpendicular to the line $y = \frac{5}{4}x + 2$, containing the point $(-10, 3)$

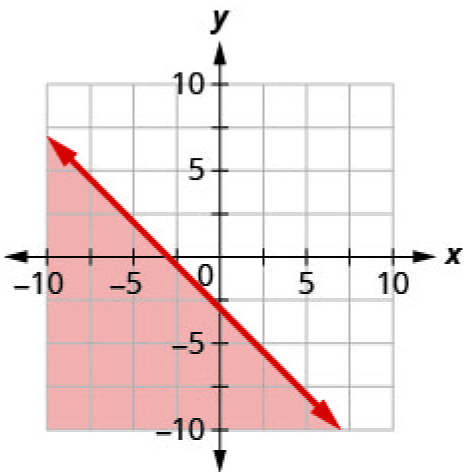
Solution:

$y = -\frac{4}{5}x - 5$

Exercise:

Problem:

Write the inequality shown by the graph with the boundary line $y = -x - 3$.

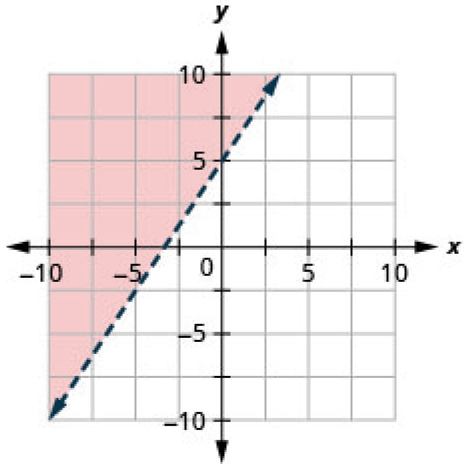


Graph each linear inequality.

Exercise:

Problem: $y > \frac{3}{2}x + 5$

Solution:



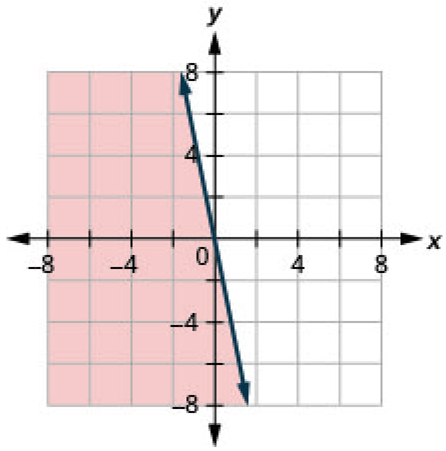
Exercise:

Problem: $x - y \geq -4$

Exercise:

Problem: $y \leq -5x$

Solution:



Exercise:

Problem:

Hiro works two part time jobs in order to earn enough money to meet her obligations of at least $450 a week. Her job at the mall pays $10 an hour and her administrative assistant job on campus pays $15 an hour. How many hours does Hiro need to work at each job to earn at least $450?

ⓐ Let *x* be the number of hours she works at the mall and let *y* be the number of hours she works as administrative assistant. Write an inequality that would model this situation.
ⓑ Graph the inequality .
ⓒ Find three ordered pairs(x, y) that would be solutions to the inequality. Then explain what that means for Hiro.

Exercise:

Problem:

Use the set of ordered pairs to ⓐ determine whether the relation is a function, ⓑ find the domain of the relation, and ⓒ find the range of the relation.

$\{(-3, 27), (-2, 8), (-1, 1), (0, 0),$
$(1, 1), (2, 8), (3, 27)\}$

Solution:

ⓐ yes ⓑ $\{-3, -2, -1, 0, 1, 2, 3\}$ ⓒ {0, 1, 8, 27}

Exercise:

Problem: Evaluate the function: ⓐ $f(-1)$ ⓑ $f(2)$ ⓒ $f(c)$.

$f(x) = 4x^2 - 2x - 3$

Exercise:

Problem: For $h(y) = 3\,|y - 1| - 3$, evaluate $h(-4)$.

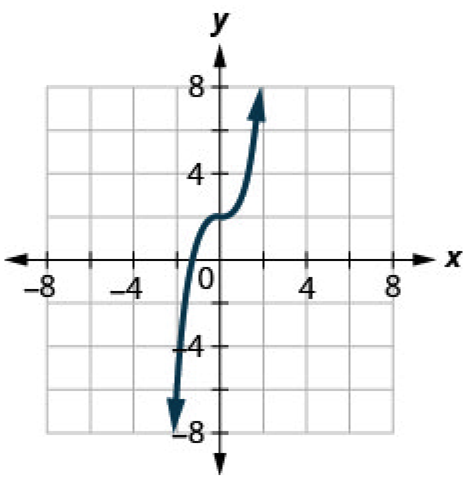
Solution:

12

Exercise:

Problem:

Determine whether the graph is the graph of a function. Explain your answer.



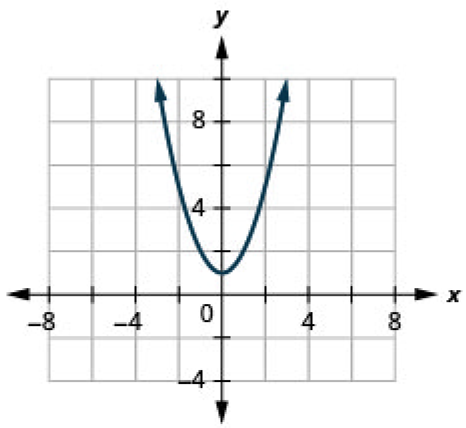
In the following exercises, ⓐ graph each function ⓑ state its domain and range.
Write the domain and range in interval notation.

Exercise:

Problem: $f\left(x\right)=x^{2}+1$

Solution:

ⓐ



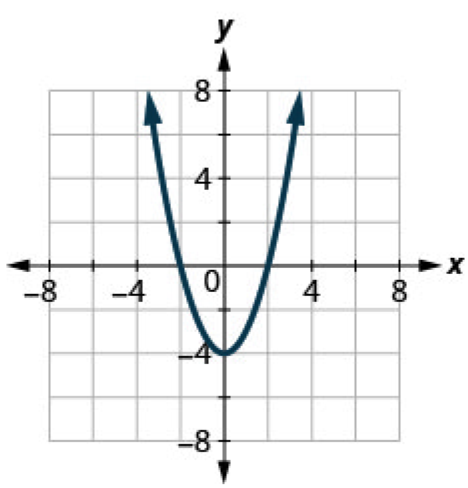
ⓑ D: (-∞,∞), R: [1,∞)

Exercise:

Problem: $f(x) = \sqrt{x+1}$

Exercise:

Problem:



ⓑ Find the y-intercepts.
ⓒ Find $f(-1)$.
ⓓ Find $f(1)$.
ⓔ Find the domain. Write it in interval notation.
ⓕ Find the range. Write it in interval notation.

Solution:

ⓐ $x = -2, 2$ ⓑ $y = -4$
ⓒ $f(-1) = -3$ ⓓ $f(1) = -3$
ⓔ D: (-∞,∞) ⓕ R: $[-4, \infty)$

Introduction
class="introduction"

Designing the number and sizes of windows in a home can pose challenges for an architect.

An architect designing a home may have restrictions on both the area and perimeter of the windows because of energy and structural concerns. The length and width chosen for each window would have to satisfy two equations: one for the area and the other for the perimeter. Similarly, a banker may have a fixed amount of money to put into two investment funds. A restaurant owner may want to increase profits, but in order to do that he will need to hire more staff. A job applicant may compare salary and costs of commuting for two job offers.

In this chapter, we will look at methods to solve situations like these using equations with two variables.

Solve Systems of Equations by Graphing: ASE
By the end of this section, you will be able to:

- Determine whether an ordered pair is a solution of a system of equations
- Solve a system of linear equations by graphing
- Determine the number of solutions of linear system
- Solve applications of systems of equations by graphing

Determine Whether an Ordered Pair is a Solution of a System of Equations

In Use a General Strategy to Solve Linear Equations we learned how to solve linear equations with one variable. Remember that the solution of an equation is a value of the variable that makes a true statement when substituted into the equation.

Now we will work with **systems of linear equations**, two or more linear equations grouped together.

Note:
System of Linear Equations
When two or more linear equations are grouped together, they form a system of linear equations.

We will focus our work here on systems of two linear equations in two unknowns. Later, you may solve larger systems of equations.

An example of a system of two linear equations is shown below. We use a brace to show the two equations are grouped together to form a system of equations.
Equation:

$$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$$

A linear equation in two variables, like $2x + y = 7$, has an infinite number of solutions. Its graph is a line. Remember, every point on the line is a solution to the equation and every solution to the equation is a point on the line.

To solve a system of two linear equations, we want to find the values of the variables that are solutions to both equations. In other words, we are looking for the ordered pairs (x, y) that make both equations true. These are called the solutions to a system of equations.

Note:
Solutions of a System of Equations

Solutions of a system of equations are the values of the variables that make all the equations true. A solution of a system of two linear equations is represented by an ordered pair (x, y).

To determine if an ordered pair is a solution to a system of two equations, we substitute the values of the variables into each equation. If the ordered pair makes both equations true, it is a solution to the system.

Let's consider the system below:

Equation:

$$\begin{cases} 3x - y = 7 \\ x - 2y = 4 \end{cases}$$

Is the ordered pair $(2, -1)$ a solution?

We substitute $x = 2$ and $y = -1$ into both equations.

$$\begin{array}{ll} 3x - y = 7 & x - 2y = 4 \\ 3(2) - (-1) \stackrel{?}{=} 7 & 2 - 2(-1) \stackrel{?}{=} 4 \\ 7 = 7 \text{ true} & 4 = 4 \text{ true} \end{array}$$

The ordered pair (2, −1) made both equations true. Therefore (2, −1) is a solution to this system.

Let's try another ordered pair. Is the ordered pair (3, 2) a solution?

We substitute $x = 3$ and $y = 2$ into both equations.

$$\begin{array}{ll} 3x - y = 7 & x - 2y = 4 \\ 3(3) - 2 \stackrel{?}{=} 7 & 2 - 2(2) \stackrel{?}{=} 4 \\ 7 = 7 \text{ true} & -2 = 4 \text{ false} \end{array}$$

The ordered pair (3, 2) made one equation true, but it made the other equation false. Since it is not a solution to **both** equations, it is not a solution to this system.

Example:
Exercise:

Problem: Determine whether the ordered pair is a solution to the system: $\begin{cases} x - y = -1 \\ 2x - y = -5 \end{cases}$

ⓐ $(-2, -1)$ ⓑ $(-4, -3)$

Solution:

Solution

ⓐ

$$\begin{cases} x - y = -1 \\ 2x - y = -5 \end{cases}$$

We substitute $x = -2$ and $y = -1$ into both equations.

$$\begin{array}{cc} x - y = -1 & 2x - y = -5 \\ -2 - (-1) \stackrel{?}{=} -1 & 2(-2) - (-1) \stackrel{?}{=} -5 \\ -1 = -1 \checkmark & 5 \neq -5 \end{array}$$

(−2, −1) does not make both equations true. (−2, −1) is not a solution.

ⓑ

We substitute $x = -4$ and $y = -3$ into both equations.

$$\begin{array}{cc} x - y = -1 & 2x - y = -5 \\ -4 - (-3) \stackrel{?}{=} -1 & 2(-4) - (-3) \stackrel{?}{=} -5 \\ -1 = -1 \checkmark & -5 = -5 \checkmark \end{array}$$

(−4, −3) does not make both equations true. (−4, −3) is a solution.

Note:
Exercise:

Problem: Determine whether the ordered pair is a solution to the system: $\begin{cases} 3x + y = 0 \\ x + 2y = -5 \end{cases}$.

ⓐ $(1, -3)$ ⓑ $(0, 0)$

Solution:

ⓐ yes ⓑ no

Note:
Exercise:

Problem: Determine whether the ordered pair is a solution to the system: $\begin{cases} x - 3y = -8 \\ -3x - y = 4 \end{cases}$.

ⓐ $(2, -2)$ ⓑ $(-2, 2)$

Solution:

ⓐ no ⓑ yes

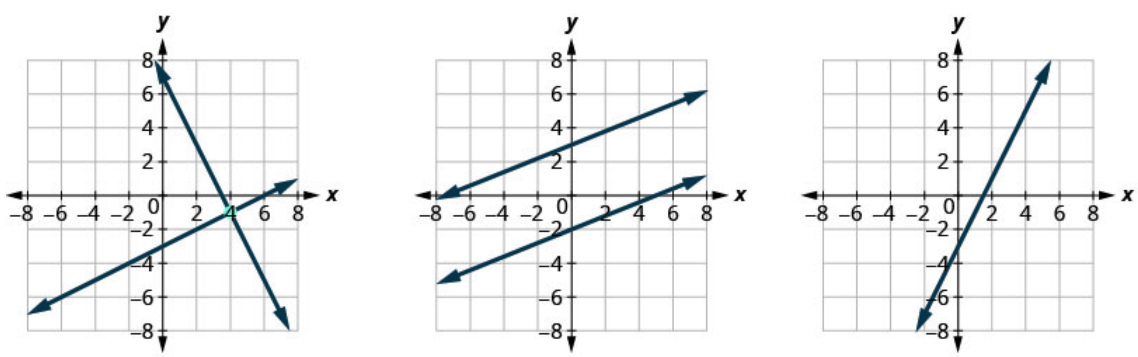
Solve a System of Linear Equations by Graphing

In this chapter we will use three methods to solve a system of linear equations. The first method we’ll use is graphing.

The graph of a linear equation is a line. Each point on the line is a solution to the equation. For a system of two equations, we will graph two lines. Then we can see all the points that are solutions to each equation. And, by finding what the lines have in common, we’ll find the solution to the system.

Most linear equations in one variable have one solution, but we saw that some equations, called contradictions, have no solutions and for other equations, called identities, all numbers are solutions.

Similarly, when we solve a system of two linear equations represented by a graph of two lines in the same plane, there are three possible cases, as shown in [link]:



The lines intersect. Intersecting lines have one point in common. There is one solution to this system.

The lines are parallel. Parallel lines have no points in common. There is no solution to this system.

Both equations give the same line. Because we have just one line, there are infinitely many solutions.

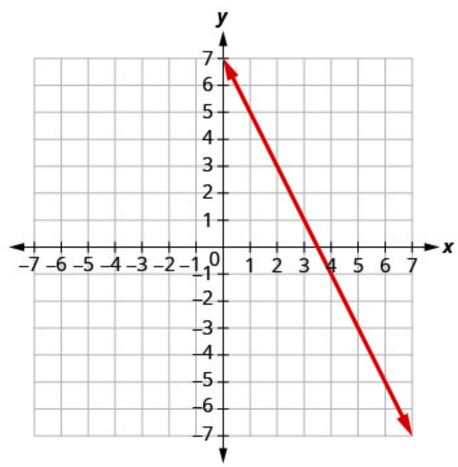
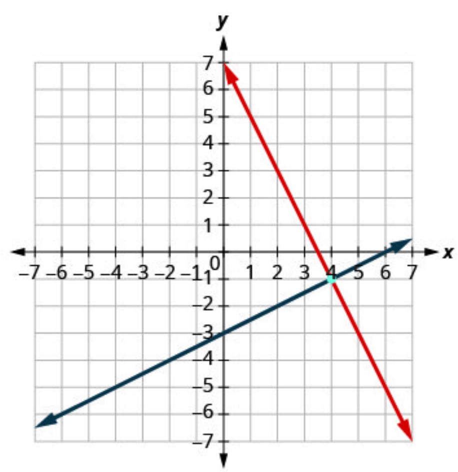
For the first example of solving a system of linear equations in this section and in the next two sections, we will solve the same system of two linear equations. But we’ll use a different method

in each section. After seeing the third method, you'll decide which method was the most convenient way to solve this system.

Example:
How to Solve a System of Linear Equations by Graphing
Exercise:

Problem: Solve the system by graphing: $\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$.

Solution:
Solution

Step 1. Graph the first equation.	To graph the first line, write the equation in slope–intercept form. $2x + y = 7$ $y = -2x + 7$ $m = -2$ $b = 7$	$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$ 
Step 2. Graph the second equation on the same rectangular coordinate system.	To graph the second line, use intercepts. $x - 2y = 6$ $(0, -3)$ $(6, 0)$	

Step 3. Determine whether the lines intersect, are parallel, or are the same line.	Look at the graph of the lines.	The lines intersect.
Step 4. Identify the solution to the system. If the lines intersect, identify the point of intersection. Check to make sure it is a solution to both equations. This is the solution to the system. If the lines are parallel, the system has no solution If the lines are the same, the system has an infinite number of solutions.	Since the lines intersect, find the point of intersection. Check the point in both equations.	The lines intersect at (4, −1). $2x + y = 7$ $2(4) + (-1) \stackrel{?}{=} 7$ $8 - 1 \stackrel{?}{=} 7$ $7 = 7$ ✓ $x - 2y = 6$ $4 - 2(-1) \stackrel{?}{=} 6$ $6 = 6$ ✓ The solution is (4, −1).

Note:
Exercise:

Problem: Solve each system by graphing: $\begin{cases} x - 3y = -3 \\ x + y = 5 \end{cases}$.

Solution:

$(3, 2)$

Note:
Exercise:

Problem: Solve each system by graphing: $\begin{cases} -x + y = 1 \\ 3x + 2y = 12 \end{cases}$.

Solution:

$(2, 3)$

The steps to use to solve a system of linear equations by graphing are shown below.

Note:
To solve a system of linear equations by graphing.

Graph the first equation.
Graph the second equation on the same rectangular coordinate system.
Determine whether the lines intersect, are parallel, or are the same line.
Identify the solution to the system.

- If the lines intersect, identify the point of intersection. Check to make sure it is a solution to both equations. This is the solution to the system.
- If the lines are parallel, the system has no solution.
- If the lines are the same, the system has an infinite number of solutions.

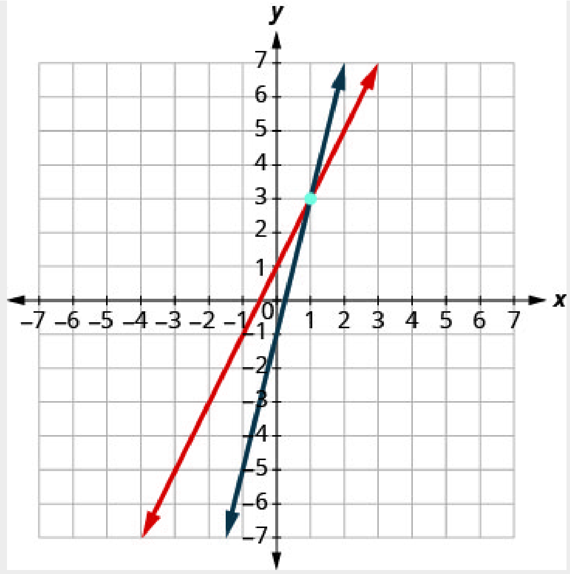
Example:
Exercise:

Problem: Solve the system by graphing: $\begin{cases} y = 2x + 1 \\ y = 4x - 1 \end{cases}$.

Solution:
Solution

Both of the equations in this system are in slope-intercept form, so we will use their slopes and *y*-intercepts to graph them. $\begin{cases} y = 2x + 1 \\ y = 4x - 1 \end{cases}$

Find the slope and *y*-intercept of the first equation.	$y = 2x + 1$ $m = 2$ $b = 1$
Find the slope and *y*-intercept of the first equation.	$y = 4x - 1$ $m = 4$ $b = -1$
Graph the two lines.	
Determine the point of intersection.	The lines intersect at (1, 3).

	
Check the solution in both equations.	$\begin{aligned} y &= 2x+1 \\ 3 &\stackrel{?}{=} 2\cdot 1+1 \\ 3 &= 3 \checkmark \end{aligned} \qquad \begin{aligned} y &= 4x-1 \\ 3 &\stackrel{?}{=} 4\cdot 1-1 \\ 3 &= 3 \checkmark \end{aligned}$
	The solution is (1, 3).

Note:
Exercise:

Problem: Solve each system by graphing: $\begin{cases} y = 2x+2 \\ y = -x-4 \end{cases}$.

Solution:

$(-2, -2)$

Note:
Exercise:

Problem: Solve each system by graphing: $\begin{cases} y = 3x+3 \\ y = -x+7 \end{cases}$.

Solution:

$(1, 6)$

Both equations in [link] were given in slope–intercept form. This made it easy for us to quickly graph the lines. In the next example, we'll first re-write the equations into slope–intercept form.

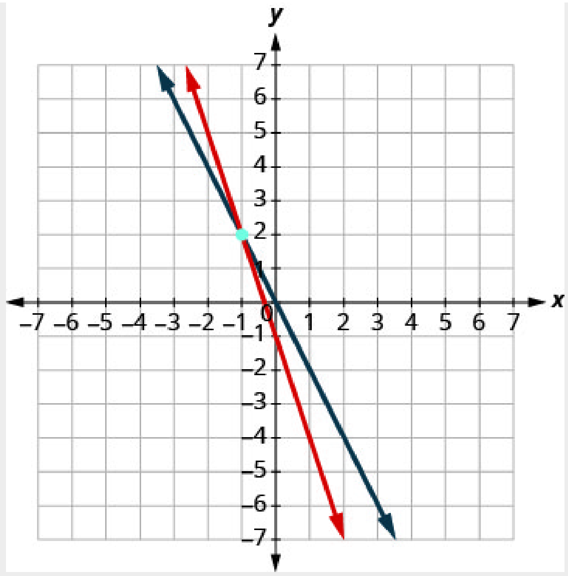
Example:
Exercise:

Problem: Solve the system by graphing: $\begin{cases} 3x + y = -1 \\ 2x + y = 0 \end{cases}$.

Solution:
Solution

We'll solve both of these equations for y so that we can easily graph them using their slopes and y-intercepts. $\begin{cases} 3x + y = -1 \\ 2x + y = 0 \end{cases}$

Solve the first equation for *y*.	$\begin{aligned} 3x + y &= -1 \\ y &= -3x - 1 \end{aligned}$
Find the slope and *y*-intercept.	$\begin{aligned} m &= -3 \\ b &= -1 \end{aligned}$
Solve the second equation for *y*.	$\begin{aligned} 2x + y &= 0 \\ y &= -2x \end{aligned}$
Find the slope and *y*-intercept.	$\begin{aligned} m &= -2 \\ b &= 0 \end{aligned}$
Graph the lines.	

	
Determine the point of intersection.	The lines intersect at (−1, 2).
Check the solution in both equations.	$\begin{aligned} 3x + y &= -1 \\ 3(-1) + 2 &\stackrel{?}{=} -1 \\ -1 &= -1 \checkmark \end{aligned} \qquad \begin{aligned} 2x + y &= 0 \\ 2(-1) + 2 &\stackrel{?}{=} 0 \\ 0 &= 0 \checkmark \end{aligned}$
	The solution is (−1, 2).

Note:
Exercise:

Problem: Solve each system by graphing: $\begin{cases} -x + y = 1 \\ 2x + y = 10 \end{cases}$.

Solution:

$(3, 4)$

Note:
Exercise:

Problem: Solve each system by graphing: $\begin{cases} 2x+y=6 \\ x+y=1 \end{cases}$.

Solution:

$(5,-4)$

Usually when equations are given in standard form, the most convenient way to graph them is by using the intercepts. We'll do this in [link].

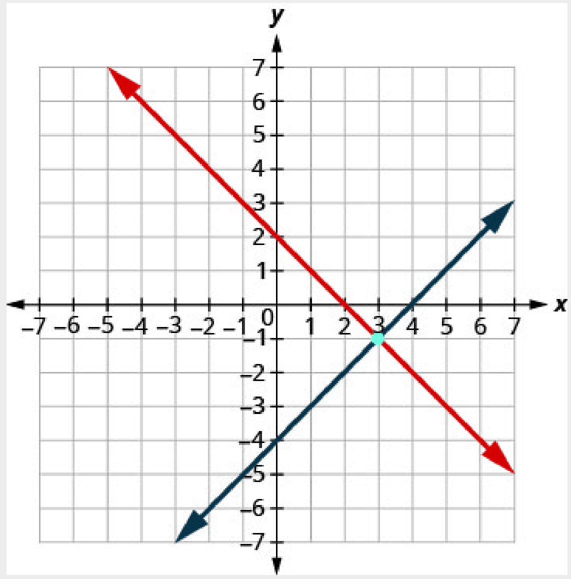
Example:
Exercise:

Problem: Solve the system by graphing: $\begin{cases} x+y=2 \\ x-y=4 \end{cases}$.

Solution:
Solution

We will find the *x*- and *y*-intercepts of both equations and use them to graph the lines.

	$x+y=2$	
To find the intercepts, let $x = 0$ and solve for y, then let $y = 0$ and solve for x.	$\begin{aligned} x+y &= 2 \\ 0+y &= 2 \\ y &= 2 \end{aligned}$ $\quad$ $\begin{aligned} x+y &= 2 \\ x+0 &= 2 \\ x &= 2 \end{aligned}$	
	$x-y=4$	
To find the intercepts, let $x = 0$ then let $y = 0$.	$\begin{aligned} x-y &= 4 \\ 0-y &= 4 \\ -y &= 4 \\ y &= -4 \end{aligned}$ $\quad$ $\begin{aligned} x-y &= 4 \\ x-0 &= 4 \\ x &= 4 \end{aligned}$	

Graph the line.		
Determine the point of intersection.	The lines intersect at (3, −1).	
Check the solution in both equations.	$\begin{aligned} x+y &= 2 \\ 3+(-1) &\stackrel{?}{=} 2 \\ 2 &= 2\checkmark \end{aligned}$ $\qquad$ $\begin{aligned} x-y &= 4 \\ 3-(-1) &\stackrel{?}{=} 4 \\ 4 &= 4\checkmark \end{aligned}$ The solution is (3, −1).	

Note:
Exercise:

Problem: Solve each system by graphing: $\begin{cases} x+y=6 \\ x-y=2 \end{cases}$.

Solution:

$(4, 2)$

Note:
Exercise:

Problem: Solve each system by graphing: $\begin{cases} x + y = 2 \\ x - y = -8 \end{cases}.$

Solution:

$(5, -3)$

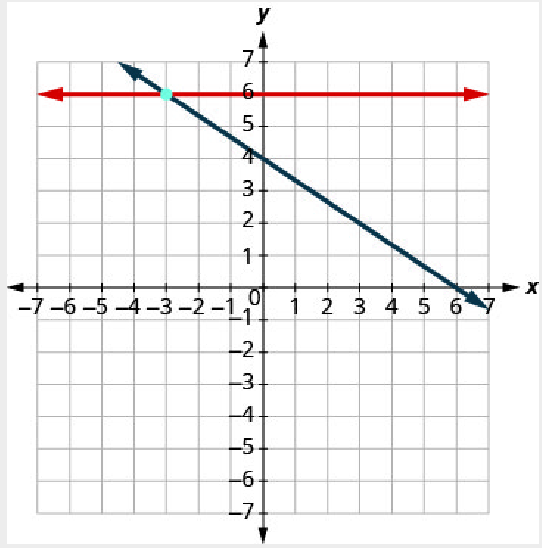
Do you remember how to graph a linear equation with just one variable? It will be either a vertical or a horizontal line.

Example:
Exercise:

Problem: Solve the system by graphing: $\begin{cases} y = 6 \\ 2x + 3y = 12 \end{cases}.$

Solution:
Solution

	$\begin{cases} y = 6 \\ 2x + 3y = 12 \end{cases}$
We know the first equation represents a horizontal line whose *y*-intercept is 6.	$y = 6$
The second equation is most conveniently graphed using intercepts.	$2x + 3y = 12$
To find the intercepts, let *x* = 0 and then *y* = 0.	<table><tr><th>x</th><th>y</th></tr><tr><td>0</td><td>4</td></tr><tr><td>6</td><td>0</td></tr></table>

Graph the lines.	
Determine the point of intersection.	The lines intersect at (−3, 6).
Check the solution to both equations.	$\begin{aligned} y &= 6 \\ 6 &\stackrel{?}{=} 6\checkmark \\ 2 &= 2 \end{aligned}$ $\quad$ $\begin{aligned} 2x + 3y &= 12 \\ 2(-3) + 3(6) &\stackrel{?}{=} 12 \\ -6 + 18 &\stackrel{?}{=} 12 \\ 12 &= 12\checkmark \end{aligned}$
	The solution is (−3, 6).

Note:
Exercise:

Problem: Solve each system by graphing: $\begin{cases} y = -1 \\ x + 3y = 6 \end{cases}$.

Solution:

$(9, -1)$

Note:
Exercise:

Problem: Solve each system by graphing: $\begin{cases} x = 4 \\ 3x - 2y = 24 \end{cases}$.
Solution: $(4, -6)$

In all the systems of linear equations so far, the lines intersected and the solution was one point. In the next two examples, we'll look at a system of equations that has no solution and at a system of equations that has an infinite number of solutions.

Example:
Exercise:

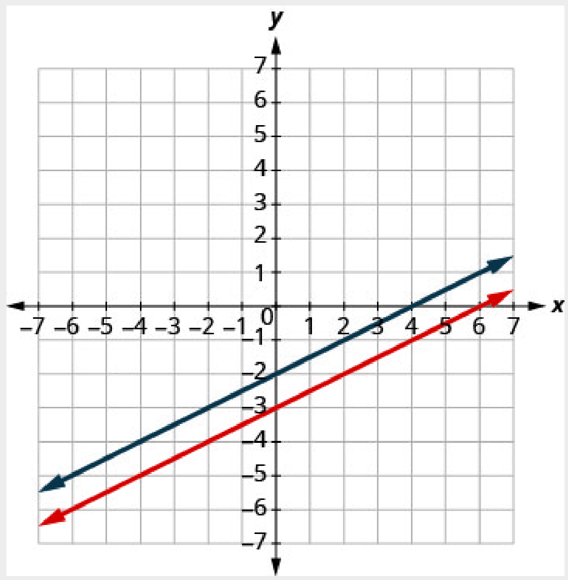
Problem: Solve the system by graphing: $\begin{cases} y = \frac{1}{2}x - 3 \\ x - 2y = 4 \end{cases}$.

Solution:
Solution

	$\begin{cases} y = \frac{1}{2}x - 3 \\ x - 2y = 4 \end{cases}$
To graph the first equation, we will use its slope and *y*-intercept.	$y = \frac{1}{2}x - 3$
	$m = \frac{1}{2}$
	$b = -3$

To graph the second equation, we will use the intercepts.	$x - 2y = 4$

x	y
0	–2
4	0

Graph the lines.	
Determine the point of intersection.	The lines are parallel.
	Since no point is on both lines, there is no ordered pair that makes both equations true. There is no solution to this system.

Note:
Exercise:

Problem: Solve each system by graphing: $\begin{cases} y = -\frac{1}{4}x + 2 \\ x + 4y = -8 \end{cases}$.

Solution:

no solution

Note:
Exercise:

Problem: Solve each system by graphing: $\begin{cases} y = 3x - 1 \\ 6x - 2y = 6 \end{cases}$.

Solution:

no solution

Example:
Exercise:

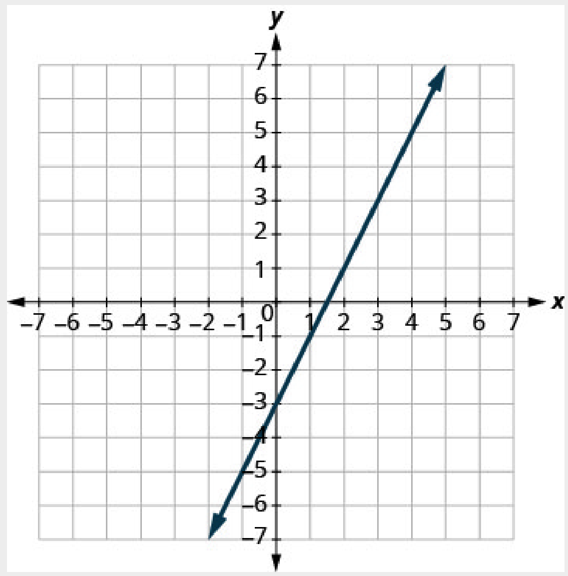
Problem: Solve the system by graphing: $\begin{cases} y = 2x - 3 \\ -6x + 3y = -9 \end{cases}$.

Solution:
Solution

	$\begin{cases} y = 2x - 3 \\ -6x + 3y = -9 \end{cases}$
Find the slope and *y*-intercept of the first equation.	$y = 2x - 3$ $m = 2$ $b = -3$
Find the intercepts of the second equation.	$-6x + 3y = -9$

<table>
<tr><td></td><td>

x	y
0	−3
$\frac{3}{2}$	0

</td></tr>
<tr><td>Graph the lines.</td><td>
</td></tr>
<tr><td>Determine the point of intersection.</td><td>The lines are the same!</td></tr>
<tr><td></td><td>Since every point on the line makes both equations
true, there are infinitely many ordered pairs that make
both equations true.</td></tr>
<tr><td></td><td>There are infinitely many solutions to this system.</td></tr>
</table>

Note:
Exercise:

Problem: Solve each system by graphing: $\begin{cases} y = -3x - 6 \\ 6x + 2y = -12 \end{cases}$.

Solution:

infinitely many solutions

Note:
Exercise:

Problem: Solve each system by graphing: $\begin{cases} y=\frac{1}{2}x-4 \\ 2x-4y=16 \end{cases}$.

Solution:

infinitely many solutions

If you write the second equation in [link] in slope-intercept form, you may recognize that the equations have the same slope and same *y*-intercept.

When we graphed the second line in the last example, we drew it right over the first line. We say the two lines are coincident. Coincident lines have the same slope and same *y*-intercept.

Note:
Coincident Lines
Coincident lines have the same slope and same *y*-intercept.

Determine the Number of Solutions of a Linear System

There will be times when we will want to know how many solutions there will be to a system of linear equations, but we might not actually have to find the solution. It will be helpful to determine this without graphing.

We have seen that two lines in the same plane must either intersect or are parallel. The systems of equations in [link] through [link] all had two intersecting lines. Each system had one solution.

A system with parallel lines, like [link], has no solution. What happened in [link]? The equations have coincident lines, and so the system had infinitely many solutions.

We’ll organize these results in [link] below:

Graph	Number of solutions
2 intersecting lines	1
Parallel lines	None
Same line	Infinitely many

Parallel lines have the same slope but different *y*-intercepts. So, if we write both equations in a system of linear equations in slope–intercept form, we can see how many solutions there will be without graphing! Look at the system we solved in [link].

Equation:

$$\begin{cases} y &= \frac{1}{2}x - 3 \\ x - 2y &= 4 \end{cases}$$

$$\begin{array}{ll} \text{The first line is in slope–intercept form.} & \text{If we solve the second equation for } y, \text{ we get} \\ & \begin{array}{rcl} x - 2y &=& 4 \\ -2y &=& -x + 4 \\ y &=& \frac{1}{2}x - 2 \end{array} \\ y = \frac{1}{2}x - 3 & \\ m = \frac{1}{2}, b = -3 & m = \frac{1}{2}, b = -2 \end{array}$$

The two lines have the same slope but different *y*-intercepts. They are parallel lines.

[link] shows how to determine the number of solutions of a linear system by looking at the slopes and intercepts.

Number of Solutions of a Linear System of Equations			
Slopes	**Intercepts**	**Type of Lines**	**Number of Solutions**
Different		Intersecting	1 point
Same	Different	Parallel	No solution
Same	Same	Coincident	Infinitely many solutions

Let’s take one more look at our equations in [link] that gave us parallel lines.

Equation:

$$\begin{cases} y = \frac{1}{2}x - 3 \\ x - 2y = 4 \end{cases}$$

When both lines were in slope-intercept form we had:

Equation:

$$y = \frac{1}{2}x - 3 \qquad y = \frac{1}{2}x - 2$$

Do you recognize that it is impossible to have a single ordered pair (x, y) that is a solution to both of those equations?

We call a system of equations like this an inconsistent system. It has no solution.

A system of equations that has at least one solution is called a consistent system.

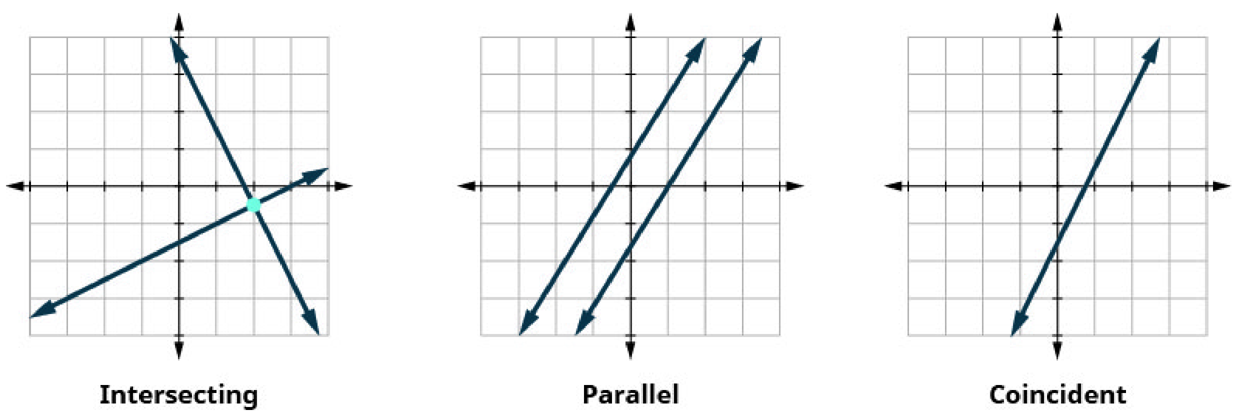
Note:
Consistent and Inconsistent Systems
A **consistent system** of equations is a system of equations with at least one solution.
An **inconsistent system** of equations is a system of equations with no solution.

We also categorize the equations in a system of equations by calling the equations *independent* or *dependent*. If two equations are **independent equations**, they each have their own set of solutions. Intersecting lines and parallel lines are independent.

If two equations are dependent, all the solutions of one equation are also solutions of the other equation. When we graph two dependent equations, we get coincident lines.

Note:
Independent and Dependent Equations
Two equations are **independent** if they have different solutions.
Two equations are **dependent** if all the solutions of one equation are also solutions of the other equation.

Let’s sum this up by looking at the graphs of the three types of systems. See [link] and [link].



Lines	Intersecting	Parallel	Coincident
Number of solutions	1 point	No solution	Infinitely many
Consistent/inconsistent	Consistent	Inconsistent	Consistent
Dependent/independent	Independent	Independent	Dependent

Example:
Exercise:

Problem:

Without graphing, determine the number of solutions and then classify the system of equations: $\begin{cases} y = 3x - 1 \\ 6x - 2y = 12 \end{cases}$.

Solution:
Solution

We will compare the slopes and intercepts of the two lines.	$\begin{cases} y = 3x - 1 \\ 6x - 2y = 12 \end{cases}$.
The first equation is already in slope-intercept form.	$y = 3x - 1$
Write the second equation in slope-intercept form.	$\begin{array}{rcl} 6x - 2y & = & 12 \\ -2y & = & -6x + 12 \\ \frac{-2y}{-2} & = & \frac{-6x+12}{-2} \\ y & = & 3x - 6 \end{array}$
Find the slope and intercept of each line.	$\begin{array}{rcl} y & = & 3x - 1 \\ m & = & 3 \\ b & = & -1 \end{array}$ $\begin{array}{rcl} y & = & 3x - 6 \\ m & = & 3 \\ b & = & -6 \end{array}$
	Since the slopes are the same and y-intercepts are different, the lines are parallel.

A system of equations whose graphs are parallel lines has no solution and is inconsistent and independent.

Note:
Exercise:

Problem:

Without graphing, determine the number of solutions and then classify the system of equations.

$$\begin{cases} y = -2x - 4 \\ 4x + 2y = 9 \end{cases}$$

Solution:

no solution, inconsistent, independent

Note:
Exercise:

Problem:

Without graphing, determine the number of solutions and then classify the system of equations.

$$\begin{cases} y = \frac{1}{3}x - 5 \\ x - 3y = 6 \end{cases}$$

Solution:

no solution, inconsistent, independent

Example:
Exercise:

Problem:

Without graphing, determine the number of solutions and then classify the system of equations: $\begin{cases} 2x + y = -3 \\ x - 5y = 5 \end{cases}$.

Solution:
Solution

We will compare the slope and intercepts of the two lines.	$\begin{cases}2x+y=-3\\x-5y=5\end{cases}$	
Write both equations in slope-intercept form.	$\begin{aligned}2x+y&=-3\\y&=-2x-3\end{aligned}$	$\begin{aligned}x-5y=5&=5\\-5y&=-x+5\\\frac{-5y}{-5}&=\frac{-x+5}{-5}\\y&=\frac{1}{5}x-1\end{aligned}$
Find the slope and intercept of each line.	$\begin{aligned}y&=-2x-3\\m&=-2\\b&=-3\end{aligned}$	$\begin{aligned}y&=\frac{1}{5}x-1\\m&=\frac{1}{5}\\b&=-1\end{aligned}$
	Since the slopes are different, the lines intersect.	

A system of equations whose graphs are intersect has 1 solution and is consistent and independent.

Note:
Exercise:

Problem:

Without graphing, determine the number of solutions and then classify the system of equations.

$$\begin{cases}3x+2y=2\\2x+y=1\end{cases}$$

Solution:

one solution, consistent, independent

Note:
Exercise:

Problem:

Without graphing, determine the number of solutions and then classify the system of equations.

$$\begin{cases} x + 4y = 12 \\ -x + y = 3 \end{cases}$$

Solution:

one solution, consistent, independent

Example:
Exercise:

Problem:

Without graphing, determine the number of solutions and then classify the system of equations. $\begin{cases} 3x - 2y = 4 \\ y = \frac{3}{2}x - 2 \end{cases}$

Solution:
Solution

We will compare the slopes and intercepts of the two lines.	$\begin{cases} 3x - 2y = 4 \\ y = \frac{3}{2}x - 2 \end{cases}$
Write the first equation in slope-intercept form.	$\begin{aligned} 3x - 2y &= 4 \\ -2y &= -3x + 4 \\ \frac{-2y}{-2} &= \frac{-3x+4}{-2} \\ y &= \frac{3}{2}x - 2 \end{aligned}$
The second equation is already in slope-intercept form.	$y = \frac{3}{2}x - 2$
	Since the slopes are the same, they have the same slope and same y-intercept and so the lines are coincident.

A system of equations whose graphs are coincident lines has infinitely many solutions and is consistent and dependent.

Note:
Exercise:

Problem:

Without graphing, determine the number of solutions and then classify the system of equations.

$$\begin{cases} 4x - 5y = 20 \\ y = \frac{4}{5}x - 4 \end{cases}$$

Solution:

infinitely many solutions, consistent, dependent

Note:
Exercise:

Problem:

Without graphing, determine the number of solutions and then classify the system of equations.

$$\begin{cases} -2x - 4y = 8 \\ y = -\frac{1}{2}x - 2 \end{cases}$$

Solution:

infinitely many solutions, consistent, dependent

Solve Applications of Systems of Equations by Graphing

Here is a problem solving strategy to set up and solve applications of systems of linear equations.

Note:
Use a problem solving strategy for systems of linear equations.

Readthe problem. Make sure all the words and ideas are understood.
Identifywhat we are looking for.
Namewhat we are looking for. Choose variables to represent those quantities.
Translateinto a system of equations.

Solve the system of equations using good algebra techniques.
Check the answer in the problem and make sure it makes sense.
Answer the question with a complete sentence.

Step 5 is where we will use the method introduced in this section. We will graph the equations and find the solution.

Example:
Exercise:

Problem:

Sondra is making 10 quarts of punch from fruit juice and club soda. The number of quarts of fruit juice is 4 times the number of quarts of club soda. How many quarts of fruit juice and how many quarts of club soda does Sondra need?

Solution:
Solution

Step 1. Read the problem.

Step 2. Identify what we are looking for.

We are looking for the number of quarts of fruit juice and the number of quarts of club soda that Sondra will need.

Step 3. Name what we are looking for. Choose variables to represent those quantities.

Let $f =$ number of quarts of fruit juice.
$c =$ number of quarts of club soda

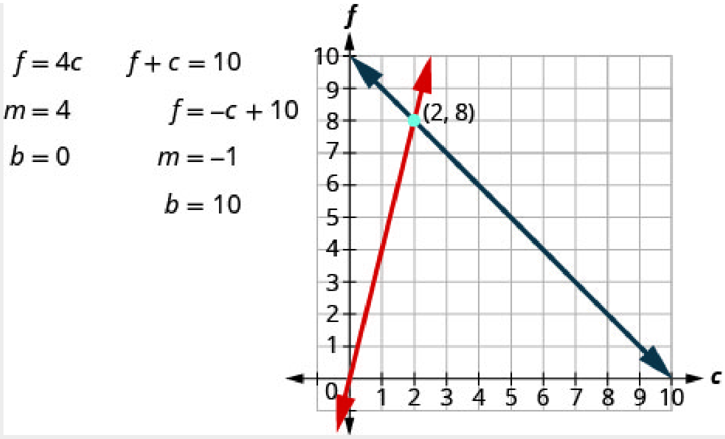
Step 4. Translate into a system of equations.

$$\text{The } \underbrace{\text{number of quarts of fruit juice}}_{f} \underbrace{\text{and}}_{+} \text{ the } \underbrace{\text{number of quarts of club soda}}_{c} \underbrace{\text{is 10}}_{=10}$$

$$\text{The } \underbrace{\text{number of quarts of fruit juice}}_{f} \underbrace{\text{is}}_{=} \underbrace{\text{four times the number of quarts of club soda}}_{4c}$$

We now have the system. $\begin{cases} f + c = 10 \\ f = 4c \end{cases}$

Step 5. Solve the system of equations using good algebra techniques.



The point of intersection (2, 8) is the solution. This means Sondra needs 2 quarts of club soda and 8 quarts of fruit juice.

Step 6. Check the answer in the problem and make sure it makes sense.

Does this make sense in the problem?

Yes, the number of quarts of fruit juice, 8 is 4 times the number of quarts of club soda, 2.

Yes, 10 quarts of punch is 8 quarts of fruit juice plus 2 quarts of club soda.

Step 7. Answer the question with a complete sentence.

Sondra needs 8 quarts of fruit juice and 2 quarts of soda.

Note:
Exercise:

Problem:

Manny is making 12 quarts of orange juice from concentrate and water. The number of quarts of water is 3 times the number of quarts of concentrate. How many quarts of concentrate and how many quarts of water does Manny need?

Solution:

Manny needs 3 quarts juice concentrate and 9 quarts water.

Note:
Exercise:

Problem:
Alisha is making an 18 ounce coffee beverage that is made from brewed coffee and milk. The number of ounces of brewed coffee is 5 times greater than the number of ounces of milk. How many ounces of coffee and how many ounces of milk does Alisha need?
Solution:
Alisha needs 15 ounces of coffee and 3 ounces of milk.

Note:
Access these online resources for additional instruction and practice with solving systems of equations by graphing.

- Instructional Video Solving Linear Systems by Graphing
- Instructional Video Solve by Graphing

Key Concepts

- **To solve a system of linear equations by graphing**

  Graph the first equation.
  Graph the second equation on the same rectangular coordinate system.
  Determine whether the lines intersect, are parallel, or are the same line.

Identify the solution to the system.	If the lines intersect, identify the point of intersection. Check to make sure it is a solution to both equations. This is the solution to the system.	If the lines are parallel, the system has no solution.	If the lines are the same, the system has an infinite number of solutions.

  Check the solution in both equations.

- Determine the number of solutions from the graph of a linear system

Graph	Number of solutions
2 intersecting lines	1
Parallel lines	None
Same line	Infinitely many

- Determine the number of solutions of a linear system by looking at the slopes and intercepts

Number of Solutions of a Linear System of Equations			
Slopes	**Intercepts**	**Type of Lines**	**Number of Solutions**
Different		Intersecting	1 point
Same	Different	Parallel	No solution
Same	Same	Coincident	Infinitely many solutions

- Determine the number of solutions and how to classify a system of equations

Lines	Intersecting	Parallel	Coincident
Number of solutions	1 point	No solution	Infinitely many
Consistent/inconsistent	Consistent	Inconsistent	Consistent
Dependent/independent	Independent	Independent	Dependent

- **Problem Solving Strategy for Systems of Linear Equations**

  Readthe problem. Make sure all the words and ideas are understood.
  Identifywhat we are looking for.
  Namewhat we are looking for. Choose variables to represent those quantities.
  Translateinto a system of equations.
  Solvethe system of equations using good algebra techniques.
  Checkthe answer in the problem and make sure it makes sense.
  Answerthe question with a complete sentence.

Practice Makes Perfect

Determine Whether an Ordered Pair is a Solution of a System of Equations. In the following exercises, determine if the following points are solutions to the given system of equations.

Exercise:

Problem: $\begin{cases}2x-6y=0\\3x-4y=5\end{cases}$

ⓐ $(3,1)$ ⓑ $(-3,4)$

Solution:

ⓐ yes ⓑ no

Exercise:

Problem: $\begin{cases}7x-4y=-1\\-3x-2y=1\end{cases}$

ⓐ ⓑ $(1,-2)$

Exercise:

Problem: $\begin{cases} 2x+y=5 \\ x+y=1 \end{cases}$

ⓐ $(4,-3)$ ⓑ $(2,0)$

Solution:

ⓐ yes ⓑ no

Exercise:

Problem: $\begin{cases} -3x+y=8 \\ -x+2y=-9 \end{cases}$

ⓐ $(-5,-7)$ ⓑ $(-5,7)$

Exercise:

Problem: $\begin{cases} x+y=2 \\ y=\frac{3}{4}x \end{cases}$

ⓐ $\left(\frac{8}{7},\frac{6}{7}\right)$ ⓑ $\left(1,\frac{3}{4}\right)$

Solution:

ⓐ yes ⓑ no

Exercise:

Problem: $\begin{cases} x+y=1 \\ y=\frac{2}{5}x \end{cases}$

ⓐ $\left(\frac{5}{7},\frac{2}{7}\right)$ ⓑ $(5,2)$

Exercise:

Problem: $\begin{cases} x+5y=10 \\ y=\frac{3}{5}x+1 \end{cases}$

ⓐ $(-10,4)$ ⓑ $\left(\frac{5}{4},\frac{7}{4}\right)$

Solution:

ⓐ no ⓑ yes

Exercise:

Problem: $\begin{cases} x + 3y = 9 \\ y = \frac{2}{3}x - 2 \end{cases}$

ⓐ $(-6, 5)$ ⓑ $\left(5, \frac{4}{3}\right)$

Solve a System of Linear Equations by Graphing In the following exercises, solve the following systems of equations by graphing.

Exercise:

Problem: $\begin{cases} 3x + y = -3 \\ 2x + 3y = 5 \end{cases}$

Solution:

$(-2, 3)$

Exercise:

Problem: $\begin{cases} -x + y = 2 \\ 2x + y = -4 \end{cases}$

Exercise:

Problem: $\begin{cases} -3x + y = -1 \\ 2x + y = 4 \end{cases}$

Solution:

$(1, 2)$

Exercise:

Problem: $\begin{cases} -2x + 3y = -3 \\ x + y = 4 \end{cases}$

Exercise:

Problem: $\begin{cases} y = x + 2 \\ y = -2x + 2 \end{cases}$

Solution:

$(0, 2)$

Exercise:

Problem: $\begin{cases} y = x - 2 \\ y = -3x + 2 \end{cases}$

Exercise:

Problem: $\begin{cases} y = \frac{3}{2}x + 1 \\ y = -\frac{1}{2}x + 5 \end{cases}$

Solution:

$(2, 4)$

Exercise:

Problem: $\begin{cases} y = \frac{2}{3}x - 2 \\ y = -\frac{1}{3}x - 5 \end{cases}$

Exercise:

Problem: $\begin{cases} -x + y = -3 \\ 4x + 4y = 4 \end{cases}$

Solution:

$(2, -1)$

Exercise:

Problem: $\begin{cases} x - y = 3 \\ 2x - y = 4 \end{cases}$

Exercise:

Problem: $\begin{cases} -3x + y = -1 \\ 2x + y = 4 \end{cases}$

Solution:

$(1, 2)$

Exercise:

Problem: $\begin{cases} -3x + y = -2 \\ 4x - 2y = 6 \end{cases}$

Exercise:

Problem: $\begin{cases} x+y=5 \\ 2x-y=4 \end{cases}$

Solution:

$(3,2)$

Exercise:

Problem: $\begin{cases} x-y=2 \\ 2x-y=6 \end{cases}$

Exercise:

Problem: $\begin{cases} x+y=2 \\ x-y=0 \end{cases}$

Solution:

$(1,1)$

Exercise:

Problem: $\begin{cases} x+y=6 \\ x-y=-8 \end{cases}$

Exercise:

Problem: $\begin{cases} x+y=-5 \\ x-y=3 \end{cases}$

Solution:

$(-1,-4)$

Exercise:

Problem: $\begin{cases} x+y=4 \\ x-y=0 \end{cases}$

Exercise:

Problem: $\begin{cases} x+y=-4 \\ -x+2y=-2 \end{cases}$

Solution:

$(3, 3)$

Exercise:

Problem: $\begin{cases} -x + 3y = 3 \\ x + 3y = 3 \end{cases}$

Exercise:

Problem: $\begin{cases} -2x + 3y = 3 \\ x + 3y = 12 \end{cases}$

Solution:

$(-5, 6)$

Exercise:

Problem: $\begin{cases} 2x - y = 4 \\ 2x + 3y = 12 \end{cases}$

Exercise:

Problem: $\begin{cases} 2x + 3y = 6 \\ y = -2 \end{cases}$

Solution:

$(6, -2)$

Exercise:

Problem: $\begin{cases} -2x + y = 2 \\ y = 4 \end{cases}$

Exercise:

Problem: $\begin{cases} x - 3y = -3 \\ y = 2 \end{cases}$

Solution:

$(3, 2)$

Exercise:

Problem: $\begin{cases} 2x - 2y = 8 \\ y = -3 \end{cases}$

Exercise:

Problem: $\begin{cases} 2x - y = -1 \\ x = 1 \end{cases}$

Solution:

$(1, 3)$

Exercise:

Problem: $\begin{cases} x + 2y = 2 \\ x = -2 \end{cases}$

Exercise:

Problem: $\begin{cases} x - 3y = -6 \\ x = -3 \end{cases}$

Solution:

$(-3, -1)$

Exercise:

Problem: $\begin{cases} x + y = 4 \\ x = 1 \end{cases}$

Exercise:

Problem: $\begin{cases} 4x - 3y = 8 \\ 8x - 6y = 14 \end{cases}$

Solution:

no solution

Exercise:

Problem: $\begin{cases} x + 3y = 4 \\ -2x - 6y = 3 \end{cases}$

Exercise:

Problem: $\begin{cases} -2x + 4y = 4 \\ y = \frac{1}{2}x \end{cases}$

Solution:

no solution

Exercise:

Problem: $\begin{cases} 3x+5y=10 \\ y=-\frac{3}{5}x+1 \end{cases}$

Exercise:

Problem: $\begin{cases} x=-3y+4 \\ 2x+6y=8 \end{cases}$

Solution:

no solution

Exercise:

Problem: $\begin{cases} 4x=3y+7 \\ 8x-6y=14 \end{cases}$

Exercise:

Problem: $\begin{cases} 2x+y=6 \\ -8x-4y=-24 \end{cases}$

Solution:

infinitely many solutions

Exercise:

Problem: $\begin{cases} 5x+2y=7 \\ -10x-4y=-14 \end{cases}$

Exercise:

Problem: $\begin{cases} x+3y=-6 \\ 4y=-\frac{4}{3}x-8 \end{cases}$

Solution:

infinitely many solutions

Exercise:

Problem: $\begin{cases} -x+2y=-6 \\ y=-\frac{1}{2}x-1 \end{cases}$

Exercise:

Problem: $\begin{cases} -3x+2y=-2 \\ y=-x+4 \end{cases}$

Solution:

$(2,2)$

Exercise:

Problem: $\begin{cases} -x+2y=-2 \\ y=-x-1 \end{cases}$

Determine the Number of Solutions of a Linear System Without graphing the following systems of equations, determine the number of solutions and then classify the system of equations.

Exercise:

Problem: $\begin{cases} y=\frac{2}{3}x+1 \\ -2x+3y=5 \end{cases}$

Solution:

0 solutions

Exercise:

Problem: $\begin{cases} y=\frac{1}{3}x+2 \\ x-3y=9 \end{cases}$

Exercise:

Problem: $\begin{cases} y=-2x+1 \\ 4x+2y=8 \end{cases}$

Solution:

0 solutions

Exercise:

Problem: $\begin{cases} y = 3x + 4 \\ 9x - 3y = 18 \end{cases}$

Exercise:

Problem: $\begin{cases} y = \frac{2}{3}x + 1 \\ 2x - 3y = 7 \end{cases}$

Solution:

0 solutions

Exercise:

Problem: $\begin{cases} 3x + 4y = 12 \\ y = -3x - 1 \end{cases}$

Exercise:

Problem: $\begin{cases} 4x + 2y = 10 \\ 4x - 2y = -6 \end{cases}$

Solution:

consistent, 1 solution

Exercise:

Problem: $\begin{cases} 5x + 3y = 4 \\ 2x - 3y = 5 \end{cases}$

Exercise:

Problem: $\begin{cases} y = -\frac{1}{2}x + 5 \\ x + 2y = 10 \end{cases}$

Solution:

infinitely many solutions

Exercise:

Problem: $\begin{cases} y = x + 1 \\ -x + y = 1 \end{cases}$

Exercise:

Problem: $\begin{cases} y = 2x + 3 \\ 2x - y = -3 \end{cases}$

Solution:

infinitely many solutions

Exercise:

Problem: $\begin{cases} 5x - 2y = 10 \\ y = \frac{5}{2}x - 5 \end{cases}$

Solve Applications of Systems of Equations by Graphing In the following exercises, solve.
Exercise:

Problem:

Molly is making strawberry infused water. For each ounce of strawberry juice, she uses three times as many ounces of water. How many ounces of strawberry juice and how many ounces of water does she need to make 64 ounces of strawberry infused water?

Solution:

Molly needs 16 ounces of strawberry juice and 48 ounces of water.

Exercise:

Problem:

Jamal is making a snack mix that contains only pretzels and nuts. For every ounce of nuts, he will use 2 ounces of pretzels. How many ounces of pretzels and how many ounces of nuts does he need to make 45 ounces of snack mix?

Exercise:

Problem:

Enrique is making a party mix that contains raisins and nuts. For each ounce of nuts, he uses twice the amount of raisins. How many ounces of nuts and how many ounces of raisins does he need to make 24 ounces of party mix?

Solution:

Enrique needs 8 ounces of nuts and 16 ounces of water.

Exercise:

Problem:

Owen is making lemonade from concentrate. The number of quarts of water he needs is 4 times the number of quarts of concentrate. How many quarts of water and how many quarts of concentrate does Owen need to make 100 quarts of lemonade?

Everyday Math

Exercise:

Problem:

Leo is planning his spring flower garden. He wants to plant tulip and daffodil bulbs. He will plant 6 times as many daffodil bulbs as tulip bulbs. If he wants to plant 350 bulbs, how many tulip bulbs and how many daffodil bulbs should he plant?

Solution:

Leo should plant 50 tulips and 300 daffodils.

Exercise:

Problem:

A marketing company surveys 1,200 people. They surveyed twice as many females as males. How many males and females did they survey?

Writing Exercises

Exercise:

Problem:

In a system of linear equations, the two equations have the same slope. Describe the possible solutions to the system.

Solution:

Given that it is only known that the slopes of both linear equations are the same, there are either no solutions (the graphs of the equations are parallel) or infinitely many.

Exercise:

Problem:

In a system of linear equations, the two equations have the same intercepts. Describe the possible solutions to the system.

Self Check

After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
determine whether an ordered pair is a solution of a system of equations.			
solve a system of linear equations by graphing.			
determine the number of solutions of a linear system.			
solve applications of systems of equations by graphing.			

If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

Glossary

coincident lines
: Coincident lines are lines that have the same slope and same *y*-intercept.

consistent system
: A consistent system of equations is a system of equations with at least one solution.

dependent equations
: Two equations are dependent if all the solutions of one equation are also solutions of the other equation.

inconsistent system
: An inconsistent system of equations is a system of equations with no solution.

independent equations

Two equations are independent if they have different solutions.

solutions of a system of equations

Solutions of a system of equations are the values of the variables that make all the equations true. A solution of a system of two linear equations is represented by an ordered pair (x, y).

system of linear equations

When two or more linear equations are grouped together, they form a system of linear equations.

Solve Systems of Equations by Substitution: ASE
By the end of this section, you will be able to:

- Solve a system of equations by substitution
- Solve applications of systems of equations by substitution

Solving systems of linear equations by graphing is a good way to visualize the types of solutions that may result. However, there are many cases where solving a system by graphing is inconvenient or imprecise. If the graphs extend beyond the small grid with *x* and *y* both between −10 and 10, graphing the lines may be cumbersome. And if the solutions to the system are not integers, it can be hard to read their values precisely from a graph.

In this section, we will solve systems of linear equations by the substitution method.

Solve a System of Equations by Substitution

We will use the same system we used first for graphing.
Equation:

$$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$$

We will first solve one of the equations for either *x* or *y*. We can choose either equation and solve for either variable—but we'll try to make a choice that will keep the work easy.

Then we substitute that expression into the other equation. The result is an equation with just one variable—and we know how to solve those!

After we find the value of one variable, we will substitute that value into one of the original equations and solve for the other variable. Finally, we check our solution and make sure it makes both equations true.

We'll fill in all these steps now in [link].

Example:
How to Solve a System of Equations by Substitution
Exercise:

Problem: Solve the system by substitution. $\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$

Solution:
Solution

Step 1. Solve one of the equations for either variable.	We'll solve the first equation for *y*.	$2x + y = 7$ $y = 7 - 2x$
Step 2. Substitute the expression from Step 1 into the other equation.	We replace *y* in the second equation with the expression $7 - 2x$.	$x - 2y = 6$ $x - 2(7 - 2x) = 6$
Step 3. Solve the resulting equation.	Now we have an equation with just 1 variable. We know how to solve this!	$x - 2(7 - 2x) = 6$ $x - 14 + 4x = 6$ $5x = 20$ $x = 4$
Step 4. Substitute the solution in Step 3 into one of the original equations to find the other variable.	We'll use the first equation and replace *x* with 4.	$2x + y = 7$ $2(4) + y = 7$ $8 + y = 7$ $y = -1$
Step 5. Write the solution as an ordered pair.	The ordered pair is (x, y).	$(4, -1)$

Step 6. Check that the ordered pair is a solution to **both** original equations.	Substitute (4, –1) into both equations and make sure they are both true.	$2x + y = 7$ $x - 2y = 6$ $2(4) + (-1) \stackrel{?}{=} 7$ $4 - 2(-1) \stackrel{?}{=} 6$ $7 = 7$ ✓ $6 = 6$ ✓ Both equations are true. (4, –1) is the solution to the system.

Note:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} -2x + y = -11 \\ x + 3y = 9 \end{cases}$

Solution:

$(6, 1)$

Note:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} x + 3y = 10 \\ 4x + y = 18 \end{cases}$

Solution:

$(4, 2)$

Note:
Solve a system of equations by substitution.

Solve one of the equations for either variable.
Substitute the expression from Step 1 into the other equation.

Solve the resulting equation.
Substitute the solution in Step 3 into one of the original equations to find the other variable.
Write the solution as an ordered pair.
Check that the ordered pair is a solution to**both**original equations.

If one of the equations in the system is given in slope–intercept form, Step 1 is already done! We’ll see this in [link].

Example:
Exercise:

Problem: Solve the system by substitution.

$$\begin{cases} x + y = -1 \\ y = x + 5 \end{cases}$$

Solution:
Solution

The second equation is already solved for *y*. We will substitute the expression in place of *y* in the first equation.

	$\begin{cases} x + y = -1 \\ \quad y = x + 5 \end{cases}$
The second equation is already solved for *y*. We will substitute into the first equation.	

Replace the *y* with *x* + 5.	$y = x + 5$ $x + y = -1$
Solve the resulting equation for *x*.	$x + x + 5 = -1$
	$2x + 5 = -1$
	$2x = -6$
Substitute *x* = −3 into *y* = *x* + 5 to find *y*.	$x = -3$ $y = x + 5$
	$y = -3 + 5$
The ordered pair is (−3, 2).	$y = 2$
Check the ordered pair in both equations: $\begin{array}{rcl} x+y & = & -1 \\ -3+2 & \stackrel{?}{=} & -1 \\ -1 & = & -1\,\checkmark \end{array}$ $\begin{array}{rcl} y & = & x+5 \\ 2 & \stackrel{?}{=} & -3+5 \\ 2 & = & 2\,\checkmark \end{array}$	
	The solution is (−3, 2).

Note:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} x+y=6 \\ y=3x-2 \end{cases}$

Solution:

$(2,4)$

Note:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} 2x-y=1 \\ y=-3x-6 \end{cases}$

Solution:

$(-1,-3)$

If the equations are given in standard form, we'll need to start by solving for one of the variables. In this next example, we'll solve the first equation for *y*.

Example:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} 3x+y=5 \\ 2x+4y=-10 \end{cases}$

Solution:
Solution

We need to solve one equation for one variable. Then we will substitute that expression into the other equation.

Solve for *y*. Substitute into the other equation.	$\begin{aligned} 3x + y &= 5 \\ y &= -3x + 5 \\ 2x + 4y &= -10 \end{aligned}$
Replace the *y* with −3*x* + 5.	$2x + 4(-3x + 5) = -10$
Solve the resulting equation for *x*.	$2x - 12x + 20 = -10$
	$-10x + 20 = -10$ $-10x = -30$
Substitute *x* = 3 into 3*x* + *y* = 5 to find *y*.	$\begin{aligned} x &= -3 \\ 3x + y &= 5 \end{aligned}$
	$3(3) + y = 5$ $9 + y = 5$
The ordered pair is (3, −4).	$y = -4$
Check the ordered pair in both equations: $\begin{aligned} 3x + y &= 5 \\ 3 \cdot 3 + (-4) &\stackrel{?}{=} 5 \\ 9 - 4 &\stackrel{?}{=} 5 \\ 5 &= 5 \checkmark \end{aligned}$ $\quad$ $\begin{aligned} 2x + 4y &= -10 \\ 2 \cdot 3 + 4(-4) &= -10 \\ 6 - 16 &\stackrel{?}{=} -10 \\ -10 &= -10 \checkmark \end{aligned}$	
	The solution is (3, −4).

Note:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} 4x + y = 2 \\ 3x + 2y = -1 \end{cases}$

Solution:

$(1, -2)$

Note:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} -x + y = 4 \\ 4x - y = 2 \end{cases}$

Solution:

$(2, 6)$

In [link] it was easiest to solve for *y* in the first equation because it had a coefficient of 1. In [link] it will be easier to solve for *x*.

Example:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} x - 2y = -2 \\ 3x + 2y = 34 \end{cases}$

Solution:

Solution

We will solve the first equation for x and then substitute the expression into the second equation.

	$x - 2y = -2$
Solve for x. Substitute into the other equation.	$x = 2y - 2$ $3x + 2y = 34$
Replace the x with $2y - 2$.	$3(2y - 2) + 2y = 34$
Solve the resulting equation for y.	$6y - 6 + 2y = 34$
Substitute $y = 5$ into $x - 2y = -2$ to find x.	$8y - 6 = 34$ $8y = 40$ $y = 5$ $x - 2y = -2$ $x - 2 \cdot 5 = -2$ $x - 10 = -2$ $x = 8$

The ordered pair is (8, 5).	
Check the ordered pair in both equations: $$\begin{aligned} x - 2y &= -2 \\ 8 - 2\cdot 5 &\stackrel{?}{=} -2 \\ 8 - 10 &\stackrel{?}{=} -2 \\ -2 &= -2 \checkmark \end{aligned} \qquad \begin{aligned} 3x + 2y &= 34 \\ 3\cdot 8 + 2\cdot 5 &\stackrel{?}{=} 34 \\ 24 + 10 &\stackrel{?}{=} 34 \\ 34 &= 34 \checkmark \end{aligned}$$	
	The solution is (8, 5).

Note:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} x - 5y = 13 \\ 4x - 3y = 1 \end{cases}$

Solution:

$(-2, -3)$

Note:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} x - 6y = -6 \\ 2x - 4y = 4 \end{cases}$

Solution:

$(6, 2)$

When both equations are already solved for the same variable, it is easy to substitute!

Example:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} y = -2x + 5 \\ y = \frac{1}{2}x \end{cases}$

Solution:
Solution

Since both equations are solved for *y*, we can substitute one into the other.

Substitute $\frac{1}{2}x$ for *y* in the first equation.	$y = \frac{1}{2}x$ $y = -2x + 5$
Replace the *y* with $\frac{1}{2}x$.	$\frac{1}{2}x = -2x + 5$
Solve the resulting equation. Start by clearing the fraction.	$2\left(\frac{1}{2}x\right) = 2(-2x + 5)$
Solve for *x*.	$x = -4x + 10$
	$5x = 10$

Substitute $x = 2$ into $y = \frac{1}{2}x$ to find y.	$x = 2$ $y = \frac{1}{2}x$ $y = \frac{1}{2} \cdot 2$ $y = 1$
The ordered pair is (2,1).	
Check the ordered pair in both equations: $\begin{aligned} y &= \tfrac{1}{2}x \\ 1 &\stackrel{?}{=} \tfrac{1}{2} \cdot 2 \\ 1 &= 1 \checkmark \end{aligned}$ $\quad$ $\begin{aligned} y &= -2x + 5 \\ 1 &\stackrel{?}{=} -2 \cdot 2 + 5 \\ 1 &= -4 + 5 \\ 1 &= 1 \checkmark \end{aligned}$	
	The solution is (2,1).

Note:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} y = 3x - 16 \\ y = \frac{1}{3}x \end{cases}$

Solution:

$(6, 2)$

Note:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} y = -x + 10 \\ y = \frac{1}{4}x \end{cases}$

Solution:

$(8, 2)$

Be very careful with the signs in the next example.

Example:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} 4x + 2y = 4 \\ 6x - y = 8 \end{cases}$

Solution:
Solution

We need to solve one equation for one variable. We will solve the first equation for y.

	$4x + 2y = 4$
Solve the first equation for y.	$2y = -4x + 4$
Substitute $-2x + 2$ for y in the second equation.	

	$y = -2x + 2$ $6x - y = 8$
Replace the y with $-2x + 2$.	$6x - (-2x + 2) = 8$
Solve the equation for x.	$6x + 2x - 2 = 8$
	$8x - 2 = 8$ $8x = 10$
Substitute $x = \frac{5}{4}$ into $4x + 2y = 4$ to find y.	$x = \frac{5}{4}$ $4x + 2y = 4$ $4\left(\frac{5}{4}\right) + 2y = 4$ $5 + 2y = 4$ $2y = -1$ $y = -\frac{1}{2}$
The ordered pair is $\left(\frac{5}{4}, -\frac{1}{2}\right)$.	
Check the ordered pair in both equations.	

$$\begin{aligned} 4x+2y &= 4 \\ 4\left(\tfrac{5}{4}\right)+2\left(-\tfrac{1}{2}\right) &\stackrel{?}{=} 4 \\ 5-1 &\stackrel{?}{=} 4 \\ 4 &= 4 \checkmark \end{aligned}$$

$$\begin{aligned} 6x-y &= 8 \\ 6\left(\tfrac{5}{4}\right)-\left(-\tfrac{1}{2}\right) &\stackrel{?}{=} 8 \\ \tfrac{15}{4}-\left(-\tfrac{1}{2}\right) &\stackrel{?}{=} 8 \\ \tfrac{16}{2} &\stackrel{?}{=} 8 \\ 8 &= 8 \checkmark \end{aligned}$$

The solution is $\left(\frac{5}{4},-\frac{1}{2}\right)$.

Note:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} x-4y=-4 \\ -3x+4y=0 \end{cases}$

Solution:

$\left(2,\frac{3}{2}\right)$

Note:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} 4x-y=0 \\ 2x-3y=5 \end{cases}$

Solution:

$\left(-\frac{1}{2},-2\right)$

In [link], it will take a little more work to solve one equation for *x* or *y*.

Example:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} 4x - 3y = 6 \\ 15y - 20x = -30 \end{cases}$

Solution:
Solution

We need to solve one equation for one variable. We will solve the first equation for *x*.

	$4x - 3y = 6$
Solve the first equation for *x*.	$4x = 3y + 6$
Substitute $\frac{3}{4}y + \frac{3}{2}$ for *x* in the second equation.	$x = \frac{3}{4}y + \frac{3}{2}$ $15y - 20x = -30$
Replace the *x* with $\frac{3}{4}y + \frac{3}{2}$.	$15y - 20\left(\frac{3}{4}y + \frac{3}{2}\right) = -30$
Solve for *y*.	$15y - 15y - 30 = -30$

	$0 - 30 = -30$
	$0 = 0$

Since 0 = 0 is a true statement, the system is consistent. The equations are dependent. The graphs of these two equations would give the same line. The system has infinitely many solutions.

Note:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} 2x - 3y = 12 \\ -12y + 8x = 48 \end{cases}$

Solution:

infinitely many solutions

Note:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} 5x + 2y = 12 \\ -4y - 10x = -24 \end{cases}$

Solution:

infinitely many solutions

Look back at the equations in [link]. Is there any way to recognize that they are the same line?

Let’s see what happens in the next example.

Example:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} 5x - 2y = -10 \\ y = \frac{5}{2}x \end{cases}$

Solution:
Solution

The second equation is already solved for *y*, so we can substitute for *y* in the first equation.

Substitute *x* for *y* in the first equation.	$y = \frac{5}{2}x$ $5x - 2y = -10$
Replace the *y* with $\frac{5}{2}x$.	$5x - 2\left(\frac{5}{2}x\right) = -10$
Solve for *x*.	$5x - 5x = -10$
	$0 \neq -10$

Since 0 = −10 is a false statement the equations are inconsistent. The graphs of the two equation would be parallel lines. The system has no solutions.

Note:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} 3x + 2y = 9 \\ y = -\frac{3}{2}x + 1 \end{cases}$

Solution:

no solution

Note:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} 5x - 3y = 2 \\ y = \frac{5}{3}x - 4 \end{cases}$

Solution:

no solution

Solve Applications of Systems of Equations by Substitution

We’ll copy here the problem solving strategy we used in the Solving Systems of Equations by Graphing section for solving systems of equations. Now that we know how to solve systems by substitution, that’s what we’ll do in Step 5.

Note:
How to use a problem solving strategy for systems of linear equations.

Readthe problem. Make sure all the words and ideas are understood.
Identifywhat we are looking for.
Namewhat we are looking for. Choose variables to represent those quantities.
Translateinto a system of equations.

Solve the system of equations using good algebra techniques.
Check the answer in the problem and make sure it makes sense.
Answer the question with a complete sentence.

Some people find setting up word problems with two variables easier than setting them up with just one variable. Choosing the variable names is easier when all you need to do is write down two letters. Think about this in the next example—how would you have done it with just one variable?

Example:
Exercise:

Problem:

The sum of two numbers is zero. One number is nine less than the other. Find the numbers.

Solution:
Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for two numbers.
Step 3. Name what we are looking for.	Let $n =$ the first number Let $m =$ the second number
Step 4. Translate into a system of equations.	The sum of two numbers is zero.
	$n + m = 0$

	One number is nine less than the other.
	$n = m - 9$
The system is:	$\begin{cases} n + m = 0 \\ n = m - 9 \end{cases}$
Step 5. Solve the system of equations. We will use substitution since the second equation is solved for *n*.	
Substitute $m - 9$ for n in the first equation.	$n = m - 9$ $n + m = 0$
Solve for *m*.	$m - 9 + m = 0$
	$2m - 9 = 0$
	$2m = 9$
Substitute $m = \frac{9}{2}$ into the second equation and then solve for *n*.	$m = \frac{9}{2}$ $n = m - 9$
	$m = \frac{9}{2} - 9$

	$m = \frac{9}{2} - \frac{18}{2}$
	$n = -\frac{9}{2}$
Step 6. Check the answer in the problem.	Do these numbers make sense in the problem? We will leave this to you!
Step 7. Answer the question.	The numbers are $\frac{9}{2}$ and $-\frac{9}{2}$.

Note:
Exercise:

Problem:

The sum of two numbers is 10. One number is 4 less than the other. Find the numbers.

Solution:

The numbers are 3 and 7.

Note:
Exercise:

Problem:

The sum of two number is −6. One number is 10 less than the other. Find the numbers.

Solution:

The numbers are 2 and −8.

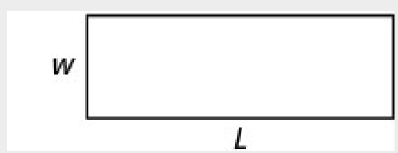
In the [link], we'll use the formula for the perimeter of a rectangle, $P = 2L + 2W$.

Example:
Exercise:

Problem:

The perimeter of a rectangle is 88. The length is five more than twice the width. Find the length and the width.

Solution:
Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	We are looking for the length and width.
Step 3. Name what we are looking for.	Let $L =$ the length $W =$ the width
Step 4. Translate into a system of equations.	The perimeter of a rectangle is 88.

	$2L + 2W = P$ $2L + 2W = 88$
	The length is five more than twice the width.
	$L = 2W + 5$
The system is:	$\begin{cases} 2L + 2W = 88 \\ L = 2W + 5 \end{cases}$
Step 5. Solve the system of equations. We will use substitution since the second equation is solved for *L*. Substitute $2W + 5$ for *L* in the first equation.	$L = 2W + 5$ $2L + 2W = 88$
Solve for *W*.	$2(2W + 5) + 2W = 88$
	$4W + 10 + 2W = 88$
	$6W + 10 = 88$
	$6W = 78$
Substitute $W = 13$ into the second equation and then solve for *L*.	$W = 13$ $L = 2W + 5$

	$L = 2 \cdot 13 + 5$
	$L = 31$
Step 6. Check the answer in the problem.	Does a rectangle with length 31 and width 13 have perimeter 88? Yes.
Step 7. Answer the equation.	The length is 31 and the width is 13.

Note:
Exercise:

Problem:

The perimeter of a rectangle is 40. The length is 4 more than the width. Find the length and width of the rectangle.

Solution:

The length is 12 and the width is 8.

Note:
Exercise:

Problem:

The perimeter of a rectangle is 58. The length is 5 more than three times the width. Find the length and width of the rectangle.

Solution:

The length is 23 and the width is 6.

For [link] we need to remember that the sum of the measures of the angles of a triangle is 180 degrees and that a right triangle has one 90 degree angle.

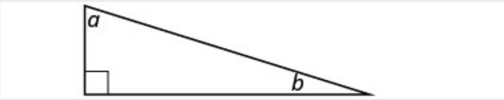
Example:
Exercise:

Problem:

The measure of one of the small angles of a right triangle is ten more than three times the measure of the other small angle. Find the measures of both angles.

Solution:
Solution

We will draw and label a figure.

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	We are looking for the measures of the angles.
Step 3. Name what we are looking for.	Let $a =$ the measure of the 1st angle $b =$ the measure of the 2nd angle
Step 4. Translate into a system	The measure of one of the small

of equations.	angles of a right triangle is ten more than three times the measure of the other small angle.
	$a = 3b + 10$
	The sum of the measures of the angles of a triangle is 180.
	$a + b + 90 = 180$
The system is:	$\begin{cases} a = 3b + 10 \\ a + b + 90 = 180 \end{cases}$
Step 5. Solve the system of equations. We will use substitution since the first equation is solved for *a*.	$a = 3b + 10$ $a + b + 90 = 180$
Substitute 3*b* + 10 for *a* in the second equation.	$(3b + 10) + b + 90 = 180$
Solve for *b*.	$4b + 100 = 180$
	$4b = 80$
	$b = 20$ $a = 3b + 10$

Substitute $b = 20$ into the first equation and then solve for a.	$a = 3 \cdot 20 + 10$ $a = 70$
Step 6. Check the answer in the problem.	We will leave this to you!
Step 7. Answer the question.	The measures of the small angles are 20 and 70.

Note:
Exercise:

Problem:

The measure of one of the small angles of a right triangle is 2 more than 3 times the measure of the other small angle. Find the measure of both angles.

Solution:

The measure of the angles are 22 degrees and 68 degrees.

Note:
Exercise:

Problem:

The measure of one of the small angles of a right triangle is 18 less than twice the measure of the other small angle. Find the measure of both angles.

Solution:

The measure of the angles are 36 degrees and 54 degrees.

Example:
Exercise:

Problem:

Heather has been offered two options for her salary as a trainer at the gym. Option A would pay her \$25,000 plus \$15 for each training session. Option B would pay her \$10,000 + \$40 for each training session. How many training sessions would make the salary options equal?

Solution:
Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	We are looking for the number of training sessions that would make the pay equal.
Step 3. Name what we are looking for.	Let $s =$ Heather's salary. $n =$ the number of training sessions
Step 4. Translate into a system of equations.	Option A would pay her \$25,000 plus \$15 for each training session.
	$s = 25{,}000 + 15n$
	Option B would pay her \$10,000 + \$40 for each training session
	$s = 10{,}000 + 40n$

The system is:	$\begin{cases} s = 25{,}000 + 15n \\ s = 10{,}000 + 40n \end{cases}$
Step 5. Solve the system of equations. We will use substitution.	$s = 25{,}000 + 15n$ $s = 10{,}000 + 40n$
Substitute 25,000 + 15*n* for *s* in the second equation.	$25{,}000 + 15n = 10{,}000 + 40n$
Solve for *n*.	$25{,}000 = 10{,}000 + 25n$
	$15{,}000 = 25n$
	$600 = n$
Step 6. Check the answer.	Are 600 training sessions a year reasonable? Are the two options equal when *n* = 600?
Step 7. Answer the question.	The salary options would be equal for 600 training sessions.

Note:
Exercise:

Problem:

Geraldine has been offered positions by two insurance companies. The first company pays a salary of $12,000 plus a commission of $100 for each policy sold. The second pays a salary of $20,000 plus a commission of $50 for each policy sold. How many policies would need to be sold to make the total pay the same?

Solution:

There would need to be 160 policies sold to make the total pay the same.

Note:
Exercise:

Problem:

Kenneth currently sells suits for company A at a salary of $22,000 plus a $10 commission for each suit sold. Company B offers him a position with a salary of $28,000 plus a $4 commission for each suit sold. How many suits would Kenneth need to sell for the options to be equal?

Solution:

Kenneth would need to sell 1,000 suits.

Note:
Access these online resources for additional instruction and practice with solving systems of equations by substitution.

- Instructional Video-Solve Linear Systems by Substitution
- Instructional Video-Solve by Substitution

Key Concepts

- **Solve a system of equations by substitution**

  Solve one of the equations for either variable.
  Substitute the expression from Step 1 into the other equation.
  Solve the resulting equation.
  Substitute the solution in Step 3 into one of the original equations to find the other variable.
  Write the solution as an ordered pair.
  Check that the ordered pair is a solution to both original equations.

Practice Makes Perfect

Solve a System of Equations by Substitution

In the following exercises, solve the systems of equations by substitution.

Exercise:

Problem: $\begin{cases} 2x + y = -4 \\ 3x - 2y = -6 \end{cases}$

Solution:

$(-2, 0)$

Exercise:

Problem: $\begin{cases} 2x + y = -2 \\ 3x - y = 7 \end{cases}$

Exercise:

Problem: $\begin{cases} x - 2y = -5 \\ 2x - 3y = -4 \end{cases}$

Solution:

$(7, 6)$

Exercise:

Problem: $\begin{cases} x-3y=-9 \\ 2x+5y=4 \end{cases}$

Exercise:

Problem: $\begin{cases} 5x-2y=-6 \\ y=3x+3 \end{cases}$

Solution:

$(0,3)$

Exercise:

Problem: $\begin{cases} -2x+2y=6 \\ y=-3x+1 \end{cases}$

Exercise:

Problem: $\begin{cases} 2x+3y=3 \\ y=-x+3 \end{cases}$

Solution:

$(6,-3)$

Exercise:

Problem: $\begin{cases} 2x+5y=-14 \\ y=-2x+2 \end{cases}$

Exercise:

Problem: $\begin{cases} 2x+5y=1 \\ y=\frac{1}{3}x-2 \end{cases}$

Solution:

$(3,-1)$

Exercise:

Problem: $\begin{cases} 3x + 4y = 1 \\ y = -\frac{2}{5}x + 2 \end{cases}$

Exercise:

Problem: $\begin{cases} 3x - 2y = 6 \\ y = \frac{2}{3}x + 2 \end{cases}$

Solution:

$(6, 6)$

Exercise:

Problem: $\begin{cases} -3x - 5y = 3 \\ y = \frac{1}{2}x - 5 \end{cases}$

Exercise:

Problem: $\begin{cases} 2x + y = 10 \\ -x + y = -5 \end{cases}$

Solution:

$(5, 0)$

Exercise:

Problem: $\begin{cases} -2x + y = 10 \\ -x + 2y = 16 \end{cases}$

Exercise:

Problem: $\begin{cases} 3x + y = 1 \\ -4x + y = 15 \end{cases}$

Solution:

$(-2, 7)$

Exercise:

Problem: $\begin{cases} x + y = 0 \\ 2x + 3y = -4 \end{cases}$

Exercise:

Problem: $\begin{cases} x + 3y = 1 \\ 3x + 5y = -5 \end{cases}$

Solution:

$(-5, 2)$

Exercise:

Problem: $\begin{cases} x + 2y = -1 \\ 2x + 3y = 1 \end{cases}$

Exercise:

Problem: $\begin{cases} 2x + y = 5 \\ x - 2y = -15 \end{cases}$

Solution:

$(-1, 7)$

Exercise:

Problem: $\begin{cases} 4x + y = 10 \\ x - 2y = -20 \end{cases}$

Exercise:

Problem: $\begin{cases} y = -2x - 1 \\ y = -\frac{1}{3}x + 4 \end{cases}$

Solution:

$(-3, 5)$

Exercise:

Problem: $\begin{cases} y = x - 6 \\ y = -\frac{3}{2}x + 4 \end{cases}$

Exercise:

Problem: $\begin{cases} y = 2x - 8 \\ y = \frac{3}{5}x + 6 \end{cases}$

Solution:

(10, 12)

Exercise:

Problem: $\begin{cases} y = -x - 1 \\ y = x + 7 \end{cases}$

Exercise:

Problem: $\begin{cases} 4x + 2y = 8 \\ 8x - y = 1 \end{cases}$

Solution:

$(\frac{1}{2}, 3)$

Exercise:

Problem: $\begin{cases} -x - 12y = -1 \\ 2x - 8y = -6 \end{cases}$

Exercise:

Problem: $\begin{cases} 15x + 2y = 6 \\ -5x + 2y = -4 \end{cases}$

Solution:

$\left(\frac{1}{2}, -\frac{3}{4}\right)$

Exercise:

Problem: $\begin{cases} 2x - 15y = 7 \\ 12x + 2y = -4 \end{cases}$

Exercise:

Problem: $\begin{cases} y = 3x \\ 6x - 2y = 0 \end{cases}$

Solution:

Infinitely many solutions

Exercise:

Problem: $\begin{cases} x = 2y \\ 4x - 8y = 0 \end{cases}$

Exercise:

Problem: $\begin{cases} 2x + 16y = 8 \\ -x - 8y = -4 \end{cases}$

Solution:

Infinitely many solutions

Exercise:

Problem: $\begin{cases} 15x + 4y = 6 \\ -30x - 8y = -12 \end{cases}$

Exercise:

Problem: $\begin{cases} y = -4x \\ 4x + y = 1 \end{cases}$

Solution:

No solution

Exercise:

Problem: $\begin{cases} y = -\frac{1}{4}x \\ x + 4y = 8 \end{cases}$

Exercise:

Problem: $\begin{cases} y = \frac{7}{8}x + 4 \\ -7x + 8y = 6 \end{cases}$

Solution:

No solution

Exercise:

Problem: $\begin{cases} y = -\frac{2}{3}x + 5 \\ 2x + 3y = 11 \end{cases}$

Solve Applications of Systems of Equations by Substitution

In the following exercises, translate to a system of equations and solve.

Exercise:

Problem:

The sum of two numbers is 15. One number is 3 less than the other. Find the numbers.

Solution:

The numbers are 6 and 9.

Exercise:

Problem:

The sum of two numbers is 30. One number is 4 less than the other. Find the numbers.

Exercise:

Problem:

The sum of two numbers is −26. One number is 12 less than the other. Find the numbers.

Solution:

The numbers are −7 and −19.

Exercise:

Problem:

The perimeter of a rectangle is 50. The length is 5 more than the width. Find the length and width.

Exercise:

Problem:

The perimeter of a rectangle is 60. The length is 10 more than the width. Find the length and width.

Solution:

The length is 20 and the width is 10.

Exercise:

Problem:

The perimeter of a rectangle is 58. The length is 5 more than three times the width. Find the length and width.

Exercise:

Problem:

The perimeter of a rectangle is 84. The length is 10 more than three times the width. Find the length and width.

Solution:

The length is 34 and the width is 8.

Exercise:

Problem:

The measure of one of the small angles of a right triangle is 14 more than 3 times the measure of the other small angle. Find the measure of both angles.

Exercise:

Problem:

The measure of one of the small angles of a right triangle is 26 more than 3 times the measure of the other small angle. Find the measure of both angles.

Solution:

The measures are 16° and 74°.

Exercise:

Problem:

The measure of one of the small angles of a right triangle is 15 less than twice the measure of the other small angle. Find the measure of both angles.

Exercise:

Problem:

The measure of one of the small angles of a right triangle is 45 less than twice the measure of the other small angle. Find the measure of both angles.

Solution:

The measures are 45° and 45°.

Exercise:

Problem:

Maxim has been offered positions by two car dealers. The first company pays a salary of $10,000 plus a commission of $1,000 for each car sold. The second pays a salary of $20,000 plus a commission of $500 for each car sold. How many cars would need to be sold to make the total pay the same?

Exercise:

Problem:

Jackie has been offered positions by two cable companies. The first company pays a salary of $ 14,000 plus a commission of $100 for each cable package sold. The second pays a salary of $20,000 plus a commission of $25 for each cable package sold. How many cable packages would need to be sold to make the total pay the same?

Solution:

80 cable packages would need to be sold.

Exercise:

Problem:

Amara currently sells televisions for company A at a salary of $17,000 plus a $100 commission for each television she sells. Company B offers her a position with a salary of $29,000 plus a $20 commission for each television she sells. How televisions would Amara need to sell for the options to be equal?

Exercise:

Problem:

Mitchell currently sells stoves for company A at a salary of $12,000 plus a $150 commission for each stove he sells. Company B offers him a position with a salary of $24,000 plus a $50 commission for each stove he sells. How many stoves would Mitchell need to sell for the options to be equal?

Solution:

Mitchell would need to sell 120 stoves.

Everyday Math

Exercise:

Problem:

When Gloria spent 15 minutes on the elliptical trainer and then did circuit training for 30 minutes, her fitness app says she burned 435 calories. When she spent 30 minutes on the elliptical trainer and 40 minutes circuit training she burned 690 calories. Solve the system $\begin{cases} 15e + 30c = 435 \\ 30e + 40c = 690 \end{cases}$ for e, the number of calories she burns for each minute on the elliptical trainer, and c, the number of calories she burns for each minute of circuit training.

Exercise:

Problem:

Stephanie left Riverside, California, driving her motorhome north on Interstate 15 towards Salt Lake City at a speed of 56 miles per hour. Half an hour later, Tina left Riverside in her car on the same route as Stephanie, driving 70 miles per hour. Solve the system $\begin{cases} 56s = 70t \\ s = t + \frac{1}{2} \end{cases}$.

ⓐ for t to find out how long it will take Tina to catch up to Stephanie.
ⓑ what is the value of s, the number of hours Stephanie will have driven before Tina catches up to her?

Solution:

ⓐ $t = 2$ hours ⓑ $s = 2\frac{1}{2}$ hours

Writing Exercises

Exercise:

Problem: Solve the system of equations $\begin{cases} x + y = 10 \\ x - y = 6 \end{cases}$

ⓐ by graphing.
ⓑ by substitution.

ⓒ Which method do you prefer? Why?

Exercise:

Problem: Solve the system of equations $\begin{cases} 3x + y = 12 \\ x = y - 8 \end{cases}$ by substitution and explain all your steps in words.

Solution:

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
solve a system of equations by substitution.			
solve applications of systems of equations by substitution.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

Solve Systems of Equations by Elimination: ASE
By the end of this section, you will be able to:

- Solve a system of equations by elimination
- Solve applications of systems of equations by elimination
- Choose the most convenient method to solve a system of linear equations

We have solved systems of linear equations by graphing and by substitution. Graphing works well when the variable coefficients are small and the solution has integer values. Substitution works well when we can easily solve one equation for one of the variables and not have too many fractions in the resulting expression.

The third method of solving systems of linear equations is called the Elimination Method. When we solved a system by substitution, we started with two equations and two variables and reduced it to one equation with one variable. This is what we'll do with the elimination method, too, but we'll have a different way to get there.

Solve a System of Equations by Elimination

The Elimination Method is based on the Addition Property of Equality. The Addition Property of Equality says that when you add the same quantity to both sides of an equation, you still have equality. We will extend the Addition Property of Equality to say that when you add equal quantities to both sides of an equation, the results are equal.

For any expressions *a*, *b*, *c*, and *d*,
Equation:

$$\begin{array}{lrcl} \text{if} & a & = & b \\ \text{and} & c & = & d \\ \text{then} & a+c & = & b+d \end{array}$$

To solve a system of equations by elimination, we start with both equations in standard form. Then we decide which variable will be easiest to eliminate. How do we decide? We want to have the coefficients of one variable be opposites, so that we can add the equations together and eliminate that variable.

Notice how that works when we add these two equations together:
Equation:

$$\begin{array}{rl} 3x + y &= 5 \\ \underline{2x - y} &\underline{= 0} \\ 5x \quad &= 5 \end{array}$$

The *y*’s add to zero and we have one equation with one variable.

Let’s try another one:
Equation:

$$\begin{cases} x + 4y = 2 \\ 2x + 5y = -2 \end{cases}$$

This time we don’t see a variable that can be immediately eliminated if we add the equations.

But if we multiply the first equation by −2, we will make the coefficients of *x* opposites. We must multiply every term on both sides of the equation by −2.

$$\begin{cases} -2(x + 4y) = -2(2) \\ 2x + 5y = -2 \end{cases}$$

$$\begin{cases} -2x - 8y = -4 \\ 2x + 5y = -2 \end{cases}$$

Now we see that the coefficients of the *x* terms are opposites, so *x* will be eliminated when we add these two equations.

Add the equations yourself—the result should be $-3y = -6$. And that looks easy to solve, doesn’t it? Here is what it would look like.

$$\begin{array}{rl} \begin{cases} -2x - 8y = -4 \\ \underline{2x + 5y = -2} \end{cases} \\ -3y = -6 \end{array}$$

We’ll do one more:
Equation:

$$\begin{cases} 4x - 3y = 10 \\ 3x + 5y = -7 \end{cases}$$

It doesn’t appear that we can get the coefficients of one variable to be opposites by multiplying one of the equations by a constant, unless we use fractions. So instead, we’ll have to multiply both equations by a constant.

We can make the coefficients of *x* be opposites if we multiply the first equation by 3 and the second by −4, so we get 12*x* and −12*x*.

$$3(4x-3y)=3(10)$$
$$-4(3x+5y)=-4(-7)$$

This gives us these two new equations:
Equation:

$$\begin{cases} 12x-9y=30 \\ -12x-20y=28 \end{cases}$$

When we add these equations,
Equation:

$$\begin{cases} 12x-9y=30 \\ \underline{-12x-20y=28} \end{cases}$$
$$-29y=58$$

the *x*’s are eliminated and we just have −29*y* = 58.

Once we get an equation with just one variable, we solve it. Then we substitute that value into one of the original equations to solve for the remaining variable. And, as always, we check our answer to make sure it is a solution to both of the original equations.

Now we’ll see how to use elimination to solve the same system of equations we solved by graphing and by substitution.

Example:
How to Solve a System of Equations by Elimination
Exercise:

Problem: Solve the system by elimination. $\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$

Solution:
Solution

Step 1. Write both equations in standard form. If any coefficients are fractions, clear them.	Both equations are in standard form, $Ax + By = C$. There are no fractions.	$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$
Step 2. Make the coefficients of one variable opposites. Decide which variable you will eliminate. Multiply one or both equations so that the coefficients of that variable are opposites.	We can eliminate the y's by multiplying the first equation by 2. Multiply both sides of $2x + y = 7$ by 2.	$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$ $\begin{cases} 2(2x + y) = 2(7) \\ x - 2y = 6 \end{cases}$
Step 3. Add the equations resulting from Step 2 to eliminate one variable.	We add the x's, y's, and constants.	$\begin{cases} 4x + 2y = 14 \\ x - 2y = 6 \end{cases}$ $5x \qquad = 20$
Step 4. Solve for the remaining variable.	Solve for x.	$x = 4$
Step 5. Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.	Substitute $x = 4$ into the second equation, $x - 2y = 6$. Then solve for y.	$x - 2y = 6$ $4 - 2y = 6$ $-2y = 2$ $y = -1$
Step 6. Write the solution as an ordered pair.	Write it as (x, y).	$(4, -1)$
Step 7. Check that the ordered pair is a solution to **both** original equations.	Substitute $(4, -1)$ into $2x + y = 7$ and $x - 2y = 6$ Do they make both equations true? Yes!	$2x + y = 7$ $2(4) + (-1) \stackrel{?}{=} 7$ $7 = 7$ ✓ $x - 2y = 6$ $4 - 2(-1) \stackrel{?}{=} 6$ $6 = 6$ ✓ The solution is $(4, -1)$.

Note:
Exercise:

Problem: Solve the system by elimination. $\begin{cases} 3x + y = 5 \\ 2x - 3y = 7 \end{cases}$

Solution:

$(2, -1)$

Note:
Exercise:

Problem: Solve the system by elimination. $\begin{cases} 4x + y = -5 \\ -2x - 2y = -2 \end{cases}$

Solution:

$(-2, 3)$

The steps are listed below for easy reference.

Note:
How to solve a system of equations by elimination.

Write both equations in standard form. If any coefficients are fractions, clear them.

Make the coefficients of one variable opposites.

- Decide which variable you will eliminate.
- Multiply one or both equations so that the coefficients of that variable are opposites.

Add the equations resulting from Step 2 to eliminate one variable.

Solve for the remaining variable. Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable. Write the solution as an ordered pair. Check that the ordered pair is a solution to**both**original equations.

First we’ll do an example where we can eliminate one variable right away.

Example:
Exercise:

Problem: Solve the system by elimination. $\begin{cases} x+y=10 \\ x-y=12 \end{cases}$

Solution:
Solution

	$\begin{cases} x+y=10 \\ x-y=12 \end{cases}$
Both equations are in standard form.	
The coefficients of *y* are already opposites.	
Add the two equations to eliminate *y*. The resulting equation has only 1 variable, *x*.	$\begin{array}{r} x+y=10 \\ x-y=12 \\ \hline 2x\quad=22 \end{array}$
Solve for *x*, the remaining variable. Substitute *x* = 11 into one of the original equations.	$x=11$ $x+y=10$

	$11 + y = 10$
Solve for the other variable, *y*.	$y = -1$
Write the solution as an ordered pair.	The ordered pair is (11, −1).
Check that the ordered pair is a solution to **both** original equations. $\begin{aligned} x + y &= 10 \\ 11 + (-1) &\stackrel{?}{=} 10 \\ 10 &= 10 \checkmark \end{aligned}$ $\begin{aligned} x - y &= 12 \\ 11 - (-1) &\stackrel{?}{=} 12 \\ 12 &= 12 \checkmark \end{aligned}$	
	The solution is (11, −1).

Note:
Exercise:

Problem: Solve the system by elimination. $\begin{cases} 2x + y = 5 \\ x - y = 4 \end{cases}$

Solution:

$(3, -1)$

Note:
Exercise:

Problem: Solve the system by elimination. $\begin{cases} x + y = 3 \\ -2x - y = -1 \end{cases}$

Solution:

$(-2, 5)$

In [link], we will be able to make the coefficients of one variable opposites by multiplying one equation by a constant.

Example:
Exercise:

Problem: Solve the system by elimination. $\begin{cases} 3x - 2y = -2 \\ 5x - 6y = 10 \end{cases}$

Solution:
Solution

	$\begin{cases} 3x - 2y = -2 \\ 5x - 6y = 10 \end{cases}$
Both equations are in standard form.	
None of the coefficients are opposites.	
We can make the coefficients of *y* opposites by multiplying the first equation by −3.	$\begin{cases} -3(3x - 2y) = -3(-2) \\ 5x - 6y = 10 \end{cases}$
Simplify.	$\begin{cases} -9x + 6y = 6 \\ 5x - 6y = 10 \end{cases}$

Add the two equations to eliminate *y*.	$\begin{aligned} -9x + 6y &= 6 \\ 5x - 6y &= 10 \\ \hline -4x \quad &= 16 \end{aligned}$
Solve for the remaining variable, *x*. Substitute *x* = −4 into one of the original equations.	$x = -4$ $3x - 2y = -2$
	$3(-4) - 2y = -2$
Solve for *y*.	$-12 - 2y = -2$ $-2y = 10$ $y = -5$
Write the solution as an ordered pair.	The ordered pair is (−4, −5).
Check that the ordered pair is a solution to both original equations. $\begin{aligned} 3x - 2y &= -2 \\ 3(-4) - 2(-5) &\stackrel{?}{=} -2 \\ -12 + 10 &\stackrel{?}{=} -2 \\ -2y &= -2 \checkmark \end{aligned} \qquad \begin{aligned} 5x - 6y &= 10 \\ 3(-4) - 6(-5) &\stackrel{?}{=} 10 \\ -20 + 30 &\stackrel{?}{=} 10 \\ 10 &= 10 \checkmark \end{aligned}$	
	The solution is (−4, −5).

Note:
Exercise:

Problem: Solve the system by elimination. $\begin{cases}4x-3y=1\\5x-9y=-4\end{cases}$

Solution:

$(1,1)$

Note:
Exercise:

Problem: Solve the system by elimination. $\begin{cases}3x+2y=2\\6x+5y=8\end{cases}$

Solution:

$(-2,4)$

Now we'll do an example where we need to multiply both equations by constants in order to make the coefficients of one variable opposites.

Example:
Exercise:

Problem: Solve the system by elimination. $\begin{cases}4x-3y=9\\7x+2y=-6\end{cases}$

Solution:
Solution

In this example, we cannot multiply just one equation by any constant to get opposite coefficients. So we will strategically multiply both equations by a

constant to get the opposites.

	$\begin{cases} 4x - 3y = 9 \\ 7x + 2y = -6 \end{cases}$
Both equations are in standard form. To get opposite coefficients of *y*, we will multiply the first equation by 2 and the second equation by 3.	$\begin{cases} 2(4x - 3y) = 2(9) \\ 3(7x + 2y) = 3(-6) \end{cases}$
Simplify.	$\begin{cases} 8x - 6y = 18 \\ 21x + 6y = -18 \end{cases}$
Add the two equations to eliminate *y*.	$\begin{array}{l} 8x - 6y = 18 \\ \underline{21x + 6y = -18} \\ 39x \quad\quad = 0 \end{array}$
Solve for *x*. Substitute $x = 0$ into one of the original equations.	$x = 0$ $7x + 2y = -6$
	$7 \cdot 0 + 2y = -6$
Solve for *y*.	$2y = -6$
	$y = -3$
Write the solution as an ordered pair.	The ordered pair is (0, −3).
Check that the ordered pair is a solution to **both** original equations.	

$$\begin{array}{rcl} 4x-3y & = & 9 \\ 4(0)-3(-3) & \stackrel{?}{=} & 9 \\ 9 & = & 9\,\checkmark \end{array} \qquad \begin{array}{rcl} 7x+2y & = & -6 \\ 7(0)+2(-3) & \stackrel{?}{=} & -6 \\ -6 & = & -6\,\checkmark \end{array}$$	
	The solution is (0, −3).

What other constants could we have chosen to eliminate one of the variables? Would the solution be the same?

Note:
Exercise:

Problem: Solve the system by elimination. $\begin{cases} 3x-4y=-9 \\ 5x+3y=14 \end{cases}$

Solution:

$(1,3)$

Note:
Exercise:

Problem: Solve the system by elimination. $\begin{cases} 7x+8y=4 \\ 3x-5y=27 \end{cases}$

Solution:

$(4,-3)$

When the system of equations contains fractions, we will first clear the fractions by multiplying each equation by its LCD.

Example:
Exercise:

Problem: Solve the system by elimination. $\begin{cases} x + \frac{1}{2}y = 6 \\ \frac{3}{2}x + \frac{2}{3}y = \frac{17}{2} \end{cases}$

Solution:
Solution

In this example, both equations have fractions. Our first step will be to multiply each equation by its LCD to clear the fractions.

	$\begin{cases} x + \frac{1}{2}y = 6 \\ \frac{3}{2}x + \frac{2}{3}y = \frac{17}{2} \end{cases}$
To clear the fractions, multiply each equation by its LCD.	$\begin{cases} 2\left(x + \frac{1}{2}y\right) = 2(6) \\ 6\left(\frac{3}{2}x + \frac{2}{3}y\right) = 6\left(\frac{17}{2}\right) \end{cases}$
Simplify.	$\begin{cases} 2x + y = 12 \\ 9x + 4y = 51 \end{cases}$
Now we are ready to eliminate one of the variables. Notice that both equations are in standard form.	
We can eliminate *y* multiplying the top equation by −4.	$\begin{cases} -4(2x + y) = -4(12) \\ 9x + 4y = 51 \end{cases}$

Simplify and add. Substitute $x = 3$ into one of the original equations.	$\begin{cases} -8x - 4y = -48 \\ 9x + 4y = 51 \end{cases}$ $x = 3$ $x + \frac{1}{2}y = 6$
Solve for y.	$3 + \frac{1}{2}y = 6$
	$\frac{1}{2}y = 3$
	$y = 6$
Write the solution as an ordered pair.	The ordered pair is (3, 6).
Check that the ordered pair is a solution to **both** original equations. $\begin{aligned} x + \tfrac{1}{2}y &= 6 \\ 3 + \tfrac{1}{2}(6) &\stackrel{?}{=} 6 \\ 3 + 6 &\stackrel{?}{=} 6 \\ 6 &= 6 \checkmark \end{aligned}$ $\begin{aligned} \tfrac{3}{2}x + \tfrac{2}{3}y &= \tfrac{17}{2} \\ \tfrac{3}{2}(3) + \tfrac{2}{3}(6) &\stackrel{?}{=} \tfrac{17}{2} \\ \tfrac{9}{2} + 4 &\stackrel{?}{=} \tfrac{17}{2} \\ \tfrac{9}{2} + \tfrac{8}{2} &\stackrel{?}{=} \tfrac{17}{2} \\ \tfrac{17}{2} &= \tfrac{17}{2} \checkmark \end{aligned}$	
	The solution is (3, 6).

Note:
Exercise:

Problem: Solve the system by elimination. $\begin{cases} \frac{1}{3}x - \frac{1}{2}y = 1 \\ \frac{3}{4}x - y = \frac{5}{2} \end{cases}$

Solution:

$(6, 2)$

Note:
Exercise:

Problem: Solve the system by elimination. $\begin{cases} x + \frac{3}{5}y = -\frac{1}{5} \\ -\frac{1}{2}x - \frac{2}{3}y = \frac{5}{6} \end{cases}$

Solution:

$(1, -2)$

In the Solving Systems of Equations by Graphing we saw that not all systems of linear equations have a single ordered pair as a solution. When the two equations were really the same line, there were infinitely many solutions. We called that a consistent system. When the two equations described parallel lines, there was no solution. We called that an inconsistent system.

Example:
Exercise:

Problem: Solve the system by elimination. $\begin{cases} 3x + 4y = 12 \\ y = 3 - \frac{3}{4}x \end{cases}$

Solution:
Solution

	$\begin{cases} 3x + 4y = 12 \\ y = 3 - \frac{3}{4}x \end{cases}$
Write the second equation in standard form.	$\begin{cases} 3x + 4y & = & 12 \\ \frac{3}{4}x + y & = & 3 \end{cases}$
Clear the fractions by multiplying the second equation by 4.	$\begin{cases} 3x + 4y & = & 12 \\ 4\left(\frac{3}{4}x + y\right) & = & 4(3) \end{cases}$
Simplify.	$\begin{cases} 3x + 4y & = & 12 \\ 3x + 4y & = & 12 \end{cases}$
To eliminate a variable, we multiply the second equation by -1. Simplify and add.	$\begin{cases} 3x + 4y & = & 12 \\ -3x - 4y & = & -12 \end{cases}$ $0 = 0$

This is a true statement. The equations are consistent but dependent. Their graphs would be the same line. The system has infinitely many solutions.

After we cleared the fractions in the second equation, did you notice that the two equations were the same? That means we have coincident lines.

Note:
Exercise:

Problem: Solve the system by elimination. $\begin{cases} 5x - 3y = 15 \\ y = -5 + \frac{5}{3}x \end{cases}$

Solution:

infinitely many solutions

Note:

Exercise:

Problem: Solve the system by elimination. $\begin{cases} x+2y=6 \\ y=-\frac{1}{2}x+3 \end{cases}$

Solution:

infinitely many solutions

Example:
Exercise:

Problem: Solve the system by elimination. $\begin{cases} -6x+15y=10 \\ 2x-5y=-5 \end{cases}$

Solution:
Solution

The equations are in standard form.	$\begin{cases} -6x+15y &= & 10 \\ 2x-5y &= & -5 \end{cases}$
Multiply the second equation by 3 to eliminate a variable.	$\begin{cases} -6x+15y &= & 10 \\ 3\left(2x-5y\right) &= & 3\left(-5\right) \end{cases}$
Simplify and add.	$\begin{cases} -6x+15y &= & 10 \\ 6x-15y &= & -15 \end{cases}$ $0 \neq -5$

This statement is false. The equations are inconsistent and so their graphs would be parallel lines.

The system does not have a solution.

Note:
Exercise:

Problem: Solve the system by elimination. $\begin{cases} -3x + 2y = 8 \\ 9x - 6y = 13 \end{cases}$

Solution:

no solution

Note:
Exercise:

Problem: Solve the system by elimination. $\begin{cases} 7x - 3y = -2 \\ -14x + 6y = 8 \end{cases}$

Solution:

no solution

Solve Applications of Systems of Equations by Elimination

Some applications problems translate directly into equations in standard form, so we will use the elimination method to solve them. As before, we use our Problem Solving Strategy to help us stay focused and organized.

Example:
Exercise:

Problem:

The sum of two numbers is 39. Their difference is 9. Find the numbers.

Solution:
Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for two numbers.
Step 3. Name what we are looking for. Choose a variable to represent that quantity.	Let $n =$ the first number. $m =$ the second number.
Step 4. Translate into a system of equations.	The sum of two numbers is 39. $n + m = 39$ Their difference is 9. $n - m = 9$
The system is:	$\begin{cases} n + m = 39 \\ n - m = 9 \end{cases}$
Step 5. Solve the system of equations. To solve the system of equations, use elimination. The equations are in standard form and the coefficients of m are opposites. Add.	$\begin{cases} n + m = 39 \\ n - m = 9 \end{cases}$ $\overline{2n \qquad = 48}$
Solve for n.	$n = 24$
Substitute $n = 24$ into one of the original equations and solve for m.	$n + m = 39$ $24 + m = 39$ $m = 15$
Step 6. Check the answer.	Since $24 + 15 = 39$ and $24 - 15 = 9$, the answers

	check.
Step 7. Answer the question.	The numbers are 24 and 15.

Note:
Exercise:

Problem:

The sum of two numbers is 42. Their difference is 8. Find the numbers.

Solution:

The numbers are 25 and 17.

Note:
Exercise:

Problem:

The sum of two numbers is −15. Their difference is −35. Find the numbers.

Solution:

The numbers are −25 and 10.

Example:
Exercise:

Problem:

Joe stops at a burger restaurant every day on his way to work. Monday he had one order of medium fries and two small sodas, which had a total of 620 calories. Tuesday he had two orders of medium fries and one small soda, for a total of 820 calories. How many calories are there in one order of medium fries? How many calories in one small soda?

Solution:
Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for the number of calories in one order of medium fries and in one small soda.
Step 3. Name what we are looking for.	Let f = the number of calories in 1 order of medium fries. s = the number of calories in 1 small soda.
Step 4. Translate into a system of equations:	one medium fries and two small sodas had a total of 620 calories
	$f + 2s = 620$
	two medium fries and one small soda had a total of 820 calories.
	$2f + s = 820$
Our system is:	$\begin{cases} f + 2s = 620 \\ 2f + s = 820 \end{cases}$
Step 5. Solve the system of equations.	

To solve the system of equations, use elimination. The equations are in standard form. To get opposite coefficients of *f*, multiply the top equation by −2.	$\begin{cases} -2(f+2s) = -2(620) \\ 2f + s = 820 \end{cases}$
Simplify and add.	$\begin{cases} -2f - 4s = -1240 \\ 2f + s = 820 \end{cases}$ $-3s = -420$
Solve for *s*.	$s = 140$
Substitute *s* = 140 into one of the original equations and then solve for *f*.	$f + 2s = 620$
	$f + 2 \cdot 140 = 620$
	$f + 280 = 620$
	$f = 340$
Step 6. Check the answer.	Verify that these numbers make sense in the problem and that they are solutions to both equations. We leave this to you!
Step 7. Answer the question.	The small soda has 140 calories and the fries have 340 calories.

Note:
Exercise:

Problem:

Malik stops at the grocery store to buy a bag of diapers and 2 cans of formula. He spends a total of $37. The next week he stops and buys 2 bags of diapers and 5 cans of formula for a total of $87. How much does a bag of diapers cost? How much is one can of formula?

Solution:

The bag of diapers costs $11 and the can of formula costs $13.

Note:
Exercise:

Problem:

To get her daily intake of fruit for the day, Sasha eats a banana and 8 strawberries on Wednesday for a calorie count of 145. On the following Wednesday, she eats two bananas and 5 strawberries for a total of 235 calories for the fruit. How many calories are there in a banana? How many calories are in a strawberry?

Solution:

There are 105 calories in a banana and 5 calories in a strawberry.

Choose the Most Convenient Method to Solve a System of Linear Equations

When you will have to solve a system of linear equations in a later math class, you will usually not be told which method to use. You will need to make that decision yourself. So you’ll want to choose the method that is easiest to do and minimizes your chance of making mistakes.

Graphing	Substitution	Elimination
Use when you need a picture of the situation.	Use when one equation is already solved for one variable.	Use when the equations are in standard form.

Example:
Exercise:

Problem:

For each system of linear equations decide whether it would be more convenient to solve it by substitution or elimination. Explain your answer.

ⓐ $\begin{cases} 3x + 8y = 40 \\ 7x - 4y = -32 \end{cases}$ ⓑ $\begin{cases} 5x + 6y = 12 \\ y = \frac{2}{3}x - 1 \end{cases}$

Solution:
Solution

ⓐ $\begin{cases} 3x + 8y = 40 \\ 7x - 4y = -32 \end{cases}$

Since both equations are in standard form, using elimination will be most convenient.

ⓑ $\begin{cases} 5x + 6y = 12 \\ y = \frac{2}{3}x - 1 \end{cases}$

Since one equation is already solved for y, using substitution will be most convenient.

Note:
Exercise:

Problem:

For each system of linear equations, decide whether it would be more convenient to solve it by substitution or elimination. Explain your answer.

ⓐ $\begin{cases} 4x - 5y = -32 \\ 3x + 2y = -1 \end{cases}$ ⓑ $\begin{cases} x = 2y - 1 \\ 3x - 5y = -7 \end{cases}$

Solution:

ⓐ Since both equations are in standard form, using elimination will be most convenient. ⓑ Since one equation is already solved for x, using substitution will be most convenient.

Note:

Exercise:

Problem:

For each system of linear equations, decide whether it would be more convenient to solve it by substitution or elimination. Explain your answer.

ⓐ $\begin{cases} y = 2x - 1 \\ 3x - 4y = -6 \end{cases}$ ⓑ $\begin{cases} 6x - 2y = 12 \\ 3x + 7y = -13 \end{cases}$

Solution:

ⓐ Since one equation is already solved for y, using substitution will be most convenient; ⓑ Since both equations are in standard form, using elimination will be most convenient.

Note:

Access these online resources for additional instruction and practice with solving systems of linear equations by elimination.

- Instructional Video-Solving Systems of Equations by Elimination
- Instructional Video-Solving by Elimination
- Instructional Video-Solving Systems by Elimination

Key Concepts

- **To Solve a System of Equations by Elimination**

Write both equations in standard form. If any coefficients are fractions, clear them.

Make the coefficients of one variable opposites.

- Decide which variable you will eliminate.
- Multiply one or both equations so that the coefficients of that variable are opposites.

Add the equations resulting from Step 2 to eliminate one variable.

Solve for the remaining variable.

Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.

Write the solution as an ordered pair.

Check that the ordered pair is a solution to**both**original equations.

Practice Makes Perfect

Solve a System of Equations by Elimination

In the following exercises, solve the systems of equations by elimination.

Exercise:

Problem: $\begin{cases} 5x + 2y = 2 \\ -3x - y = 0 \end{cases}$

Exercise:

Problem: $\begin{cases} -3x + y = -9 \\ x - 2y = -12 \end{cases}$

Solution:

(6, 9)

Exercise:

Problem: $\begin{cases} 6x - 5y = -1 \\ 2x + y = 13 \end{cases}$

Exercise:

Problem: $\begin{cases} 3x - y = -7 \\ 4x + 2y = -6 \end{cases}$

Solution:

$(-2, 1)$

Exercise:

Problem: $\begin{cases} x + y = -1 \\ x - y = -5 \end{cases}$

Exercise:

Problem: $\begin{cases} x + y = -8 \\ x - y = -6 \end{cases}$

Solution:

$(-7, -1)$

Exercise:

Problem: $\begin{cases} 3x - 2y = 1 \\ -x + 2y = 9 \end{cases}$

Exercise:

Problem: $\begin{cases} -7x + 6y = -10 \\ x - 6y = 22 \end{cases}$

Solution:

$(-2, -4)$

Exercise:

Problem: $\begin{cases} 3x + 2y = -3 \\ -x - 2y = -19 \end{cases}$

Exercise:

Problem: $\begin{cases} 5x + 2y = 1 \\ -5x - 4y = -7 \end{cases}$

Solution:

$(-1, 3)$

Exercise:

Problem: $\begin{cases} 6x + 4y = -4 \\ -6x - 5y = 8 \end{cases}$

Exercise:

Problem: $\begin{cases} 3x - 4y = -11 \\ x - 2y = -5 \end{cases}$

Solution:

$(-1, 2)$

Exercise:

Problem: $\begin{cases} 5x - 7y = 29 \\ x + 3y = -3 \end{cases}$

Exercise:

Problem: $\begin{cases} 6x - 5y = -75 \\ -x - 2y = -13 \end{cases}$

Solution:

$(-5, 9)$

Exercise:

Problem: $\begin{cases} -x + 4y = 8 \\ 3x + 5y = 10 \end{cases}$

Exercise:

Problem: $\begin{cases} 2x - 5y = 7 \\ 3x - y = 17 \end{cases}$

Solution:

(6, 1)

Exercise:

Problem: $\begin{cases} 5x - 3y = -1 \\ 2x - y = 2 \end{cases}$

Exercise:

Problem: $\begin{cases} 7x + y = -4 \\ 13x + 3y = 4 \end{cases}$

Solution:

$(-2, 10)$

Exercise:

Problem: $\begin{cases} -3x + 5y = -13 \\ 2x + y = -26 \end{cases}$

Exercise:

Problem: $\begin{cases} 3x - 5y = -9 \\ 5x + 2y = 16 \end{cases}$

Solution:

(2, 3)

Exercise:

Problem: $\begin{cases} 4x - 3y = 3 \\ 2x + 5y = -31 \end{cases}$

Exercise:

Problem: $\begin{cases} 4x + 7y = 14 \\ -2x + 3y = 32 \end{cases}$

Solution:

$(-7, 6)$

Exercise:

Problem: $\begin{cases} 5x + 2y = 21 \\ 7x - 4y = 9 \end{cases}$

Exercise:

Problem: $\begin{cases} 3x + 8y = -3 \\ 2x + 5y = -3 \end{cases}$

Solution:

$(-9, 3)$

Exercise:

Problem: $\begin{cases} 11x + 9y = -5 \\ 7x + 5y = -1 \end{cases}$

Exercise:

Problem: $\begin{cases} 3x + 8y = 67 \\ 5x + 3y = 60 \end{cases}$

Solution:

(9, 5)

Exercise:

Problem: $\begin{cases} 2x + 9y = -4 \\ 3x + 13y = -7 \end{cases}$

Exercise:

Problem: $\begin{cases} \frac{1}{3}x - y = -3 \\ x + \frac{5}{2}y = 2 \end{cases}$

Solution:

$(-3, 2)$

Exercise:

Problem: $\begin{cases} x + \frac{1}{2}y = \frac{3}{2} \\ \frac{1}{5}x - \frac{1}{5}y = 3 \end{cases}$

Exercise:

Problem: $\begin{cases} x + \frac{1}{3}y = -1 \\ \frac{1}{2}x - \frac{1}{3}y = -2 \end{cases}$

Solution:

$(-2, 3)$

Exercise:

Problem: $\begin{cases} \frac{1}{3}x - y = -3 \\ \frac{2}{3}x + \frac{5}{2}y = 3 \end{cases}$

Exercise:

Problem: $\begin{cases} 2x + y = 3 \\ 6x + 3y = 9 \end{cases}$

Solution:

infinitely many solutions

Exercise:

Problem: $\begin{cases} x - 4y = -1 \\ -3x + 12y = 3 \end{cases}$

Exercise:

Problem: $\begin{cases} -3x - y = 8 \\ 6x + 2y = -16 \end{cases}$

Solution:

infinitely many solutions

Exercise:

Problem: $\begin{cases} 4x + 3y = 2 \\ 20x + 15y = 10 \end{cases}$

Exercise:

Problem: $\begin{cases} 3x + 2y = 6 \\ -6x - 4y = -12 \end{cases}$

Solution:

infinitely many solutions

Exercise:

Problem: $\begin{cases} 5x - 8y = 12 \\ 10x - 16y = 20 \end{cases}$

Exercise:

Problem: $\begin{cases} -11x + 12y = 60 \\ -22x + 24y = 90 \end{cases}$

Solution:

inconsistent, no solution

Exercise:

Problem: $\begin{cases} 7x - 9y = 16 \\ -21x + 27y = -24 \end{cases}$

Exercise:

Problem: $\begin{cases} 5x - 3y = 15 \\ y = \frac{5}{3}x - 2 \end{cases}$

Solution:

inconsistent, no solution

Exercise:

Problem: $\begin{cases} 2x + 4y = 7 \\ y = -\frac{1}{2}x - 4 \end{cases}$

Solve Applications of Systems of Equations by Elimination

In the following exercises, translate to a system of equations and solve.

Exercise:

Problem:

The sum of two numbers is 65. Their difference is 25. Find the numbers.

Solution:

The numbers are 20 and 45.

Exercise:

Problem:

The sum of two numbers is 37. Their difference is 9. Find the numbers.

Exercise:

Problem:

The sum of two numbers is −27. Their difference is −59. Find the numbers.

Solution:

The numbers are 16 and −43.

Exercise:

Problem:

The sum of two numbers is −45. Their difference is −89. Find the numbers.

Exercise:

Problem:

Andrea is buying some new shirts and sweaters. She is able to buy 3 shirts and 2 sweaters for $114 or she is able to buy 2 shirts and 4 sweaters for $164. How much does a shirt cost? How much does a sweater cost?

Solution:

A shirt costs $16 and a sweater costs $33.

Exercise:

Problem:

Peter is buying office supplies. He is able to buy 3 packages of paper and 4 staplers for $40 or he is able to buy 5 packages of paper and 6 staplers for $62. How much does a package of paper cost? How much does a stapler cost?

Exercise:

Problem:

The total amount of sodium in 2 hot dogs and 3 cups of cottage cheese is 4720 mg. The total amount of sodium in 5 hot dogs and 2 cups of cottage cheese is 6300 mg. How much sodium is in a hot dog? How much sodium is in a cup of cottage cheese?

Solution:

There are 860 mg in a hot dog. There are 1,000 mg in a cup of cottage cheese.

Exercise:

Problem:

The total number of calories in 2 hot dogs and 3 cups of cottage cheese is 960 calories. The total number of calories in 5 hot dogs and 2 cups of cottage cheese is 1190 calories. How many calories are in a hot dog? How many calories are in a cup of cottage cheese?

Choose the Most Convenient Method to Solve a System of Linear Equations

In the following exercises, decide whether it would be more convenient to solve the system of equations by substitution or elimination.

Exercise:

Problem: ⓐ $\begin{cases} 8x - 15y = -32 \\ 6x + 3y = -5 \end{cases}$ ⓑ $\begin{cases} x = 4y - 3 \\ 4x - 2y = -6 \end{cases}$

Solution:

ⓐ elimination ⓑ substitution

Exercise:

Problem: ⓐ $\begin{cases} y = 7x - 5 \\ 3x - 2y = 16 \end{cases}$ ⓑ $\begin{cases} 12x - 5y = -42 \\ 3x + 7y = -15 \end{cases}$

Exercise:

Problem: ⓐ $\begin{cases} y = 4x + 9 \\ 5x - 2y = -21 \end{cases}$ ⓑ $\begin{cases} 9x - 4y = 24 \\ 3x + 5y = -14 \end{cases}$

Solution:

ⓐ substitution ⓑ elimination

Exercise:

Problem: ⓐ $\begin{cases} 14x - 15y = -30 \\ 7x + 2y = 10 \end{cases}$ ⓑ $\begin{cases} x = 9y - 11 \\ 2x - 7y = -27 \end{cases}$

Everyday Math

Exercise:

Problem:

Norris can row 3 miles upstream against the current in the same amount of time it takes him to row 5 miles downstream, with the current. Solve the system. $\begin{cases} r - c = 3 \\ r + c = 5 \end{cases}$

ⓐ for r, his rowing speed in still water.
ⓑ Then solve for c, the speed of the river current.

Solution:

ⓐ $r = 4$ ⓑ $c = 1$

Exercise:

Problem:

Josie wants to make 10 pounds of trail mix using nuts and raisins, and she wants the total cost of the trail mix to be $54. Nuts cost $6 per pound and raisins cost $3 per pound. Solve the system $\begin{cases} n + r = 10 \\ 6n + 3r = 54 \end{cases}$ to find n, the number of pounds of nuts, and r, the number of pounds of raisins she should use.

Writing Exercises

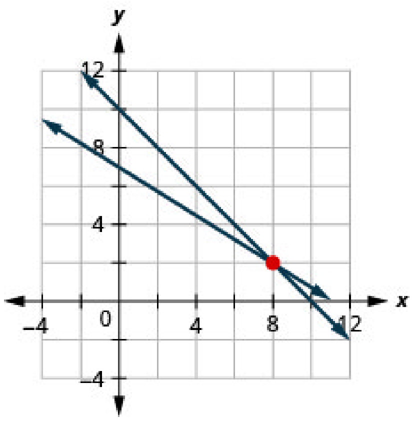
Exercise:

Problem: Solve the system
$$\begin{cases} x + y = 10 \\ 5x + 8y = 56 \end{cases}$$

ⓐ by substitution
ⓑ by graphing
ⓒ Which method do you prefer? Why?

Solution:

ⓐ (8, 2)
ⓑ



ⓒ Answers will vary.

Exercise:

Problem: Solve the system
$$\begin{cases} x + y = -12 \\ y = 4 - \frac{1}{2}x \end{cases}$$

ⓐ by substitution
ⓑ by graphing
ⓒ Which method do you prefer? Why?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve a system of equations by elimination.			
solve applications of systems of equations by elimination.			
choose the most convenient method to solve a system of linear equations.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Solve Applications with Systems of Equations: ASE
By the end of this section, you will be able to:

- Translate to a system of equations
- Solve direct translation applications
- Solve geometry applications
- Solve uniform motion applications

Previously in this chapter we solved several applications with systems of linear equations. In this section, we’ll look at some specific types of applications that relate two quantities. We’ll translate the words into linear equations, decide which is the most convenient method to use, and then solve them.

We will use our Problem Solving Strategy for Systems of Linear Equations.

Note:
Use a problem solving strategy for systems of linear equations.

Read the problem. Make sure all the words and ideas are understood.
Identify what we are looking for.
Name what we are looking for. Choose variables to represent those quantities.
Translate into a system of equations.
Solve the system of equations using good algebra techniques.
Check the answer in the problem and make sure it makes sense.
Answer the question with a complete sentence.

Translate to a System of Equations

Sometimes application problems relate two quantities. Often it is easier to use two variables to solve those problems. Here are two examples:

- The sum of two numbers is negative fourteen. One number is four less than the other. Find the numbers.
- A married couple together earns $110,000 a year. The wife earns $16,000 less than twice what her husband earns. What does the husband earn?

Here is how we can translate these two problems into a system of equations with two variables. We'll focus on Steps 1 through 4 of our Problem Solving Strategy.

Example:
How to Translate to a System of Equations
Exercise:

Problem: Translate to a system of equations:

The sum of two numbers is negative fourteen. One number is four less than the other. Find the numbers.

Solution:
Solution

Step 1. Read the problem. Make sure you understand all the words and ideas.	This is a number problem.	The sum of two numbers is negative fourteen. One number is four less than the other. Find the numbers.
Step 2. Identify what you are looking for.	"Find the numbers."	We are looking for 2 numbers.
Step 3. Name what you are looking for. Choose variables to represent those quantities.	We will use two variables, m and n.	Let m = one number n = second number

Step 4. Translate into a system of equations.	We will write one equation for each sentence.	$\underbrace{\text{The sum of the numbers}}_{m+n}\ \underbrace{\text{is}}_{=}\ \underbrace{-14}_{-14}$ $\underbrace{\text{One number}}_{m}\ \underbrace{\text{is}}_{=}\ \underbrace{\text{four less than the other}}_{n-4}$ The system is: $\begin{cases} m+n=-14 \\ m=n-4 \end{cases}$

Note:
Exercise:

Problem: Translate to a system of equations:

The sum of two numbers is negative twenty-three. One number is 7 less than the other. Find the numbers.

Solution:

$$\begin{cases} m+n=-23 \\ m=n-7 \end{cases}$$

Note:
Exercise:

Problem: Translate to a system of equations:

The sum of two numbers is negative eighteen. One number is 40 more than the other. Find the numbers.

Solution:

$$\begin{cases} m + n = -18 \\ m = n + 40 \end{cases}$$

We’ll do another example where we stop after we write the system of equations.

Example:
Exercise:

Problem: Translate to a system of equations:

A married couple together earns $110,000 a year. The wife earns $16,000 less than twice what her husband earns. What does the husband earn?

Solution:
Solution

We are looking for the amount that the husband and wife each earn.	Let $h =$ the amount the husband earns. $w =$ the amount the wife earns.
Translate.	A married couple together earns $110,000.
	$w + h = 110{,}000$

	The wife earns \$16,000 less than twice what husband earns.
	$w = 2h - 16{,}000$
The system of equations is:	$\begin{cases} w + h = 110{,}000 \\ w = 2h - 16{,}000 \end{cases}$

Note:
Exercise:

Problem: Translate to a system of equations:

A couple has a total household income of \$84,000. The husband earns \$18,000 less than twice what the wife earns. How much does the wife earn?

Solution:

$$\begin{cases} w + h = 84{,}000 \\ h = 2w - 18{,}000 \end{cases}$$

Note:
Exercise:

Problem: Translate to a system of equations:

A senior employee makes \$5 less than twice what a new employee makes per hour. Together they make \$43 per hour. How much does

each employee make per hour?

Solution:

$$\begin{cases} s = 2n - 5 \\ s + n = 43 \end{cases}$$

Solve Direct Translation Applications

We set up, but did not solve, the systems of equations in [link] and [link] Now we'll translate a situation to a system of equations and then solve it.

Example:
Exercise:

Problem: Translate to a system of equations and then solve:

Devon is 26 years older than his son Cooper. The sum of their ages is 50. Find their ages.

Solution:
Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for the ages of Devon and Cooper.

Step 3. Name what we are looking for.	Let d = Devon's age. c = Cooper's age
Step 4. Translate into a system of equations.	Devon is 26 years older than Cooper.
	$d = c + 26$
	The sum of their ages is 50.
	$d + c = 50$
The system is:	$\begin{cases} d = c + 26 \\ d + c = 50 \end{cases}$
Step 5. Solve the system of equations. Solve by substitution.	$\begin{cases} d = c + 26 \\ d + c = 50 \end{cases}$ $d + c = 50$
Substitute c + 26 into the second equation.	$c + 26 + c = 50$
Solve for c.	$2c + 26 = 50$
	$2c = 24$

	$c = 12$ $d = c + 26$
Substitute c = 12 into the first equation and then solve for d.	$d = 12 + 26$
	$d = 38$
Step 6. Check the answer in the problem.	Is Devon's age 26 more than Cooper's? Yes, 38 is 26 more than 12. Is the sum of their ages 50? Yes, 38 plus 12 is 50.
Step 7. Answer the question.	Devon is 38 and Cooper is 12 years old.

Note:
Exercise:

Problem: Translate to a system of equations and then solve:

Ali is 12 years older than his youngest sister, Jameela. The sum of their ages is 40. Find their ages.

Solution:

Ali is 28 and Jameela is 16.

Note:
Exercise:

Problem: Translate to a system of equations and then solve:

Jake's dad is 6 more than 3 times Jake's age. The sum of their ages is 42. Find their ages.

Solution:

Jake is 9 and his dad is 33.

Example:
Exercise:

Problem: Translate to a system of equations and then solve:

When Jenna spent 10 minutes on the elliptical trainer and then did circuit training for 20 minutes, her fitness app says she burned 278 calories. When she spent 20 minutes on the elliptical trainer and 30 minutes circuit training she burned 473 calories. How many calories does she burn for each minute on the elliptical trainer? How many calories does she burn for each minute of circuit training?

Solution:
Solution

Step 1. Read the problem.	
Step 2. Identify what we	We are looking for the number

are looking for.	of calories burned each minute on the elliptical trainer and each minute of circuit training.
Step 3. Name what we are looking for.	Let $e =$ number of calories burned per minute on the elliptical trainer. $c =$ number of calories burned per minute while circuit training
Step 4. Translate into a system of equations.	10 minutes on the elliptical and circuit training for 20 minutes, burned 278 calories
	$10e + 20c = 278$
	20 minutes on the elliptical and 30 minutes of circuit training burned 473 calories
	$20e + 30c = 473$
The system is:	$\begin{cases} 10e + 20c = 278 \\ 20e + 30c = 473 \end{cases}$
Step 5. Solve the system of	

equations.	
Multiply the first equation by −2 to get opposite coefficients of *e*.	$\begin{cases} -2(10e + 20c) = -2(278) \\ 20e + 30c = 473 \end{cases}$
Simplify and add the equations. Solve for *c*.	$\begin{cases} -20e - 40c = -556 \\ 20e + 30c = 473 \end{cases}$ $-10c = -83$ $c = 8.3$
Substitute *c* = 8.3 into one of the original equations to solve for *e*.	$10e + 20c = 278$
	$10e + 20(8.3) = 278$
	$10e + 166 = 278$
	$10e = 112$
	$e = 11.2$
Step 6. Check the answer in the problem.	Check the math on your own.
$\begin{cases} 10(11.2) + 20(8.3) \stackrel{?}{=} 278 \\ 20(11.2) + 30(8.3) \stackrel{?}{=} 473 \end{cases}$	
Step 7. Answer the	Jenna burns 8.3 calories per

question.	minute circuit training and 11.2 calories per minute while on the elliptical trainer.

Note:
Exercise:

Problem: Translate to a system of equations and then solve:

Mark went to the gym and did 40 minutes of Bikram hot yoga and 10 minutes of jumping jacks. He burned 510 calories. The next time he went to the gym, he did 30 minutes of Bikram hot yoga and 20 minutes of jumping jacks burning 470 calories. How many calories were burned for each minute of yoga? How many calories were burned for each minute of jumping jacks?

Solution:

Mark burned 11 calories for each minute of yoga and 7 calories for each minute of jumping jacks.

Note:
Exercise:

Problem: Translate to a system of equations and then solve:

Erin spent 30 minutes on the rowing machine and 20 minutes lifting weights at the gym and burned 430 calories. During her next visit to the gym she spent 50 minutes on the rowing machine and 10 minutes

lifting weights and burned 600 calories. How many calories did she burn for each minutes on the rowing machine? How many calories did she burn for each minute of weight lifting?

Solution:

Erin burned 11 calories for each minute on the rowing machine and 5 calories for each minute of weight lifting.

Solve Geometry Applications

Here are some properties of angles.

The measures of two complementary angles add to 90 degrees. The measures of two supplementary angles add to 180 degrees.

Note:
Complementary and Supplementary Angles
Two angles are **complementary** if the sum of the measures of their angles is 90 degrees.
Two angles are **supplementary** if the sum of the measures of their angles is 180 degrees.

If two angles are complementary, we say that *one angle is the complement of the other.*

If two angles are supplementary, we say that *one angle is the supplement of the other.*

Example:

Exercise:

Problem: Translate to a system of equations and then solve:

The difference of two complementary angles is 26 degrees. Find the measures of the angles.

Solution:
Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for the measure of each angle.
Step 3. Name what we are looking for.	Let $x =$ the measure of the first angle. Let $y =$ the measure of the second angle.
Step 4. Translate into a system of equations.	The angles are complementary. $x + y = 90$
	The difference of the two angles is 26 degrees. $x - y = 26$
The system is	$\begin{cases} x + y = 90 \\ x - y = 26 \end{cases}$
Step 5. Solve the system of	

equations by elimination.	$\begin{cases} x+y=90 \\ \underline{x-y=26} \end{cases}$ $2x \quad = 116$ $x \quad = 58$
Substitute $x = 58$ into the first equation.	$x+y = 90$ $58+y = 90$ $y = 32$
Step 6. Check the answer in the problem. $58+32 = 90✓$ $58-32 = 26✓$	
Step 7. Answer the question.	The angle measures are 58 degrees and 42 degrees.

Note:
Exercise:

Problem: Translate to a system of equations and then solve:

The difference of two complementary angles is 20 degrees. Find the measures of the angles.

Solution:

The angle measures are 55 degrees and 35 degrees.

Note:
Exercise:

Problem: Translate to a system of equations and then solve:

The difference of two complementary angles is 80 degrees. Find the measures of the angles.

Solution:

The angle measures are 5 degrees and 85 degrees.

Example:
Exercise:

Problem: Translate to a system of equations and then solve:

Two angles are supplementary. The measure of the larger angle is twelve degrees less than five times the measure of the smaller angle. Find the measures of both angles.

Solution:
Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for the measure of each angle.
Step 3. Name what we are	Let $x =$ the measure of the

looking for.	first angle. $y=$ the measure of the second angle
Step 4. Translate into a system of equations.	The angles are supplementary.
	$x+y=180$
	The larger angle is twelve less than five times the smaller angle
	$y=5x-12$
The system is: **Step 5. Solve** the system of equations substitution.	$\begin{cases} x+y=180 \\ y=5x-12 \end{cases}$ $x+y=180$
Substitute $5x-12$ for y in the first equation.	$x+5x-12=180$
Solve for x.	$6x-12=180$
	$6x=192$
	$x=32$ $y=5x-12$

Substitute 32 for in the second equation, then solve for *y*.	$y = 5 \cdot 32 - 12$
	$y = 160 - 12$
	$y = 148$
Step 6. Check the answer in the problem. $$\begin{aligned} 32 + 158 &= 180 \checkmark \\ 5 \cdot 32 - 12 &= 147 \checkmark \end{aligned}$$	
Step 7. Answer the question.	The angle measures are 148 and 32.

Note:
Exercise:

Problem: Translate to a system of equations and then solve:

Two angles are supplementary. The measure of the larger angle is 12 degrees more than three times the smaller angle. Find the measures of the angles.

Solution:

The angle measures are 42 degrees and 138 degrees.

Note:
Exercise:

Problem: Translate to a system of equations and then solve:

Two angles are supplementary. The measure of the larger angle is 18 less than twice the measure of the smaller angle. Find the measures of the angles.

Solution:

The angle measures are 66 degrees and 114 degrees.

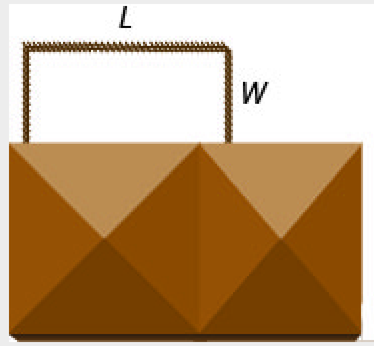
Example:
Exercise:

Problem: Translate to a system of equations and then solve:

Randall has 125 feet of fencing to enclose the rectangular part of his backyard adjacent to his house. He will only need to fence around three sides, because the fourth side will be the wall of the house. He wants the length of the fenced yard (parallel to the house wall) to be 5 feet more than four times as long as the width. Find the length and the width.

Solution:
Solution

Step 1. Read the problem.	

Step 2. Identify what you are looking for.	We are looking for the length and width.
	
Step 3. Name what we are looking for.	Let L = the length of the fenced yard. W = the width of the fenced yard
Step 4. Translate into a system of equations.	One length and two widths equal 125.
	$L + 2W = 125$
	The length will be 5 feet more than four times the width.
	$L = 4W + 5$
The system is: **Step 5. Solve** the system of equations by substitution.	$\begin{cases} L + 2W = 125 \\ L = 4W + 5 \end{cases}$ $L + 2W = 125$

Substitute $L = 4W + 5$ into the first equation, then solve for W.	$4W + 5 + 2W = 125$
	$6W + 5 = 125$
	$6W = 120$
Substitute 20 for W in the second equation, then solve for L.	$W = 20$ $L = 4W + 5$
	$L = 4 \cdot 20 + 5$
	$L = 80 + 5$
	$L = 85$
Step 6. Check the answer in the problem. $\begin{aligned} 20 + 28 + 20 &= 125\ \checkmark \\ 85 &= 4 \cdot 20 + 5\ \checkmark \end{aligned}$	
Step 7. Answer the equation.	The length is 85 feet and the width is 20 feet.

Note:
Exercise:

Problem: Translate to a system of equations and then solve:

Mario wants to put a rectangular fence around the pool in his backyard. Since one side is adjacent to the house, he will only need to fence three sides. There are two long sides and the one shorter side is parallel to the house. He needs 155 feet of fencing to enclose the pool. The length of the long side is 10 feet less than twice the width. Find the length and width of the pool area to be enclosed.

Solution:

The length is 60 feet and the width is 35 feet.

Note:
Exercise:

Problem: Translate to a system of equations and then solve:

Alexis wants to build a rectangular dog run in her yard adjacent to her neighbor's fence. She will use 136 feet of fencing to completely enclose the rectangular dog run. The length of the dog run along the neighbor's fence will be 16 feet less than twice the width. Find the length and width of the dog run.

Solution:

The length is 60 feet and the width is 38 feet.

Solve Uniform Motion Applications

We used a table to organize the information in uniform motion problems when we introduced them earlier. We'll continue using the table here. The basic equation was $D = rt$ where D is the distance travelled, r is the rate, and t is the time.

Our first example of a uniform motion application will be for a situation similar to some we have already seen, but now we can use two variables and two equations.

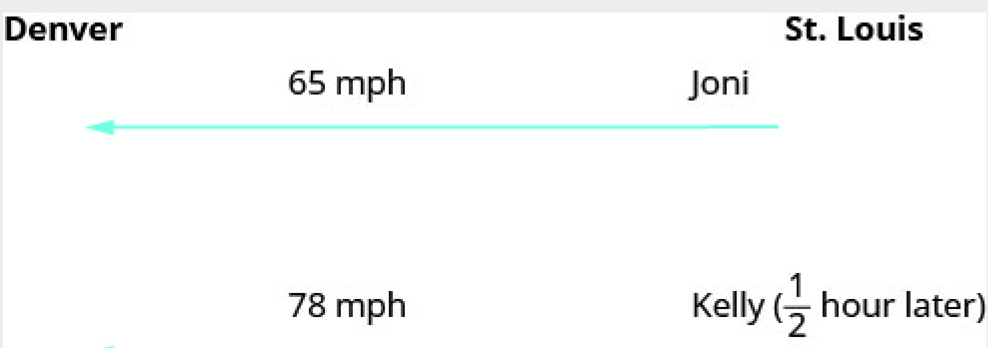
Example:
Exercise:

Problem: Translate to a system of equations and then solve:

Joni left St. Louis on the interstate, driving west towards Denver at a speed of 65 miles per hour. Half an hour later, Kelly left St. Louis on the same route as Joni, driving 78 miles per hour. How long will it take Kelly to catch up to Joni?

Solution:
Solution

A diagram is useful in helping us visualize the situation.



Identify and name what we are looking for. A chart will help us organize the data. We know the rates of both Joni and Kelly, and so we enter them in the chart.	
We are looking for the length of time Kelly, *k*, and Joni, *j*, will each drive. Since $D = r \cdot t$ we can fill in the Distance column.	see chart below
Translate into a system of equations. To make the system of equations, we must recognize that Kelly and Joni will drive the same distance. So, $65j = 78k$. Also, since Kelly left later, her time will be $\frac{1}{2}$ hour less than Joni's time. So, $k = j - \frac{1}{2}$.	
Now we have the system.	$\begin{cases} k = j - \frac{1}{2} \\ 65j = 78k \end{cases}$
Solve the system of equations by substitution.	$65j = 78k$
Substitute $k = j - \frac{1}{2}$ into the second equation, then solve for *j*.	$65j = 78\left(j - \frac{1}{2}\right)$
	$65j = 78j - 39$

Type	Rate	•	Time	=	Distance
Joni	65		j		$65j$
Kelly	78		k		$78k$

	$-13j = -39$
	$j = 3$
To find Kelly's time, substitute $j = 3$ into the first equation, then solve for k.	$k = j - \frac{1}{2}$
	$k = 3 - \frac{1}{2}$
	$k = \frac{5}{2}$ or $k = 2\frac{1}{2}$
Check the answer in the problem. Joni 3 hours (65 mph) = 195 miles. Kelly $2\frac{1}{2}$ hours (78 mph) = 195 miles. Yes, they will have traveled the same distance when they meet.	
Answer the question.	Kelly will catch up to Joni in $2\frac{1}{2}$ hours. By then, Joni will have traveled 3 hours.

Note:
Exercise:

Problem:

Translate to a system of equations and then solve: Mitchell left Detroit on the interstate driving south towards Orlando at a speed of 60 miles per hour. Clark left Detroit 1 hour later traveling at a speed of 75 miles per hour, following the same route as Mitchell. How long will it take Clark to catch Mitchell?

Solution:

It will take Clark 4 hours to catch Mitchell.

Note:
Exercise:

Problem:

Translate to a system of equations and then solve: Charlie left his mother's house traveling at an average speed of 36 miles per hour. His sister Sally left 15 minutes (1/4 hour) later traveling the same route at an average speed of 42 miles per hour. How long before Sally catches up to Charlie?

Solution:

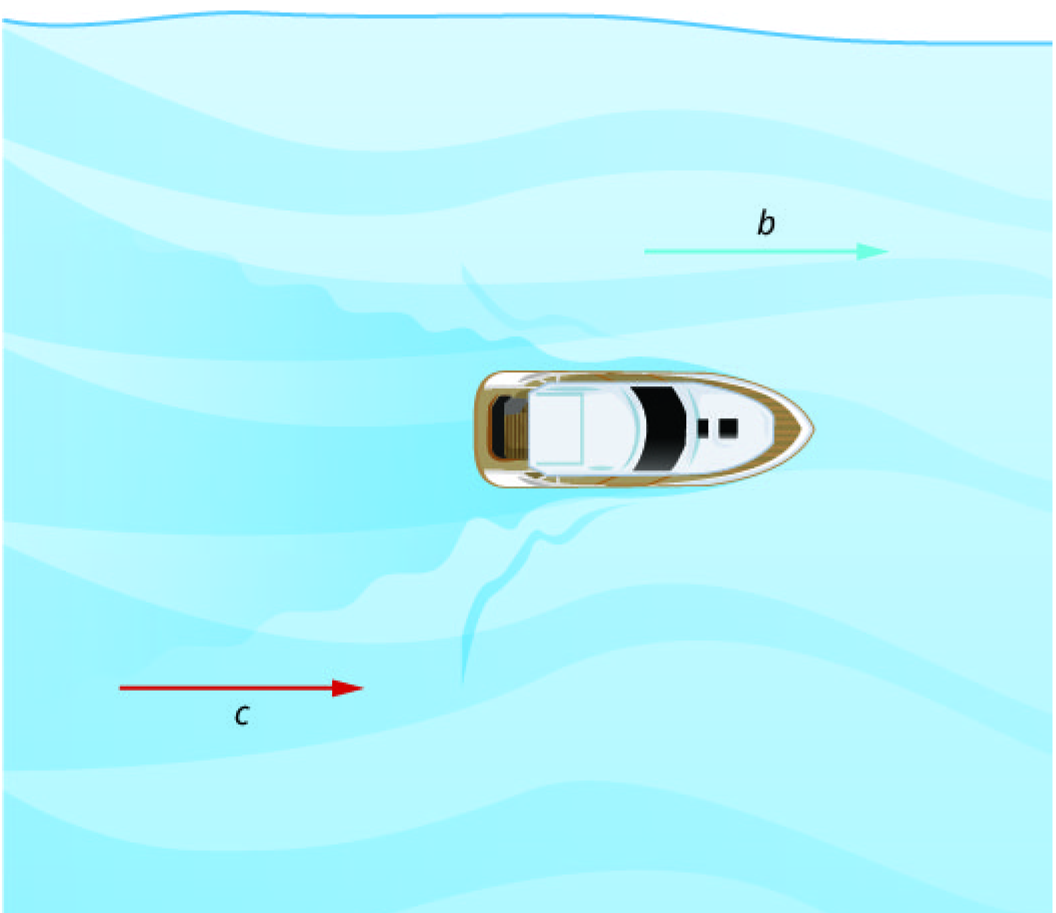
It will take Sally $1\frac{1}{2}$ hours to catch up to Charlie.

Many real-world applications of uniform motion arise because of the effects of currents—of water or air—on the actual speed of a vehicle. Cross-country airplane flights in the United States generally take longer going west than going east because of the prevailing wind currents.

Let’s take a look at a boat travelling on a river. Depending on which way the boat is going, the current of the water is either slowing it down or speeding it up.

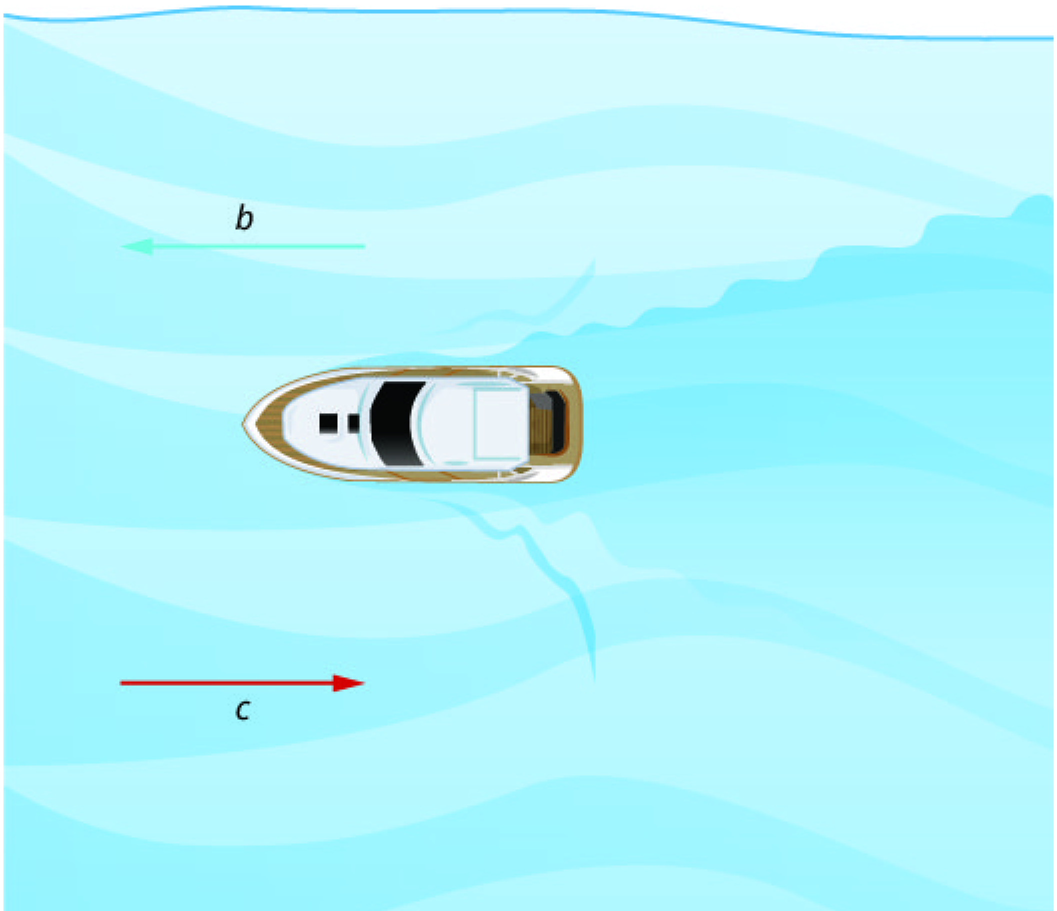
[link] and [link] show how a river current affects the speed at which a boat is actually travelling. We’ll call the speed of the boat in still water b and the speed of the river current c.

In [link] the boat is going downstream, in the same direction as the river current. The current helps push the boat, so the boat’s actual speed is faster than its speed in still water. The actual speed at which the boat is moving is $b + c$.



In [link] the boat is going upstream, opposite to the river current. The current is going against the boat, so the boat’s actual speed is slower than its

speed in still water. The actual speed of the boat is $b - c$.



We'll put some numbers to this situation in [link].

Example:
Exercise:

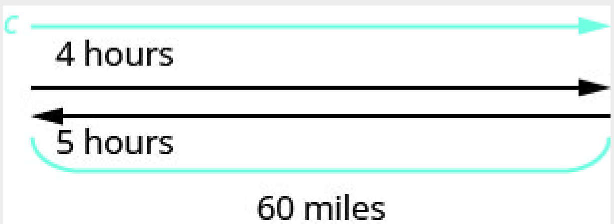
Problem: Translate to a system of equations and then solve:

A river cruise ship sailed 60 miles downstream for 4 hours and then took 5 hours sailing upstream to return to the dock. Find the speed of the ship in still water and the speed of the river current.

Solution:
Solution

Read the problem.

This is a uniform motion problem and a picture will help us visualize the situation.



<table>
<tr><td>Identify what we are looking for.</td><td>We are looking for the speed of the ship
in still water and the speed of the current.</td></tr>
<tr><td>Name what we are looking for.</td><td>Let $s =$ the rate of the ship in still water.
$c =$ the rate of the current</td></tr>
<tr><td>A chart will help us organize the information.
The ship goes downstream and then upstream.
Going downstream, the current helps the
ship; therefore, the ship’s actual rate is $s + c$.</td><td>
<table>
<tr><th></th><th>Rate</th><th>• Time</th><th>= Distance</th></tr>
<tr><td>downstream</td><td>$s + c$</td><td>4</td><td>60</td></tr>
<tr><td>upstream</td><td>$s - c$</td><td>5</td><td>60</td></tr>
</table>
</td></tr>
</table>

Going upstream, the current slows the ship; therefore, the actual rate is $s - c$.	
Downstream it takes 4 hours. Upstream it takes 5 hours. Each way the distance is 60 miles.	
Translate into a system of equations. Since rate times time is distance, we can write the system of equations.	$\begin{cases} 4(s+c)=60 \\ 5(s-c)=60 \end{cases}$
Solve the system of equations. Distribute to put both equations in standard form, then solve by elimination.	$\begin{cases} 4s+4c=60 \\ 5s-5c=60 \end{cases}$
Multiply the top equation by 5 and the bottom equation by 4. Add the equations, then solve for s.	$\begin{array}{l} 20s+20c=300 \\ \underline{20s-20c=240} \\ 40s \qquad = 540 \end{array}$
Substitute $s = 13.5$ into one of the original equations.	$s = 13.5$ $4(s+c)=60$
	$4(13.5+c)=60$
	$54+4c=60$
	$4c=6$

	$4c = 1.5$
Check the answer in the problem. The downstream rate would be $13.5 + 1.5 = 15$ mph. In 4 hours the ship would travel $15 \cdot 4 = 60$ miles. The upstream rate would be $13.5 - 1.5 = 12$ mph. In 5 hours the ship would travel $12 \cdot 5 = 60$ miles.	
Answer the question.	The rate of the ship is 13.5 mph and the rate of the current is 1.5 mph.

Note:
Exercise:

Problem:

Translate to a system of equations and then solve: A Mississippi river boat cruise sailed 120 miles upstream for 12 hours and then took 10 hours to return to the dock. Find the speed of the river boat in still water and the speed of the river current.

Solution:

The rate of the boat is 11 mph and the rate of the current is 1 mph.

Note:
Exercise:

Problem:

Translate to a system of equations and then solve: Jason paddled his canoe 24 miles upstream for 4 hours. It took him 3 hours to paddle back. Find the speed of the canoe in still water and the speed of the river current.

Solution:

The speed of the canoe is 7 mph and the speed of the current is 1 mph.

Wind currents affect airplane speeds in the same way as water currents affect boat speeds. We'll see this in [link]. A wind current in the same direction as the plane is flying is called a *tailwind*. A wind current blowing against the direction of the plane is called a *headwind*.

Example:
Exercise:

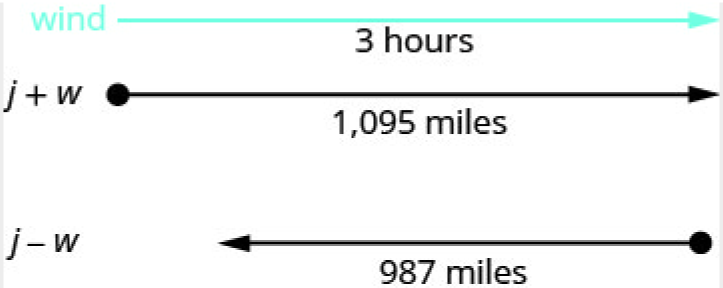
Problem: Translate to a system of equations and then solve:

A private jet can fly 1095 miles in three hours with a tailwind but only 987 miles in three hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Solution:
Solution

Read the problem.

This is a uniform motion problem and a picture will help us visualize.



<table>
<tr><td>Identify what we are looking for.</td><td>We are looking for the speed of the jet
in still air and the speed of the wind.</td></tr>
<tr><td>Name what we are looking for.</td><td>Let $j =$ the speed of the jet in still air.
$w =$ the speed of the wind</td></tr>
<tr><td>A chart will help us organize the information.
The jet makes two trips-one in a tailwind
and one in a headwind.
In a tailwind, the wind helps the jet and so
the rate is $j + w$.
In a headwind, the wind slows the jet and
so the rate is $j - w$.</td><td><table><tr><th></th><th>Rate</th><th>• Time</th><th>= Distance</th></tr><tr><td>tailwind</td><td>$j + w$</td><td>3</td><td>1095</td></tr><tr><td>headwind</td><td>$j - w$</td><td>3</td><td>987</td></tr></table></td></tr>
<tr><td>Each trip takes 3 hours.</td><td></td></tr>
</table>

In a tailwind the jet flies 1095 miles. In a headwind the jet flies 987 miles.	
Translate into a system of equations. Since rate times time is distance, we get the system of equations.	$\begin{cases} 3(j+w)=1095 \\ 3(j-w)=\ \ 987 \end{cases}$
Solve the system of equations. Distribute, then solve by elimination.	$\begin{cases} 3j+3w=1095 \\ 3j-3w=\ \ 987 \end{cases}$ $6j \quad = 2082$
Add, and solve for *j*. Substitute $j = 347$ into one of the original equations, then solve for *w*.	$j \quad = 347$ $3(j+w)=1095$
	$3(347+w)=1095$
	$1041+3w=1095$
	$3w=54$
	$w=18$
Check the answer in the problem.	

With the tailwind, the actual rate of the jet would be $347 + 18 = 365$ mph. In 3 hours the jet would travel $365 \cdot 3 = 1095$ miles. Going into the headwind, the jet's actual rate would be $347 - 18 = 329$ mph. In 3 hours the jet would travel $329 \cdot 3 = 987$ miles.	
Answer the question.	The rate of the jet is 347 mph and the rate of the wind is 18 mph.

Note:
Exercise:

Problem:

Translate to a system of equations and then solve: A small jet can fly 1,325 miles in 5 hours with a tailwind but only 1025 miles in 5 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Solution:

The speed of the jet is 235 mph and the speed of the wind is 30 mph.

Note:
Exercise:

Problem:

Translate to a system of equations and then solve: A commercial jet can fly 1728 miles in 4 hours with a tailwind but only 1536 miles in 4 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Solution:

The speed of the jet is 408 mph and the speed of the wind is 24 mph.

Practice Makes Perfect

Translate to a System of Equations

In the following exercises, translate to a system of equations and solve the system.

Exercise:

Problem:

The sum of two numbers is fifteen. One number is three less than the other. Find the numbers.

Solution:

The numbers are 6 and 9.

Exercise:

Problem:

The sum of two numbers is twenty-five. One number is five less than the other. Find the numbers.

Exercise:

Problem:

The sum of two numbers is negative thirty. One number is five times the other. Find the numbers.

Solution:

The numbers are -5 and -25.

Exercise:

Problem:

The sum of two numbers is negative sixteen. One number is seven times the other. Find the numbers.

Exercise:

Problem:

Twice a number plus three times a second number is twenty-two. Three times the first number plus four times the second is thirty-one. Find the numbers.

Solution:

The numbers are 5 and 4.

Exercise:

Problem:

Six times a number plus twice a second number is four. Twice the first number plus four times the second number is eighteen. Find the numbers.

Exercise:

Problem:

Three times a number plus three times a second number is fifteen. Four times the first plus twice the second number is fourteen. Find the numbers.

Solution:

The numbers are 2 and 3.

Exercise:

Problem:

Twice a number plus three times a second number is negative one. The first number plus four times the second number is two. Find the numbers.

Exercise:

Problem:

A married couple together earn $75,000. The husband earns $15,000 more than five times what his wife earns. What does the wife earn?

Solution:

$10,000

Exercise:

Problem:

During two years in college, a student earned $9,500. The second year she earned $500 more than twice the amount she earned the first year. How much did she earn the first year?

Exercise:

Problem:

Daniela invested a total of $50,000, some in a certificate of deposit (CD) and the remainder in bonds. The amount invested in bonds was $5000 more than twice the amount she put into the CD. How much did she invest in each account?

Solution:

She put $15,000 into a CD and $35,000 in bonds.

Exercise:

Problem:

Jorge invested $28,000 into two accounts. The amount he put in his money market account was $2,000 less than twice what he put into a CD. How much did he invest in each account?

Exercise:

Problem:

In her last two years in college, Marlene received $42,000 in loans. The first year she received a loan that was $6,000 less than three times the amount of the second year’s loan. What was the amount of her loan for each year?

Solution:

The amount of the first year’s loan was $30,000 and the amount of the second year’s loan was $12,000.

Exercise:

Problem:

Jen and David owe $22,000 in loans for their two cars. The amount of the loan for Jen’s car is $2000 less than twice the amount of the loan for David’s car. How much is each car loan?

Solve Direct Translation Applications

In the following exercises, translate to a system of equations and solve.

Exercise:

Problem:

Alyssa is twelve years older than her sister, Bethany. The sum of their ages is forty-four. Find their ages.

Solution:

Bethany is 16 years old and Alyssa is 28 years old.

Exercise:

Problem:

Robert is 15 years older than his sister, Helen. The sum of their ages is sixty-three. Find their ages.

Exercise:

Problem:

The age of Noelle's dad is six less than three times Noelle's age. The sum of their ages is seventy-four. Find their ages.

Solution:

Noelle is 20 years old and her dad is 54 years old.

Exercise:

Problem:

The age of Mark's dad is 4 less than twice Marks's age. The sum of their ages is ninety-five. Find their ages.

Exercise:

Problem:

Two containers of gasoline hold a total of fifty gallons. The big container can hold ten gallons less than twice the small container. How many gallons does each container hold?

Solution:

The small container holds 20 gallons and the large container holds 30 gallons.

Exercise:

Problem:

June needs 48 gallons of punch for a party and has two different coolers to carry it in. The bigger cooler is five times as large as the smaller cooler. How many gallons can each cooler hold?

Exercise:

Problem:

Shelly spent 10 minutes jogging and 20 minutes cycling and burned 300 calories. The next day, Shelly swapped times, doing 20 minutes of jogging and 10 minutes of cycling and burned the same number of calories. How many calories were burned for each minute of jogging and how many for each minute of cycling?

Solution:

There were 10 calories burned jogging and 10 calories burned cycling.

Exercise:

Problem:

Drew burned 1800 calories Friday playing one hour of basketball and canoeing for two hours. Saturday he spent two hours playing basketball and three hours canoeing and burned 3200 calories. How many calories did he burn per hour when playing basketball?

Exercise:

Problem:

Troy and Lisa were shopping for school supplies. Each purchased different quantities of the same notebook and thumb drive. Troy bought four notebooks and five thumb drives for $116. Lisa bought two notebooks and three thumb dives for $68. Find the cost of each notebook and each thumb drive.

Solution:

Notebooks are $4 and thumb drives are $20.

Exercise:

Problem:

Nancy bought seven pounds of oranges and three pounds of bananas for $17. Her husband later bought three pounds of oranges and six pounds of bananas for $12. What was the cost per pound of the oranges and the bananas?

Solve Geometry Applications In the following exercises, translate to a system of equations and solve.

Exercise:

Problem:

The difference of two complementary angles is 30 degrees. Find the measures of the angles.

Solution:

The measures are 60 degrees and 30 degrees.

Exercise:

Problem:

The difference of two complementary angles is 68 degrees. Find the measures of the angles.

Exercise:

Problem:

The difference of two supplementary angles is 70 degrees. Find the measures of the angles.

Solution:

The measures are 125 degrees and 55 degrees.

Exercise:

Problem:

The difference of two supplementary angles is 24 degrees. Find the measure of the angles.

Exercise:

Problem:

The difference of two supplementary angles is 8 degrees. Find the measures of the angles.

Solution:

94 degrees and 86 degrees

Exercise:

Problem:

The difference of two supplementary angles is 88 degrees. Find the measures of the angles.

Exercise:

Problem:

The difference of two complementary angles is 55 degrees. Find the measures of the angles.

Solution:

72.5 degrees and 17.5 degrees

Exercise:

Problem:

The difference of two complementary angles is 17 degrees. Find the measures of the angles.

Exercise:

Problem:

Two angles are supplementary. The measure of the larger angle is four more than three times the measure of the smaller angle. Find the measures of both angles.

Solution:

The measures are 44 degrees and 136 degrees.

Exercise:

Problem:

Two angles are supplementary. The measure of the larger angle is five less than four times the measure of the smaller angle. Find the measures of both angles.

Exercise:

Problem:

Two angles are complementary. The measure of the larger angle is twelve less than twice the measure of the smaller angle. Find the measures of both angles.

Solution:

The measures are 34 degrees and 56 degrees.

Exercise:

Problem:

Two angles are complementary. The measure of the larger angle is ten more than four times the measure of the smaller angle. Find the measures of both angles.

Exercise:

Problem:

Wayne is hanging a string of lights 45 feet long around the three sides of his rectangular patio, which is adjacent to his house. The length of his patio, the side along the house, is five feet longer than twice its width. Find the length and width of the patio.

Solution:

The width is 10 feet and the length is 25 feet.

Exercise:

Problem:

Darrin is hanging 200 feet of Christmas garland on the three sides of fencing that enclose his rectangular front yard. The length, the side along the house, is five feet less than three times the width. Find the length and width of the fencing.

Exercise:

Problem:

A frame around a rectangular family portrait has a perimeter of 60 inches. The length is fifteen less than twice the width. Find the length and width of the frame.

Solution:

The width is 15 feet and the length is 15 feet.

Exercise:

Problem:

The perimeter of a rectangular toddler play area is 100 feet. The length is ten more than three times the width. Find the length and width of the play area.

Solve Uniform Motion Applications In the following exercises, translate to a system of equations and solve.

Exercise:

Problem:

Sarah left Minneapolis heading east on the interstate at a speed of 60 mph. Her sister followed her on the same route, leaving two hours later and driving at a rate of 70 mph. How long will it take for Sarah's sister to catch up to Sarah?

Solution:

It took Sarah's sister 12 hours.

Exercise:

Problem:

College roommates John and David were driving home to the same town for the holidays. John drove 55 mph, and David, who left an hour later, drove 60 mph. How long will it take for David to catch up to John?

Exercise:

Problem:

At the end of spring break, Lucy left the beach and drove back towards home, driving at a rate of 40 mph. Lucy's friend left the beach for home 30 minutes (half an hour) later, and drove 50 mph. How long did it take Lucy's friend to catch up to Lucy?

Solution:

It took Lucy's friend 2 hours.

Exercise:

Problem:

Felecia left her home to visit her daughter driving 45 mph. Her husband waited for the dog sitter to arrive and left home twenty minutes (1/3 hour) later. He drove 55 mph to catch up to Felecia. How long before he reaches her?

Exercise:

Problem:

The Jones family took a 12 mile canoe ride down the Indian River in two hours. After lunch, the return trip back up the river took three hours. Find the rate of the canoe in still water and the rate of the current.

Solution:

The canoe rate is 5 mph and the current rate is 1 mph.

Exercise:

Problem:

A motor boat travels 60 miles down a river in three hours but takes five hours to return upstream. Find the rate of the boat in still water and the rate of the current.

Exercise:

Problem:

A motor boat traveled 18 miles down a river in two hours but going back upstream, it took 4.5 hours due to the current. Find the rate of the motor boat in still water and the rate of the current. (Round to the nearest hundredth.).

Solution:

The boat rate is 6.5 mph and the current rate is 2.5 mph.

Exercise:

Problem:

A river cruise boat sailed 80 miles down the Mississippi River for four hours. It took five hours to return. Find the rate of the cruise boat in still water and the rate of the current. (Round to the nearest hundredth.).

Exercise:

Problem:

A small jet can fly 1,072 miles in 4 hours with a tailwind but only 848 miles in 4 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Solution:

The jet rate is 240 mph and the wind speed is 28 mph.

Exercise:

Problem:

A small jet can fly 1,435 miles in 5 hours with a tailwind but only 1215 miles in 5 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Exercise:

Problem:

A commercial jet can fly 868 miles in 2 hours with a tailwind but only 792 miles in 2 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Solution:

The jet rate is 415 mph and the wind speed is 19 mph.

Exercise:

Problem:

A commercial jet can fly 1,320 miles in 3 hours with a tailwind but only 1,170 miles in 3 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Everyday Math

Exercise:

Problem:

At a school concert, 425 tickets were sold. Student tickets cost $5 each and adult tickets cost $8 each. The total receipts for the concert were $2,851. Solve the system

$$\begin{cases} s + a = 425 \\ 5s + 8a = 2{,}851 \end{cases}$$

to find s, the number of student tickets and a, the number of adult tickets.

Solution:

$s = 183, a = 242$

Exercise:

Problem:

The first graders at one school went on a field trip to the zoo. The total number of children and adults who went on the field trip was 115. The number of adults was $\frac{1}{4}$ the number of children. Solve the system

$$\begin{cases} c + a = 115 \\ a = \frac{1}{4}c \end{cases}$$

to find c, the number of children and a, the number of adults.

Writing Exercises

Exercise:

Problem:

Write an application problem similar to [link] using the ages of two of your friends or family members. Then translate to a system of equations and solve it.

Solution:

Answers will vary.

Exercise:

Problem:

Write a uniform motion problem similar to [link] that relates to where you live with your friends or family members. Then translate to a system of equations and solve it.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
translate to a system of equations.			
solve direct translation applications.			
solve geometry applications.			
solve uniform motion applications.			

ⓑ On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Glossary

complementary angles
: Two angles are complementary if the sum of the measures of their angles is 90 degrees.

supplementary angles
: Two angles are supplementary if the sum of the measures of their angles is 180 degrees.

Solve Mixture Applications with Systems of Equations: ASE
By the end of this section, you will be able to:

- Solve mixture applications
- Solve interest applications

Solve Mixture Applications

When we solved mixture applications with coins and tickets earlier, we started by creating a table so we could organize the information. For a coin example with nickels and dimes, the table looked like this:

Type	Number •	Value($) =	Total Value($)
nickels		0.05	
dimes		0.10	

Using one variable meant that we had to relate the number of nickels and the number of dimes. We had to decide if we were going to let n be the number of nickels and then write the number of dimes in terms of n, or if we would let d be the number of dimes and write the number of nickels in terms of d.

Now that we know how to solve systems of equations with two variables, we'll just let n be the number of nickels and d be the number of dimes. We'll write one equation based on the total value column, like we did before, and the other equation will come from the number column.

For the first example, we'll do a ticket problem where the ticket prices are in whole dollars, so we won't need to use decimals just yet.

Example:
Exercise:

Problem: Translate to a system of equations and solve:

The box office at a movie theater sold 147 tickets for the evening show, and receipts totaled $1,302. How many $11 adult and how many $8 child tickets were sold?

Solution:
Solution

Step 1. Read the problem.	We will create a table to organize the information.
Step 2. Identify what we are looking for.	We are looking for the number of adult tickets and the number of child tickets sold.
Step 3. Name what we are looking for.	Let $a =$ the number of adult tickets. $c =$ the number of child tickets
A table will help us organize the data. We have two types of tickets: adult and child.	Write a and c for the number of tickets.
Write the total number of tickets sold at the bottom of the Number column.	Altogether 147 were sold.
Write the value of each type of ticket in the	The value of each adult ticket is $11.

Value column.	The value of each child tickets is $8.
The number times the value gives the total value, so the total value of adult tickets is $a \cdot 11 = 11a$, and the total value of child tickets is $c \cdot 8 = 8c$.	<table><tr><th>Type</th><th>Number</th><th>•</th><th>Value ($)</th><th>=</th><th>Total Value ($)</th></tr><tr><td>adult</td><td>a</td><td></td><td>11</td><td></td><td>$11a$</td></tr><tr><td>child</td><td>c</td><td></td><td>8</td><td></td><td>$8c$</td></tr><tr><td></td><td>147</td><td></td><td></td><td></td><td>1302</td></tr></table>
Altogether the total value of the tickets was $1,302.	Fill in the Total Value column.
Step 4. Translate into a system of equations.	
The Number column and the Total Value column give us the system of equations. We will use the elimination method to solve this system.	$\begin{cases} a + c = 147 \\ 11a + 8c = 1302 \end{cases}$
Multiply the first equation by −8.	$\begin{cases} -8(a + c) = -8(147) \\ 11a + 8c = 1302 \end{cases}$
Simplify and add, then solve for a.	$\begin{array}{l} -8a + 8c = -1176 \\ \underline{11a + 8c = 1302} \\ 3a \qquad = 126 \end{array}$

	$a = 42$ $a + c = 147$
Substitute *a* = 42 into the first equation, then solve for *c*.	$42 + c = 147$
	$c = 105$
Step 5. Check the answer in the problem. 42 adult tickets at $11 per ticket makes $462 105 child tickets at $8 per ticket makes $840. The total receipts are $1,302.✓	
Step 6. Answer the question.	The movie theater sold 42 adult tickets and 105 child tickets.

Note:
Exercise:

Problem: Translate to a system of equations and solve:

The ticket office at the zoo sold 553 tickets one day. The receipts totaled $3,936. How many $9 adult tickets and how many $6 child tickets were sold?

Solution:

There were 206 adult tickets sold and 347 children tickets sold.

Note:
Exercise:

Problem: Translate to a system of equations and solve:

A science center sold 1,363 tickets on a busy weekend. The receipts totaled $12,146. How many $12 adult tickets and how many $7 child tickets were sold?

Solution:

There were 521 adult tickets sold and 842 children tickets sold.

In [link] we’ll solve a coin problem. Now that we know how to work with systems of two variables, naming the variables in the ‘number’ column will be easy.

Example:
Exercise:

Problem: Translate to a system of equations and solve:

Priam has a collection of nickels and quarters, with a total value of $7.30. The number of nickels is six less than three times the number of quarters. How many nickels and how many quarters does he have?

Solution:
Solution

<table>
<tr><td>Step 1. Read the problem.</td><td>We will create a table to organize the information.</td></tr>
<tr><td>Step 2. Identify what we are looking for.</td><td>We are looking for the number of nickels and the number of quarters.</td></tr>
<tr><td>Step 3. Name what we are looking for.</td><td>Let $n =$ the number of nickels.
$q =$ the number of quarters</td></tr>
<tr><td>A table will help us organize the data.
We have two types of coins, nickels and quarters.</td><td>Write n and q for the number of each type of coin.</td></tr>
<tr><td>Fill in the Value column with the value of each
type of coin.</td><td>The value of each nickel is $0.05.
The value of each quarter is $0.25.</td></tr>
<tr><td>The number times the value gives the total
value, so, the total value of the nickels is
$n(0.05) = 0.05n$ and the total value of
quarters is $q(0.25) = 0.25q$.
Altogether the total value of the coins
is $7.30.</td><td>
<table>
<tr><th>Type</th><th>Number •</th><th>Value ($) =</th><th>Total Value ($)</th></tr>
<tr><td>nickels</td><td>n</td><td>0.05</td><td>$0.05n$</td></tr>
<tr><td>quarters</td><td>q</td><td>0.25</td><td>$0.25q$</td></tr>
<tr><td></td><td></td><td></td><td>7.30</td></tr>
</table>
</td></tr>
<tr><td>Step 4. Translate into a system of equations.</td><td></td></tr>
</table>

The Total value column gives one equation.	$0.05n + 0.25q = 7.30$
We also know the number of nickels is six less than three times the number of quarters. Translate to get the second equation.	$n = 3q - 6$
Now we have the system to solve.	$\begin{cases} 0.05n + 0.25q = 7.30 \\ n = 3q - 6 \end{cases}$
Step 5. Solve the system of equations We will use the substitution method. Substitute $n = 3q - 6$ into the first equation. Simplify and solve for q.	$0.05n + 0.25q = 7.30$
	$0.05(3q - 6) + 0.25q = 7.3$
	$0.15q - 0.3 + 0.25q = 7.3$
	$0.4q - 0.3 = 7.3$
	$0.4q = 7.6$
	$q = 19$
To find the number of nickels,	$n = 3q - 6$

substitute $q = 19$ into the second equation.	
	$n = 3 \cdot 19 - 6$
	$n = 51$
Step 6. Check the answer in the problem. $\begin{aligned} 19 \text{ quarters at } \$0.25 &= \$4.75 \\ 51 \text{nickels at } \$0.05 &= \$2.55 \\ \text{Total} &= \$7.30 \checkmark \\ 3 \cdot 19 - 16 &= 51 \checkmark \end{aligned}$	
Step 7. Answer the question.	Priam has 19 quarters and 51 nickels.

Note:
Exercise:

Problem: Translate to a system of equations and solve:

Matilda has a handful of quarters and dimes, with a total value of \$8.55. The number of quarters is 3 more than twice the number of dimes. How many dimes and how many quarters does she have?

Solution:

Matilda has 13 dimes and 29 quarters.

Note:
Exercise:

Problem: Translate to a system of equations and solve:

Juan has a pocketful of nickels and dimes. The total value of the coins is $8.10. The number of dimes is 9 less than twice the number of nickels. How many nickels and how many dimes does Juan have?

Solution:

Juan has 36 nickels and 63 dimes.

Some mixture applications involve combining foods or drinks. Example situations might include combining raisins and nuts to make a trail mix or using two types of coffee beans to make a blend.

Example:
Exercise:

Problem: Translate to a system of equations and solve:

Carson wants to make 20 pounds of trail mix using nuts and chocolate chips. His budget requires that the trail mix costs him $7.60 per pound. Nuts cost $9.00 per pound and chocolate chips cost $2.00 per pound. How many pounds of nuts and how many pounds of chocolate chips should he use?

Solution:
Solution

Step 1. Read the problem.	We will create a table to organize the information.
Step 2. Identify what we are looking for.	We are looking for the number of pounds of nuts and the number of pounds of chocolate chips.
Step 3. Name what we are looking for.	Let $n =$ the number of pound of nuts. $c =$ the number of pounds of chips
Carson will mix nuts and chocolate chips to get trail mix. Write in *n* and *c* for the number of pounds of nuts and chocolate chips. There will be 20 pounds of trail mix. Put the price per pound of each item in the Value column. Fill in the last column using	<table><tr><th>Type</th><th>Number of pounds</th><th>• Value ($) =</th><th>Total Value ($)</th></tr><tr><td>nuts</td><td>n</td><td>9.00</td><td>$9n$</td></tr><tr><td>chocolate chips</td><td>c</td><td>2.00</td><td>$2c$</td></tr><tr><td>trail mix</td><td>20</td><td>7.60</td><td>$7.60(20) = 152$</td></tr></table>
Number · Value = Total Value	
Step 4. Translate into a system of equations. We get the equations from the Number and Total Value columns.	$\begin{cases} n + c = 20 \\ 9n + 2c = 152 \end{cases}$

Step 5. Solve the system of equations We will use elimination to solve the system.	
Multiply the first equation by −2 to eliminate *c*.	$\begin{cases} -2(n+c) = -2(20) \\ 9n + 2c = 152 \end{cases}$
Simplify and add. Solve for *n*.	$\begin{cases} -2n - 2c = -40 \\ 9n + 2c = 152 \end{cases}$ $7n = 112$
	$n = 16$
To find the number of pounds of chocolate chips, substitute $n = 16$ into the first equation, then solve for *c*.	$n + c = 20$ $16 + c = 20$
	$c = 4$
Step 6. Check the answer in the problem. $$\begin{aligned} 16 + 4 &= 20 \checkmark \\ 9 \cdot 16 + 2 \cdot 4 &= 152 \checkmark \end{aligned}$$	
Step 7. Answer the question.	Carson should mix 16 pounds of nuts with 4 pounds of chocolate chips to create the trail mix.

Note:
Exercise:

Problem: Translate to a system of equations and solve:

Greta wants to make 5 pounds of a nut mix using peanuts and cashews. Her budget requires the mixture to cost her $6 per pound. Peanuts are $4 per pound and cashews are $9 per pound. How many pounds of peanuts and how many pounds of cashews should she use?

Solution:

Greta should use 3 pounds of peanuts and 2 pounds of cashews.

Note:
Exercise:

Problem: Translate to a system of equations and solve:

Sammy has most of the ingredients he needs to make a large batch of chili. The only items he lacks are beans and ground beef. He needs a total of 20 pounds combined of beans and ground beef and has a budget of $3 per pound. The price of beans is $1 per pound and the price of ground beef is $5 per pound. How many pounds of beans and how many pounds of ground beef should he purchase?

Solution:

Sammy should purchase 10 pounds of beans and 10 pounds of ground beef.

Another application of mixture problems relates to concentrated cleaning supplies, other chemicals, and mixed drinks. The concentration is given as a percent. For example, a 20% concentrated household cleanser means that 20% of the total amount is cleanser, and the rest is water. To make 35 ounces of a 20% concentration, you mix 7 ounces (20% of 35) of the cleanser with 28 ounces of water.

For these kinds of mixture problems, we'll use percent instead of value for one of the columns in our table.

Example:
Exercise:

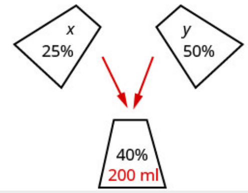
Problem: Translate to a system of equations and solve:

Sasheena is a lab assistant at her community college. She needs to make 200 milliliters of a 40% solution of sulfuric acid for a lab experiment. The lab has only 25% and 50% solutions in the storeroom. How much should she mix of the 25% and the 50% solutions to make the 40% solution?

Solution:
Solution

Step 1. Read the problem.	A figure may help us visualize the situation, then we will create a table to organize the information.

Sasheena must mix some of the 25% solution and some of the 50% solution together to get 200 ml of the 40% solution.



Step 2. Identify what we are looking for.

We are looking for how much of each solution she needs.

Step 3. Name what we are looking for.

Let $x =$ number of ml of 25% solution.
$y =$ number of ml of 50% solution

A table will help us organize the data.

She will mix *x* ml of 25% with *y* ml of 50% to get 200 ml of 40% solution.

We write the percents as decimals in the chart.

We multiply the number of units times the concentration to get the total amount of sulfuric acid in each solution.

Type	Number of units	• Concentration %	= Amount
25%	x	0.25	$0.25x$
50%	y	0.50	$0.50y$
40%	200	0.40	0.40(200)

Step 4. Translate into a system of equations. We get the equations from the Number column and the

Amount column.	
Now we have the system.	$\begin{cases} x + y = 200 \\ 0.25x + 0.50y = 0.40(200) \end{cases}$
Step 5. Solve the system of equations. We will solve the system by elimination. Multiply the first equation by −0.5 to eliminate *y*.	$\begin{cases} -0.5(x + y) = -0.5(200) \\ 0.25x + 0.50y = 80 \end{cases}$
Simplify and add to solve for *x*.	$\begin{array}{rcl} -0.5x - 0.5y & = & -100 \\ 0.25x + 0.5y & = & 80 \\ \hline -0.25x & = & -20 \\ x & = & 80 \end{array}$
To solve for *y*, substitute *x* = 80 into the first equation.	$x + y = 200$
	$80 + y = 200$
	$y = 120$
Step 6. Check the answer in the problem. $\begin{array}{rcl} 80 + 120 & = & 120 \checkmark \\ 0.25(80) + 0.50(120) & = & 80 \checkmark \\ & & \text{Yes!} \end{array}$	

Step 7. Answer the question.	Sasheena should mix 80 ml of the 25% solution with 120 ml of the 50% solution to get the 200 ml of the 40% solution.

Note:
Exercise:

Problem: Translate to a system of equations and solve:

LeBron needs 150 milliliters of a 30% solution of sulfuric acid for a lab experiment but only has access to a 25% and a 50% solution. How much of the 25% and how much of the 50% solution should he mix to make the 30% solution?

Solution:

LeBron needs 120 ml of the 25% solution and 30 ml of the 50% solution.

Note:
Exercise:

Problem: Translate to a system of equations and solve:

Anatole needs to make 250 milliliters of a 25% solution of hydrochloric acid for a lab experiment. The lab only has a 10% solution and a 40% solution in the storeroom. How much of the 10%

and how much of the 40% solutions should he mix to make the 25% solution?

Solution:

Anatole should mix 125 ml of the 10% solution and 125 ml of the 40% solution.

Solve Interest Applications

The formula to model interest applications is $I = Prt$. Interest, I, is the product of the principal, P, the rate, r, and the time, t. In our work here, we will calculate the interest earned in one year, so t will be 1.

We modify the column titles in the mixture table to show the formula for interest, as you'll see in [link].

Example:
Exercise:

Problem: Translate to a system of equations and solve:

Adnan has $40,000 to invest and hopes to earn 7.1% interest per year. He will put some of the money into a stock fund that earns 8% per year and the rest into bonds that earns 3% per year. How much money should he put into each fund?

Solution:
Solution

<table>
<tr><td>Step 1. Read the problem.</td><td>A chart will help us organize the information.</td></tr>
<tr><td>Step 2. Identify what we are looking for.</td><td>We are looking for the amount to invest in each fund.</td></tr>
<tr><td>Step 3. Name what we are looking for.</td><td>Let $s =$ the amount invested in stocks.
$b =$ the amount invested in bonds.</td></tr>
<tr><td>Write the interest rate as a decimal for
each fund.
Multiply:
Principal · Rate · Time
to get the Interest.</td><td>
<table>
<tr><th>Account</th><th>Principal •</th><th>Rate</th><th>• Time =</th><th>Interest</th></tr>
<tr><td>stock fund</td><td>s</td><td>0.08</td><td>1</td><td>$0.08s$</td></tr>
<tr><td>bonds</td><td>b</td><td>0.03</td><td>1</td><td>$0.03b$</td></tr>
<tr><td>Total</td><td>40,000</td><td>0.071</td><td></td><td>0.071(40,000)</td></tr>
</table>
</td></tr>
<tr><td>Step 4. Translate into a system of
equations.
We get our system of equations from
the Principal column and the Interest column.</td><td>$\begin{cases} s + b = 40{,}000 \\ 0.08s + 0.03b = 0.071(40{,}000) \end{cases}$</td></tr>
<tr><td>Step 5. Solve the system of equations
Solve by elimination.
Multiply the top equation by −0.03.</td><td>$\begin{cases} {\color{red}-0.03}(s + b) = {\color{red}-0.03}(40{,}000) \\ 0.08s + 0.03b = 2{,}840 \end{cases}$</td></tr>
<tr><td>Simplify and add to solve for s.</td><td>$\begin{cases} -0.03s - 0.03b = -1{,}200 \\ 0.08s + 0.03b = 2{,}840 \end{cases}$
$\overline{0.05s = 1{,}640}$</td></tr>
<tr><td></td><td></td></tr>
</table>

	$s = 32{,}800$
To find b, substitute s = 32,800 into the first equation.	$s + b = 40{,}000$ $32{,}800 + b = 40{,}000$
	$b = 7{,}200$
Step 6. Check the answer in the problem.	We leave the check to you.
Step 7. Answer the question.	Adnan should invest $32,800 in stock and $7,200 in bonds.

Did you notice that the Principal column represents the total amount of money invested while the Interest column represents only the interest earned? Likewise, the first equation in our system, $s + b = 40{,}000$, represents the total amount of money invested and the second equation, $0.08s + 0.03b = 0.071(40{,}000)$, represents the interest earned.

Note:
Exercise:

Problem: Translate to a system of equations and solve:

Leon had $50,000 to invest and hopes to earn 6.2 % interest per year. He will put some of the money into a stock fund that earns 7% per year and the rest in to a savings account that earns 2% per year. How much money should he put into each fund?

Solution:

Leon should put $42,000 in the stock fund and $8000 in the savings account.

Note:
Exercise:

Problem: Translate to a system of equations and solve:

Julius invested $7,000 into two stock investments. One stock paid 11% interest and the other stock paid 13% interest. He earned 12.5% interest on the total investment. How much money did he put in each stock?

Solution:

Julius invested $1,750 at 11% and $5,250 at 13%.

Example:
Exercise:

Problem: Translate to a system of equations and solve:

Rosie owes $21,540 on her two student loans. The interest rate on her bank loan is 10.5% and the interest rate on the federal loan is 5.9%. The total amount of interest she paid last year was $1,669.68. What was the principal for each loan?

Solution:
Solution

Step 1. Read the problem.	A chart will help us organize the information.
Step 2. Identify what we are looking for.	We are looking for the principal of each loan.
Step 3. Name what we are looking for.	Let $b =$ the principal for the bank loan. $f =$ the principal on the federal loan
The total loans are \$21,540.	
Record the interest rates as decimals in the chart.	
Multiply using the formula $I = \Pr t$ to get the Interest.	
Step 4. Translate into a system of equations. The system of equations comes from the Principal column and the Interest column.	$\begin{cases} b+f=21{,}540 \\ 0.105b+0.059f=1669.68 \end{cases}$
Step 5. Solve the system of equations We will use substitution to	$b+f=21{,}540$ $b=-f+21{,}540$

Account	Principal •	Rate •	Time =	Interest
bank	b	0.105	1	$0.105b$
federal	f	0.059	1	$0.059f$
Total	21,540			1669.68

solve. Solve the first equation for *b*.	
Substitute $b = -f + 21{,}540$ into the second equation.	$0.105b + 0.059f = 1669.68$ $0.105(-f + 21{,}540) + 0.059f = 1669.68$
Simplify and solve for *f*.	$-0.105f + 2261.70 + 0.059f = 1669.68$
	$-0.046f + 2261.70 = 1669.68$
	$-0.046f = -592.02$
	$f = 12{,}870$
To find *b*, substitute $f = 12{,}870$ into the first equation.	$b + f = 21{,}540$ $12{,}870 + f = 21{,}540$
	$f = 8{,}670$
Step 6. Check the answer in the problem.	We leave the check to you.
Step 7. Answer the question.	The principal of the bank loan is \$12,870 and

	the principal for the federal loan is $8,670.

Note:
Exercise:

Problem: Translate to a system of equations and solve:

Laura owes $18,000 on her student loans. The interest rate on the bank loan is 2.5% and the interest rate on the federal loan is 6.9 %. The total amount of interest she paid last year was $1,066. What was the principal for each loan?

Solution:

The principal amount for the bank loan was $4,000. The principal amount for the federal loan was $14,000.

Note:
Exercise:

Problem: Translate to a system of equations and solve:

Jill's Sandwich Shoppe owes $65,200 on two business loans, one at 4.5% interest and the other at 7.2% interest. The total amount of interest owed last year was $3,582. What was the principal for each loan?

Solution:

The principal amount for was $41,200 at 4.5%. The principal amount was, $24,000 at 7.2%.

Note:
Access these online resources for additional instruction and practice with solving application problems with systems of linear equations.

- Cost and Mixture Word Problems
- Mixture Problems

Key Concepts

- **Table for coin and mixture applications**

Type	Number •	Value($) =	Total Value($)
Total			

- **Table for concentration applications**

Type	Number of units •	Concentration % =	Amount
Total			

- **Table for interest applications**

Account	Principal •	Rate •	Time =	Interest
			1	
			1	
Total				

Practice Makes Perfect

Solve Mixture Applications

In the following exercises, translate to a system of equations and solve.

Exercise:

Problem:

Tickets to a Broadway show cost $35 for adults and $15 for children. The total receipts for 1650 tickets at one performance were $47,150. How many adult and how many child tickets were sold?

Solution:

There 1120 adult tickets and 530 child tickets sold.

Exercise:

Problem:

Tickets for a show are $70 for adults and $50 for children. One evening performance had a total of 300 tickets sold and the receipts totaled $17,200. How many adult and how many child tickets were sold?

Exercise:

Problem:

Tickets for a train cost $10 for children and $22 for adults. Josie paid $1,200 for a total of 72 tickets. How many children's tickets and how many adult tickets did Josie buy?

Solution:

Josie bought 40 adult tickets and 32 children tickets.

Exercise:

Problem:

Tickets for a baseball game are $69 for Main Level seats and $39 for Terrace Level seats. A group of sixteen friends went to the game and spent a total of $804 for the tickets. How many of Main Level and how many Terrace Level tickets did they buy?

Exercise:

Problem:

Tickets for a dance recital cost $15 for adults and $7 for children. The dance company sold 253 tickets and the total receipts were $2,771. How many adult tickets and how many child tickets were sold?

Solution:

There were 125 adult tickets and 128 children tickets sold.

Exercise:

Problem:

Tickets for the community fair cost $12 for adults and $5 dollars for children. On the first day of the fair, 312 tickets were sold for a total of $2,204. How many adult tickets and how many child tickets were sold?

Exercise:

Problem:

Brandon has a cup of quarters and dimes with a total value of $3.80. The number of quarters is four less than twice the number of dimes. How many quarters and how many dimes does Brandon have?

Solution:

Brandon has 12 quarters and 8 dimes.

Exercise:

Problem:

Sherri saves nickels and dimes in a coin purse for her daughter. The total value of the coins in the purse is $0.95. The number of nickels is two less than five times the number of dimes. How many nickels and how many dimes are in the coin purse?

Exercise:

Problem:

Peter has been saving his loose change for several days. When he counted his quarters and dimes, he found they had a total value $13.10. The number of quarters was fifteen more than three times the number of dimes. How many quarters and how many dimes did Peter have?

Solution:

Peter had 11 dimes and 48 quarters.

Exercise:

Problem:

Lucinda had a pocketful of dimes and quarters with a value of $ $6.20. The number of dimes is eighteen more than three times the number of quarters. How many dimes and how many quarters does Lucinda have?

Exercise:

Problem:

A cashier has 30 bills, all of which are $10 or $20 bills. The total value of the money is $460. How many of each type of bill does the cashier have?

Solution:

The cashier has fourteen \$10 bills and sixteen \$20 bills.

Exercise:

Problem:

A cashier has 54 bills, all of which are \$10 or \$20 bills. The total value of the money is \$910. How many of each type of bill does the cashier have?

Exercise:

Problem:

Marissa wants to blend candy selling for \$1.80 per pound with candy costing \$1.20 per pound to get a mixture that costs her \$1.40 per pound to make. She wants to make 90 pounds of the candy blend. How many pounds of each type of candy should she use?

Solution:

Marissa should use 60 pounds of the \$1.20/lb candy and 30 pounds of the \$1.80/lb candy.

Exercise:

Problem:

How many pounds of nuts selling for \$6 per pound and raisins selling for \$3 per pound should Kurt combine to obtain 120 pounds of trail mix that cost him \$5 per pound?

Exercise:

Problem:

Hannah has to make twenty-five gallons of punch for a potluck. The punch is made of soda and fruit drink. The cost of the soda is $1.79 per gallon and the cost of the fruit drink is $2.49 per gallon. Hannah's budget requires that the punch cost $2.21 per gallon. How many gallons of soda and how many gallons of fruit drink does she need?

Solution:

Hannah needs 10 gallons of soda and 15 gallons of fruit drink.

Exercise:

Problem:

Joseph would like to make 12 pounds of a coffee blend at a cost of $6.25 per pound. He blends Ground Chicory at $4.40 a pound with Jamaican Blue Mountain at $8.84 per pound. How much of each type of coffee should he use?

Exercise:

Problem:

Julia and her husband own a coffee shop. They experimented with mixing a City Roast Columbian coffee that cost $7.80 per pound with French Roast Columbian coffee that cost $8.10 per pound to make a 20 pound blend. Their blend should cost them $7.92 per pound. How much of each type of coffee should they buy?

Solution:

Julia and her husband should buy 12 pounds of City Roast Columbian coffee and 8 pounds of French Roast Columbian coffee.

Exercise:

Problem:

Melody wants to sell bags of mixed candy at her lemonade stand. She will mix chocolate pieces that cost \$4.89 per bag with peanut butter pieces that cost \$3.79 per bag to get a total of twenty-five bags of mixed candy. Melody wants the bags of mixed candy to cost her \$4.23 a bag to make. How many bags of chocolate pieces and how many bags of peanut butter pieces should she use?

Exercise:

Problem:

Jotham needs 70 liters of a 50% alcohol solution. He has a 30% and an 80% solution available. How many liters of the 30% and how many liters of the 80% solutions should he mix to make the 50% solution?

Solution:

Jotham should mix 2 liters of the 30% solution and 28 liters of the 80% solution.

Exercise:

Problem:

Joy is preparing 15 liters of a 25% saline solution. She only has 40% and 10% solution in her lab. How many liters of the 40% and how many liters of the 10% should she mix to make the 25% solution?

Exercise:

Problem:

A scientist needs 65 liters of a 15% alcohol solution. She has available a 25% and a 12% solution. How many liters of the 25% and how many liters of the 12% solutions should she mix to make the 15% solution?

Solution:

The scientist should mix 15 liters of the 25% solution and 50 liters of the 12% solution.

Exercise:

Problem:

A scientist needs 120 liters of a 20% acid solution for an experiment. The lab has available a 25% and a 10% solution. How many liters of the 25% and how many liters of the 10% solutions should the scientist mix to make the 20% solution?

Exercise:

Problem:

A 40% antifreeze solution is to be mixed with a 70% antifreeze solution to get 240 liters of a 50% solution. How many liters of the 40% and how many liters of the 70% solutions will be used?

Solution:

160 liters of the 40% solution and 80 liters of the 70% solution will be used.

Exercise:

Problem:

A 90% antifreeze solution is to be mixed with a 75% antifreeze solution to get 360 liters of a 85% solution. How many liters of the 90% and how many liters of the 75% solutions will be used?

Solve Interest Applications

In the following exercises, translate to a system of equations and solve.

Exercise:

Problem:

Hattie had $3,000 to invest and wants to earn 10.6% interest per year. She will put some of the money into an account that earns 12% per year and the rest into an account that earns 10% per year. How much money should she put into each account?

Solution:

Hattie should invest $900 at 12% and $2,100 at 10%.

Exercise:

Problem:

Carol invested $2,560 into two accounts. One account paid 8% interest and the other paid 6% interest. She earned 7.25% interest on the total investment. How much money did she put in each account?

Exercise:

Problem:

Sam invested $48,000, some at 6% interest and the rest at 10%. How much did he invest at each rate if he received $4,000 in interest in one year?

Solution:

Sam invested $28,000 at 10% and $20,000 at 6%.

Exercise:

Problem:

Arnold invested $64,000, some at 5.5% interest and the rest at 9%. How much did he invest at each rate if he received $4,500 in interest in one year?

Exercise:

Problem:

After four years in college, Josie owes $65,800 in student loans. The interest rate on the federal loans is 4.5% and the rate on the private bank loans is 2%. The total interest she owed for one year was $2,878.50. What is the amount of each loan?

Solution:

The federal loan is $62,500 and the bank loan is $3,300.

Exercise:

Problem:

Mark wants to invest $10,000 to pay for his daughter's wedding next year. He will invest some of the money in a short term CD that pays 12% interest and the rest in a money market savings account that pays 5% interest. How much should he invest at each rate if he wants to earn $1,095 in interest in one year?

Exercise:

Problem:

A trust fund worth $25,000 is invested in two different portfolios. This year, one portfolio is expected to earn 5.25% interest and the other is expected to earn 4%. Plans are for the total interest on the fund to be $1150 in one year. How much money should be invested at each rate?

Solution:

$12,000 should be invested at 5.25% and $13,000 should be invested at 4%.

Exercise:

Problem:

A business has two loans totaling $85,000. One loan has a rate of 6% and the other has a rate of 4.5%. This year, the business expects to pay $4650 in interest on the two loans. How much is each loan?

Everyday Math

In the following exercises, translate to a system of equations and solve.

Exercise:

Problem:

Laurie was completing the treasurer's report for her son's Boy Scout troop at the end of the school year. She didn't remember how many boys had paid the $15 full-year registration fee and how many had paid the $10 partial-year fee. She knew that the number of boys who paid for a full-year was ten more than the number who paid for a partial-year. If $250 was collected for all the registrations, how many boys had paid the full-year fee and how many had paid the partial-year fee?

Solution:

14 boys paid the full-year fee. 4 boys paid the partial-year fee,

Exercise:

Problem:

As the treasurer of her daughter's Girl Scout troop, Laney collected money for some girls and adults to go to a three-day camp. Each girl paid $75 and each adult paid $30. The total amount of money collected for camp was $765. If the number of girls is three times the number of adults, how many girls and how many adults paid for camp?

Writing Exercises

Exercise:

Problem:

Take a handful of two types of coins, and write a problem similar to [link] relating the total number of coins and their total value. Set up a system of equations to describe your situation and then solve it.

Solution:

Answers will vary.

Exercise:

Problem:

In [link] we solved the system of equations $\begin{array}{l} b+f=21{,}540 \\ 0.105b+0.059f=1669.68 \end{array}$ by substitution. Would you have used substitution or elimination to solve this system? Why?

Self Check

After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve mixture applications.			
solve interest applications.			

After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

Graphing Systems of Linear Inequalities: ASE
By the end of this section, you will be able to:

- Determine whether an ordered pair is a solution of a system of linear inequalities
- Solve a system of linear inequalities by graphing
- Solve applications of systems of inequalities

Determine Whether an Ordered Pair is a Solution of a System of Linear Inequalities

The definition of a system of linear inequalities is very similar to the definition of a system of linear equations.

Note:
System of Linear Inequalities
Two or more linear inequalities grouped together form a **system of linear inequalities**.

A system of linear inequalities looks like a system of linear equations, but it has inequalities instead of equations. A system of two linear inequalities is shown below.
Equation:

$$\begin{cases} x + 4y \geq 10 \\ 3x - 2y < 12 \end{cases}$$

To solve a system of linear inequalities, we will find values of the variables that are solutions to both inequalities. We solve the system by using the graphs of each inequality and show the solution as a graph. We will find the region on the plane that contains all ordered pairs (x, y) that make both inequalities true.

Note:
Solutions of a System of Linear Inequalities
Solutions of a system of linear inequalities are the values of the variables that make all the inequalities true.

The solution of a system of linear inequalities is shown as a shaded region in the *x*-*y* coordinate system that includes all the points whose ordered pairs make the inequalities true.

To determine if an ordered pair is a solution to a system of two inequalities, we substitute the values of the variables into each inequality. If the ordered pair makes both inequalities true, it is a solution to the system.

Example:
Exercise:

Problem:

Determine whether the ordered pair is a solution to the system. $\begin{cases} x+4y\geq 10 \\ 3x-2y<12 \end{cases}$

ⓐ (−2, 4) ⓑ (3,1)

Solution:
Solution

ⓐ Is the ordered pair (−2, 4) a solution?

We substitute $x=-2$ and $y=4$ into both inequalities.

$x+4y\geq 10$	$3x-2y<12$
$-2+4(4)\overset{?}{\geq}10$	$3(-2)-2(4)\overset{?}{<}12$
$14\geq 10$ true	$-14<12$ true

The ordered pair (−2, 4) made both inequalities true. Therefore (−2, 4) is a solution to this system.

ⓑ Is the ordered pair (3,1) a solution?

We substitute $x=3$ and $y=1$ into both inequalities.

$x+4y\geq 10$	$3x-2y<12$
$3+4(1)\overset{?}{\geq}10$	$3(3)-2(1)\overset{?}{<}12$
$7\geq 10$ false	$7<12$ true

The ordered pair (3,1) made one inequality true, but the other one false. Therefore (3,1) is not a solution to this system.

Note:
Exercise:

Determine whether the ordered pair is a solution to the system.

Problem: $\begin{cases} x - 5y > 10 \\ 2x + 3y > -2 \end{cases}$

ⓐ $(3, -1)$ ⓑ $(6, -3)$

Solution:

ⓐ no ⓑ yes

Note:
Exercise:

Determine whether the ordered pair is a solution to the system.

Problem: $\begin{cases} y > 4x - 2 \\ 4x - y < 20 \end{cases}$

ⓐ $(2, 1)$ ⓑ $(4, -1)$

Solution:

ⓐ no ⓑ no

Solve a System of Linear Inequalities by Graphing

The solution to a single linear inequality is the region on one side of the boundary line that contains all the points that make the inequality true. The solution to a system of two linear inequalities is a region that contains the solutions to both inequalities. To find this

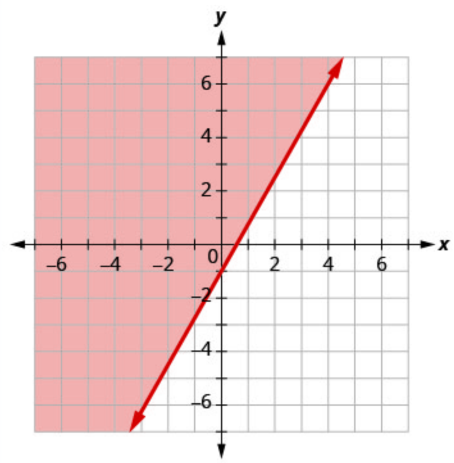
region, we will graph each inequality separately and then locate the region where they are both true. The solution is always shown as a graph.

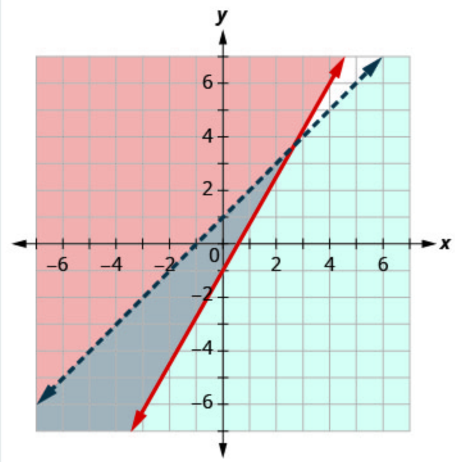
Example:
How to Solve a System of Linear inequalities
Exercise:

Problem: Solve the system by graphing.

$$\begin{cases} y \geq 2x - 1 \\ y < x + 1 \end{cases}$$

Solution:
Solution

Step 1. Graph the first inequality.	We will graph $y \geq 2x - 1$.	$\begin{cases} y \geq 2x - 1 \\ y < x + 1 \end{cases}$
Graph the boundary line.	We graph the line $y = 2x - 1$. It is a solid line because the inequality sign is $\geq$.	
Shade in the side of the boundary line where the inequality is true.	We choose (0,0) as a test point. It is a solution to $y \geq 2x - 1$, so we shade in the left side of the boundary line.	

Step 2. On the same grid, graph the second inequality. Graph the boundary line. Shade in the side of that boundary line where the inequality is true.	We will graph $y < x + 1$ on the same grid. We graph the line $y = x + 1$. It is a dashed line because the inequality sign is $<$. Again, we use (0,0) as a test point. It is a solution so we shade in that side of the line $y = x + 1$.	

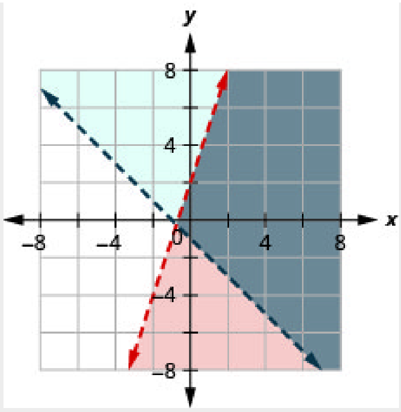
Step 3. The solution is the region where the shading overlaps.	The point where the boundary lines intersect is not a solution because it is not a solution to $y < x + 1$.	The solution is all points in the darker shaded region.

Step 4. Check by choosing a test point.	We'll use (–1, –1) as a test point.	Is (–1, –1) a solution to $y \geq 2x - 1$? $-1 \overset{?}{\geq} 2(-1) - 1$ $-1 \geq -3$ true Is (–1, –1) a solution to $y < x + 1$? $-1 \overset{?}{<} -1 + 1$ $-1 < 0$ true The region containing (–1, –1) is the solution to this system.

Note:
Exercise:

Problem: Solve the system by graphing. $\begin{cases} y < 3x + 2 \\ y > -x - 1 \end{cases}$

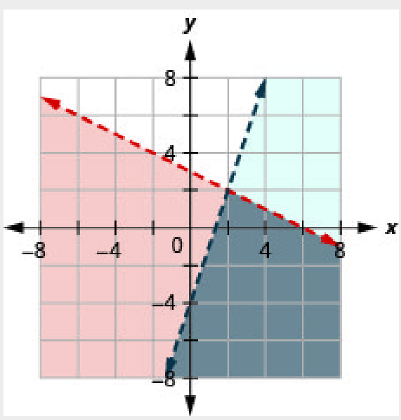
Solution:



Note:
Exercise:

Problem: Solve the system by graphing. $\begin{cases} y < -\frac{1}{2}x + 3 \\ y < 3x - 4 \end{cases}$

Solution:



Note:
Solve a system of linear inequalities by graphing.

Graph the first inequality.

- Graph the boundary line.

- Shade in the side of the boundary line where the inequality is true.

On the same grid, graph the second inequality.

- Graph the boundary line.
- Shade in the side of that boundary line where the inequality is true.

The solution is the region where the shading overlaps.
Check by choosing a test point.

Example:
Exercise:

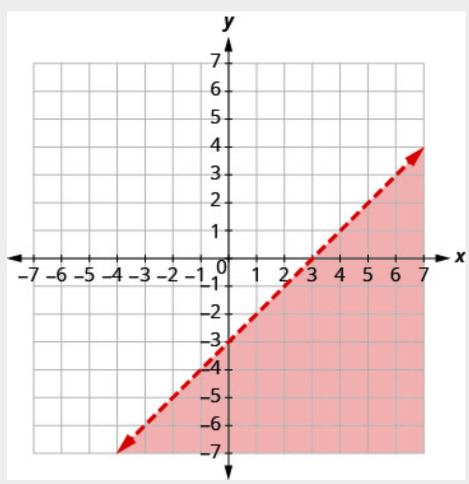
Problem: Solve the system by graphing. $\begin{cases} x - y > 3 \\ y < -\frac{1}{5}x + 4 \end{cases}$

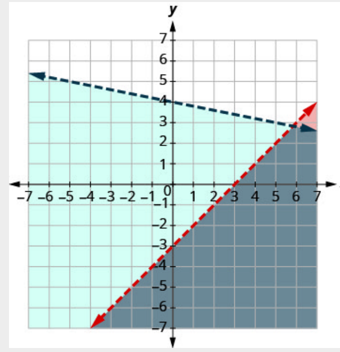
Solution:
Solution

Graph $x - y > 3$, by graphing $x - y = 3$ and
testing a point.

The intercepts are $x = 3$ and $y = -3$ and the boundary
line will be dashed.

Test (0, 0). It makes the inequality false. So,
shade the side that does not contain (0, 0) red.



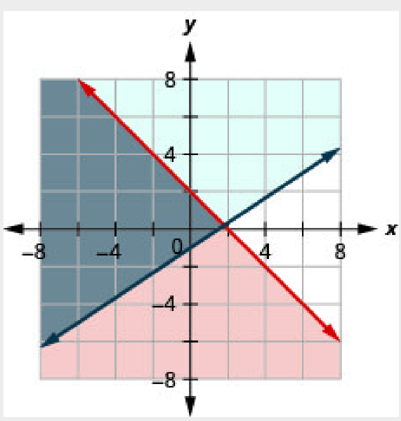
Graph $y < -\frac{1}{5}x + 4$ by graphing $y = -\frac{1}{5}x + 4$ using the slope $m = -\frac{1}{5}$ and y−intercept b = 4. The boundary line will be dashed. Test (0, 0). It makes the inequality true, so shade the side that contains (0, 0) blue. Choose a test point in the solution and verify that it is a solution to both inequalities.	

The point of intersection of the two lines is not included as both boundary lines were dashed. The solution is the area shaded twice which is the darker-shaded region.

Note:
Exercise:

Problem: Solve the system by graphing. $\begin{cases} x + y \leq 2 \\ y \geq \frac{2}{3}x - 1 \end{cases}$

Solution:

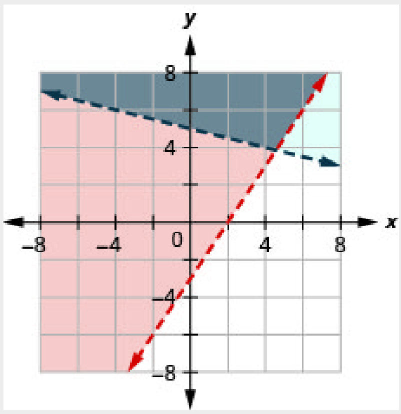


Note:

Exercise:

Problem: Solve the system by graphing. $\begin{cases} 3x - 2y \le 6 \\ y > -\frac{1}{4}x + 5 \end{cases}$

Solution:



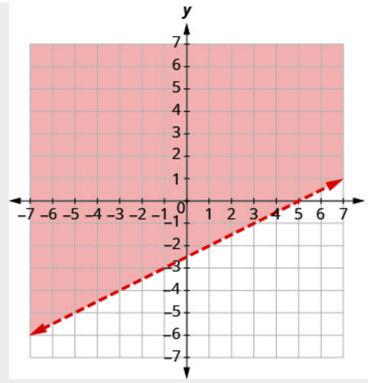
Example:
Exercise:

Problem: Solve the system by graphing. $\begin{cases} x - 2y < 5 \\ y > -4 \end{cases}$

Solution:
Solution

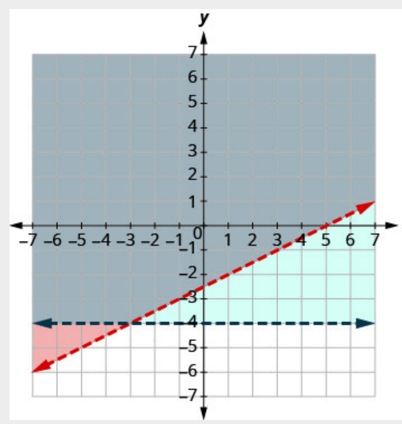
Graph $x - 2y < 5$, by graphing $x - 2y = 5$ and testing a point. The intercepts are *x* = 5 and *y* = −2.5 and the boundary line will be dashed. Test (0, 0). It makes the inequality true. So,	

shade the side
that contains (0, 0) red.



Graph $y > -4$, by graphing $y = -4$ and recognizing that it is a horizontal line through $y = -4$. The boundary line will be dashed.

Test (0, 0). It makes the inequality true. So, shade (blue)
the side that contains (0, 0) blue.



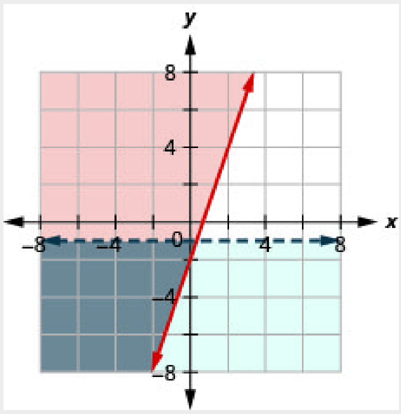
The point (0, 0) is in the solution and we have already found it to be a solution of each inequality. The point of intersection of the two lines is not included as both boundary lines were dashed.

The solution is the area shaded twice which is the darker-shaded region.

Note:
Exercise:

Problem: Solve the system by graphing. $\begin{cases} y \geq 3x - 2 \\ y < -1 \end{cases}$

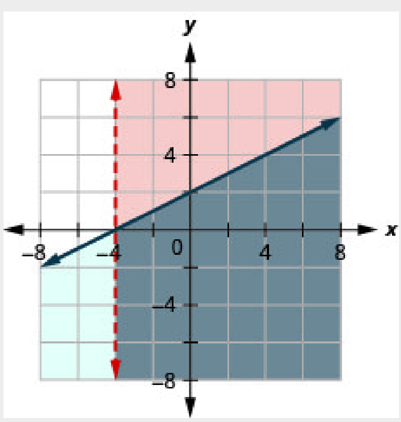
Solution:



Note:
Exercise:

Problem: Solve the system by graphing. $\begin{cases} x > -4 \\ x - 2y \leq -4 \end{cases}$

Solution:

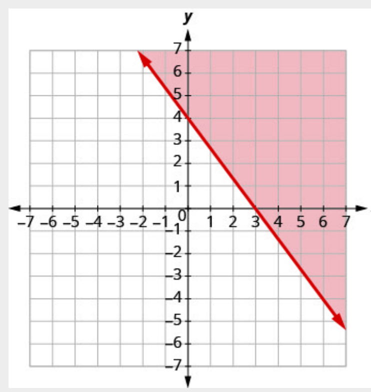
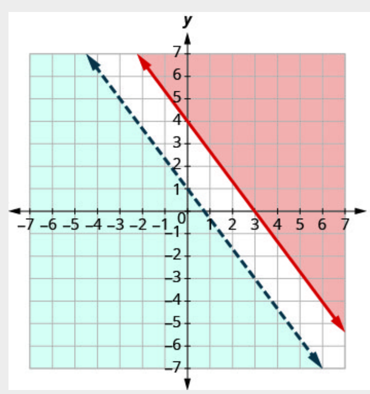


Systems of linear inequalities where the boundary lines are parallel might have no solution. We'll see this in [link].

Example:
Exercise:

Problem: Solve the system by graphing. $\begin{cases} 4x + 3y \geq 12 \\ y < -\frac{4}{3}x + 1 \end{cases}$

Solution:
Solution

Graph $4x + 3y \geq 12$, by graphing $4x + 3y = 12$ and testing a point. The intercepts are *x* = 3 and *y* = 4 and the boundary line will be solid. Test (0, 0). It makes the inequality false. So, shade the side that does not contain (0, 0) red.	
Graph $y < -\frac{4}{3}x + 1$ by graphing $y = -\frac{4}{3}x + 1$ using the slope $m = \frac{4}{3}$ and the *y*-intercept *b* = 1. The boundary line will be dashed. Test (0, 0). It makes the inequality true. So, shade the side that contains (0, 0) blue.	

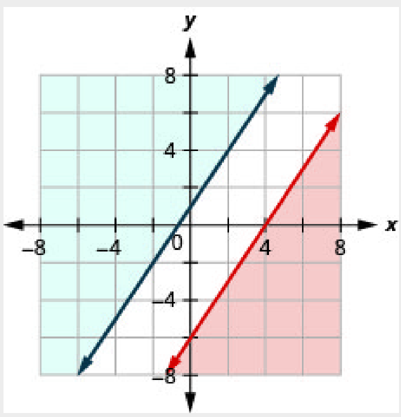
There is no point in both shaded regions, so the system has no solution. This system has no solution.

Note:
Exercise:

Problem: Solve the system by graphing. $\begin{cases} 3x - 2y \leq 12 \\ y \geq \frac{3}{2}x + 1 \end{cases}$

Solution:

no solution

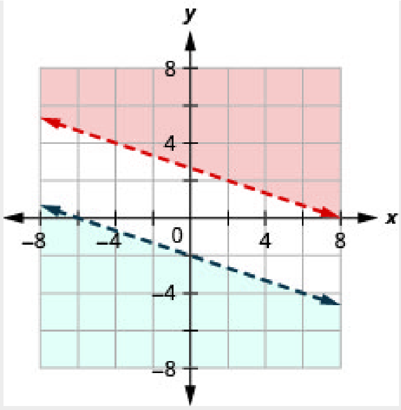


Note:
Exercise:

Problem: Solve the system by graphing. $\begin{cases} x + 3y > 8 \\ y < -\frac{1}{3}x - 2 \end{cases}$

Solution:

no solution

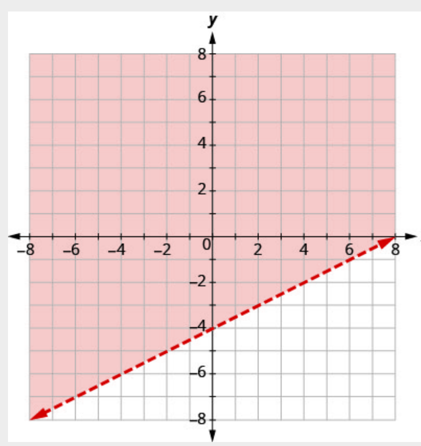


Example:
Exercise:

Problem: Solve the system by graphing. $\begin{cases} y > \frac{1}{2}x - 4 \\ x - 2y < -4 \end{cases}$

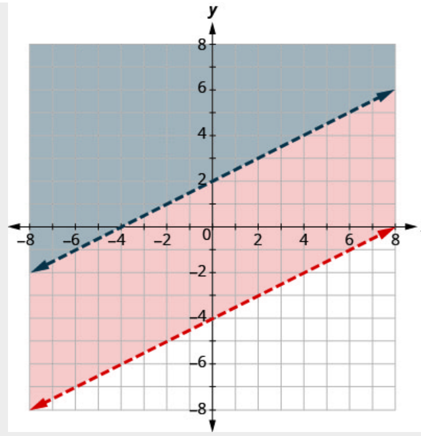
Solution:
Solution

Graph $y > \frac{1}{2}x - 4$ by graphing
$y = \frac{1}{2}x - 4$
using the slope $m = \frac{1}{2}$ and the intercept b = −4. The boundary line will be dashed.
Test (0, 0). It makes the inequality true. So,
shade the side that contains (0, 0) red.



Graph $x - 2y < -4$ by graphing
$x - 2y = -4$ and testing a point.
The intercepts are $x = -4$ and $y = 2$ and the boundary
line will be dashed.

Choose a test point in the solution and verify
that it is a solution to both inequalities.



No point on the boundary lines is included in the solution as both lines are dashed.

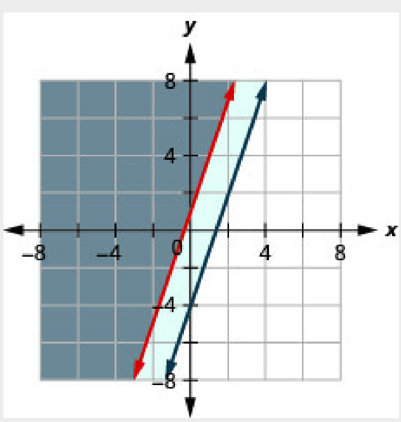
The solution is the region that is shaded twice, which is also the solution to $x - 2y < -4$.

Note:
Exercise:

Problem: Solve the system by graphing. $\begin{cases} y \geq 3x + 1 \\ -3x + y \geq -4 \end{cases}$

Solution:

$y \geq 3x + 1$

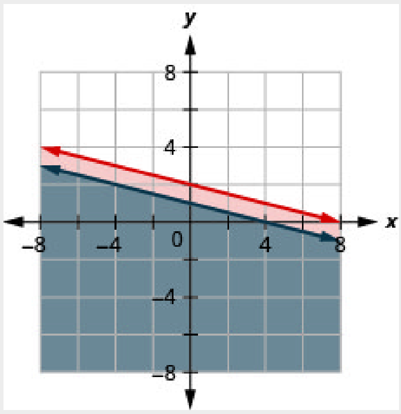


Note:
Exercise:

Problem: Solve the system by graphing. $\begin{cases} y \leq -\frac{1}{4}x + 2 \\ x + 4y \leq 4 \end{cases}$

Solution:

$x + 4y \leq 4$



Solve Applications of Systems of Inequalities

The first thing we'll need to do to solve applications of systems of inequalities is to translate each condition into an inequality. Then we graph the system as we did above to see the region that contains the solutions. Many situations will be realistic only if both variables are positive, so their graphs will only show Quadrant I.

Example:
Exercise:

Problem:

Christy sells her photographs at a booth at a street fair. At the start of the day, she wants to have at least 25 photos to display at her booth. Each small photo she displays costs her $4 and each large photo costs her $10. She doesn't want to spend more than $200 on photos to display.

ⓐ Write a system of inequalities to model this situation.

ⓑ Graph the system.

ⓒ Could she display 15 small and 5 large photos?

ⓓ Could she display 3 large and 22 small photos?

Solution:
Solution

ⓐ Let $x=$ the number of small photos.
$y=$ the number of large photos

To find the system of inequalities, translate the information.

She wants to have at least 25 photos.

The number of small plus the number of large should be at least 25.

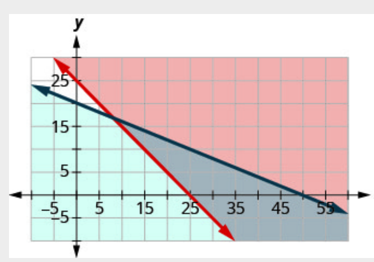
$$x+y\geq 25$$

\$4 for each small and \$10 for each large must be no more than \$200

$$4x+10y\leq 200$$

We have our system of inequalities. $\begin{cases} x+y\geq 25 \\ 4x+10y\leq 200 \end{cases}$

ⓑ

To graph $x+y\geq 25$, graph $x+y=25$ as a solid line. Choose (0, 0) as a test point. Since it does not make the inequality true, shade the side that does not include the point (0, 0) red. To graph $4x+10y\leq 200$, graph $4x+10y=200$ as a solid line. Choose (0, 0) as a test point. Since it does not make the inequality true, shade the side that includes the point (0, 0) blue.	

The solution of the system is the region of the graph that is double shaded and so is shaded darker.
ⓒ To determine if 10 small and 20 large photos would work, we see if the point (10, 20) is in the solution region. It is not. Christy would not display 10 small and 20 large photos.
ⓓ To determine if 20 small and 10 large photos would work, we see if the point (20, 10) is in the solution region. It is. Christy could choose to display 20 small and 10 large photos.

Notice that we could also test the possible solutions by substituting the values into each inequality.

Note:
Exercise:

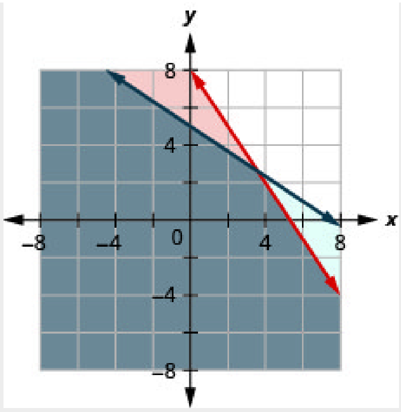
Problem:

A trailer can carry a maximum weight of 160 pounds and a maximum volume of 15 cubic feet. A microwave oven weighs 30 pounds and has 2 cubic feet of volume, while a printer weighs 20 pounds and has 3 cubic feet of space.

ⓐ Write a system of inequalities to model this situation.
ⓑ Graph the system.
ⓒ Could 4 microwaves and 2 printers be carried on this trailer?
ⓓ Could 7 microwaves and 3 printers be carried on this trailer?

Solution:

ⓐ $\begin{cases} 30m + 20p \leq 160 \\ 2m + 3p \leq 15 \end{cases}$
ⓑ



ⓒ yes
ⓓ no

Note:
Exercise:

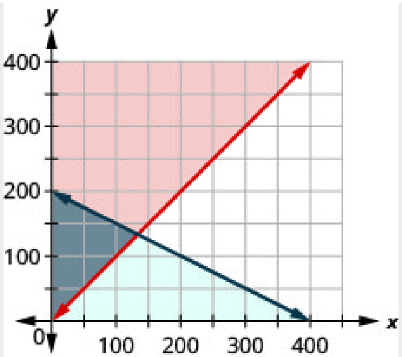
Problem:

Mary needs to purchase supplies of answer sheets and pencils for a standardized test to be given to the juniors at her high school. The number of the answer sheets needed is at least 5 more than the number of pencils. The pencils cost $2 and the answer sheets cost $1. Mary's budget for these supplies allows for a maximum cost of $400.

ⓐ Write a system of inequalities to model this situation.
ⓑ Graph the system.
ⓒ Could Mary purchase 100 pencils and 100 answer sheets?
ⓓ Could Mary purchase 150 pencils and 150 answer sheets?

Solution:

ⓐ $\begin{cases} a \geq p + 5 \\ a + 2p \leq 400 \end{cases}$
ⓑ



ⓒ no
ⓓ no

Example:
Exercise:

Problem:

Omar needs to eat at least 800 calories before going to his team practice. All he wants is hamburgers and cookies, and he doesn't want to spend more than \$5. At the hamburger restaurant near his college, each hamburger has 240 calories and costs \$1.40. Each cookie has 160 calories and costs \$0.50.

ⓐ Write a system of inequalities to model this situation.
ⓑ Graph the system.
ⓒ Could he eat 3 hamburgers and 1 cookie?
ⓓ Could he eat 2 hamburgers and 4 cookies?

Solution:
Solution

ⓐ Let $h =$ the number of hamburgers.
$c =$ the number of cookies
To find the system of inequalities, translate the information.
The calories from hamburgers at 240 calories each, plus the calories from cookies at 160 calories each must be more that 800.
Equation:

$$240h + 160c \geq 800$$

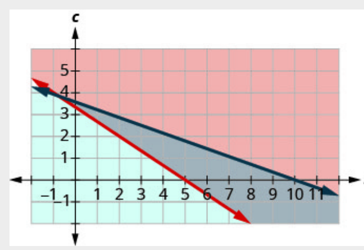
The amount spent on hamburgers at \$1.40 each, plus the amount spent on cookies at \$0.50 each must be no more than \$5.00.

Equation:

$$1.40h + 0.50c \leq 5$$

We have our system of inequalities. $\begin{cases} 240h + 160c \geq 800 \\ 1.40h + 0.50c \leq 5 \end{cases}$

ⓑ

To graph $240h + 160c \geq 800$ graph $240h + 160c = 800$ as a solid line. Choose (0, 0) as a test point. it does not make the inequality true. So, shade (red) the side that does not include the point (0, 0). To graph $1.40h + 0.50c \leq 5$, graph $1.40h + 0.50c = 5$ as a solid line. Choose (0,0) as a test point. It makes the inequality true. So, shade (blue) the side that includes the point.	

The solution of the system is the region of the graph that is double shaded and so is shaded darker.

ⓒ To determine if 3 hamburgers and 2 cookies would meet Omar's criteria, we see if the point (3, 1) is in the solution region. It is. He might choose to eat 3 hamburgers and 2 cookies.
ⓓ To determine if 2 hamburgers and 4 cookies would meet Omar's criteria, we see if the point (2, 4) is in the solution region. It is. He might choose to eat 2 hamburgers and 4 cookies.

We could also test the possible solutions by substituting the values into each inequality.

Note:
Exercise:

Problem:

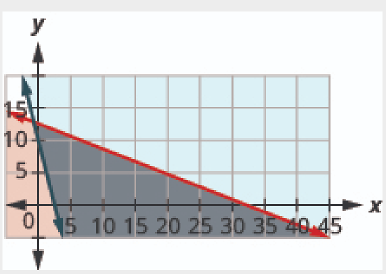
Tension needs to eat at least an extra 1,000 calories a day to prepare for running a marathon. He has only $25 to spend on the extra food he needs and will spend it on $0.75 donuts which have 360 calories each and $2 energy drinks which have 110 calories.

ⓐ Write a system of inequalities that models this situation.
ⓑ Graph the system.
ⓒ Can he buy 8 donuts and 4 energy drinks?
ⓓ Can he buy 1 donut and 3 energy drinks?

Solution:

ⓐ $\begin{cases} 0.75d + 2e \le 25 \\ 360d + 110e \ge 1000 \end{cases}$

ⓑ



ⓒ yes
ⓓ no

Note:
Exercise:

Problem:

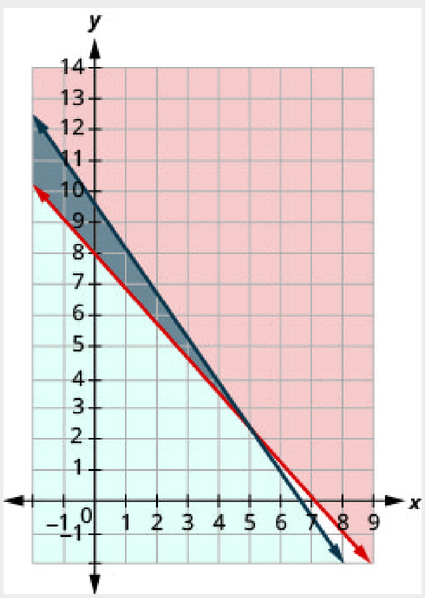
Philip's doctor tells him he should add at least 1000 more calories per day to his usual diet. Philip wants to buy protein bars that cost \$1.80 each and have 140 calories and juice that costs \$1.25 per bottle and have 125 calories. He doesn't want to spend more than \$12.

ⓐ Write a system of inequalities that models this situation.
ⓑ Graph the system.
ⓒ Can he buy 3 protein bars and 5 bottles of juice?
ⓓ Can he buy 5 protein bars and 3 bottles of juice?

Solution:

ⓐ $\begin{cases} 140p + 125j \geq 1000 \\ 1.80p + 1.25j \leq 12 \end{cases}$

ⓑ



ⓒ yes
ⓓ no

Note:
Access these online resources for additional instruction and practice with graphing systems of linear inequalities.

- Graphical System of Inequalities
- Systems of Inequalities
- Solving Systems of Linear Inequalities by Graphing

Key Concepts

- **To Solve a System of Linear Inequalities by Graphing**

  Graph the first inequality.
  - Graph the boundary line.
  - Shade in the side of the boundary line where the inequality is true.

  On the same grid, graph the second inequality.
  - Graph the boundary line.
  - Shade in the side of that boundary line where the inequality is true.

  The solution is the region where the shading overlaps.
  Check by choosing a test point.

Section Exercises

Practice Makes Perfect

Determine Whether an Ordered Pair is a Solution of a System of Linear Inequalities

In the following exercises, determine whether each ordered pair is a solution to the system.
Exercise:

Problem: $\begin{cases} 3x + y > 5 \\ 2x - y \leq 10 \end{cases}$

ⓐ $(3, -3)$ ⓑ $(7, 1)$

Solution:

ⓐ true ⓑ false

Exercise:

Problem: $\begin{cases} 4x - y < 10 \\ -2x + 2y > -8 \end{cases}$

ⓐ $(5, -2)$ ⓑ $(-1, 3)$

Exercise:

Problem: $\begin{cases} y > \frac{2}{3}x - 5 \\ x + \frac{1}{2}y \leq 4 \end{cases}$

ⓐ $(6, -4)$ ⓑ $(3, 0)$

Solution:

ⓐ false ⓑ true

Exercise:

Problem: $\begin{cases} y < \frac{3}{2}x + 3 \\ \frac{3}{4}x - 2y < 5 \end{cases}$

ⓐ $(-4, -1)$ ⓑ $(8, 3)$

Exercise:

Problem: $\begin{cases} 7x + 2y > 14 \\ 5x - y \leq 8 \end{cases}$

ⓐ $(2, 3)$ ⓑ $(7, -1)$

Solution:

ⓐtrue ⓑfalse

Exercise:

Problem: $\begin{cases} 6x - 5y < 20 \\ -2x + 7y > -8 \end{cases}$

ⓐ $(1, -3)$ ⓑ $(-4, 4)$

Exercise:

Problem: $\begin{cases} 2x + 3y \geq 2 \\ 4x - 6y < -1 \end{cases}$

ⓐ $\left(\frac{3}{2}, \frac{4}{3}\right)$ ⓑ $\left(\frac{1}{4}, \frac{7}{6}\right)$

Solution:

ⓐ true ⓑtrue

Exercise:

Problem: $\begin{cases} 5x - 3y < -2 \\ 10x + 6y > 4 \end{cases}$

ⓐ $\left(\frac{1}{5}, \frac{2}{3}\right)$ ⓑ $\left(-\frac{3}{10}, \frac{7}{6}\right)$

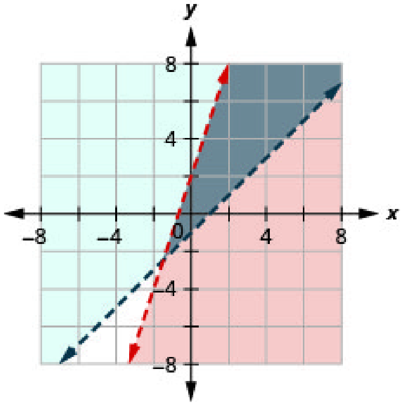
Solve a System of Linear Inequalities by Graphing

In the following exercises, solve each system by graphing.

Exercise:

Problem: $\begin{cases} y \leq 3x + 2 \\ y > x - 1 \end{cases}$

Solution:



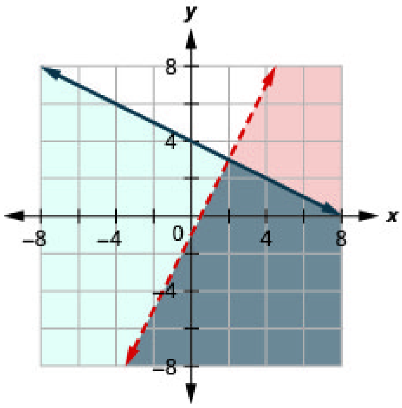
Exercise:

Problem: $\begin{cases} y < -2x + 2 \\ y \geq -x - 1 \end{cases}$

Exercise:

Problem: $\begin{cases} y < 2x - 1 \\ y \leq -\frac{1}{2}x + 4 \end{cases}$

Solution:



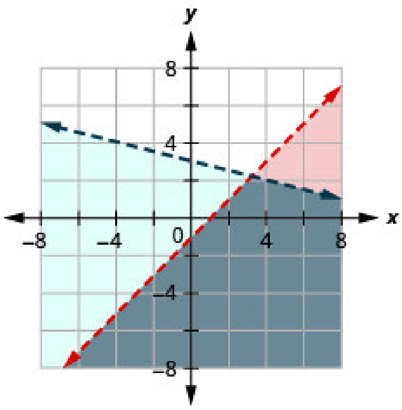
Exercise:

Problem: $\begin{cases} y \geq -\frac{2}{3}x + 2 \\ y > 2x - 3 \end{cases}$

Exercise:

Problem: $\begin{cases} x - y > 1 \\ y < -\frac{1}{4}x + 3 \end{cases}$

Solution:



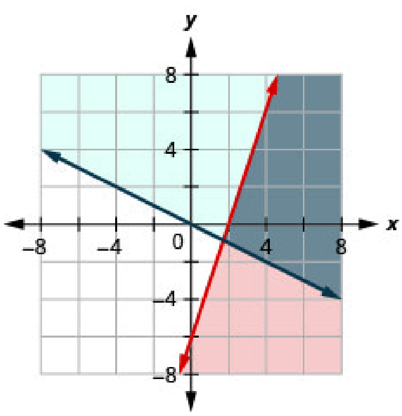
Exercise:

Problem: $\begin{cases} x + 2y < 4 \\ y < x - 2 \end{cases}$

Exercise:

Problem: $\begin{cases} 3x - y \leq 6 \\ y \geq -\frac{1}{2}x \end{cases}$

Solution:



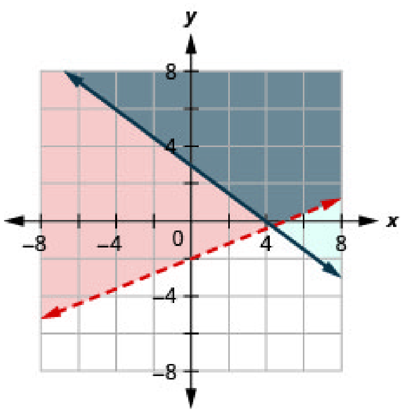
Exercise:

Problem: $\begin{cases} 2x + 4y \geq 8 \\ y \leq \frac{3}{4}x \end{cases}$

Exercise:

Problem: $\begin{cases} 2x - 5y < 10 \\ 3x + 4y \geq 12 \end{cases}$

Solution:



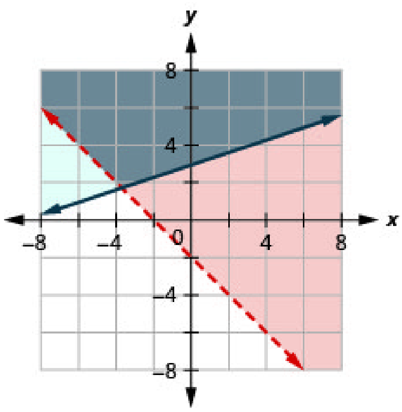
Exercise:

Problem: $\begin{cases} 3x - 2y \leq 6 \\ -4x - 2y > 8 \end{cases}$

Exercise:

Problem: $\begin{cases} 2x + 2y > -4 \\ -x + 3y \geq 9 \end{cases}$

Solution:



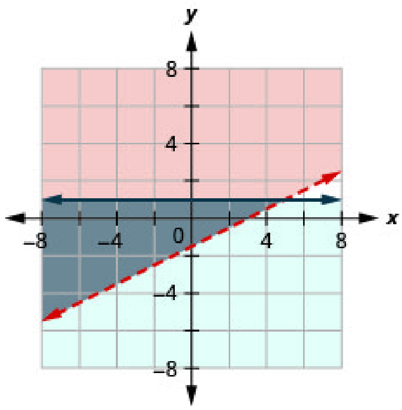
Exercise:

Problem: $\begin{cases} 2x + y > -6 \\ -x + 2y \geq -4 \end{cases}$

Exercise:

Problem: $\begin{cases} x - 2y < 3 \\ y \leq 1 \end{cases}$

Solution:



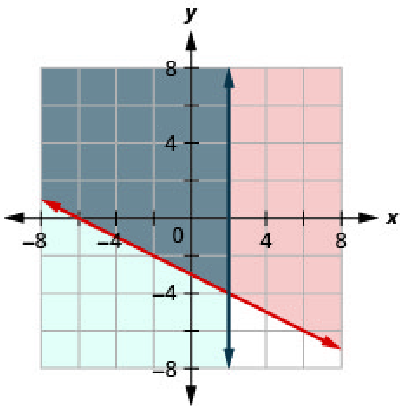
Exercise:

Problem: $\begin{cases} x - 3y > 4 \\ y \leq -1 \end{cases}$

Exercise:

Problem: $\begin{cases} y \geq -\frac{1}{2}x - 3 \\ x \leq 2 \end{cases}$

Solution:



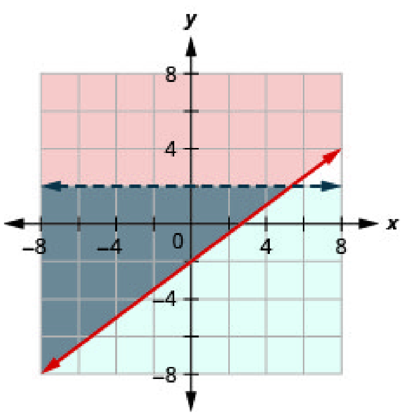
Exercise:

Problem: $\begin{cases} y \leq -\frac{2}{3}x + 5 \\ x \geq 3 \end{cases}$

Exercise:

Problem: $\begin{cases} y \geq \frac{3}{4}x - 2 \\ y < 2 \end{cases}$

Solution:



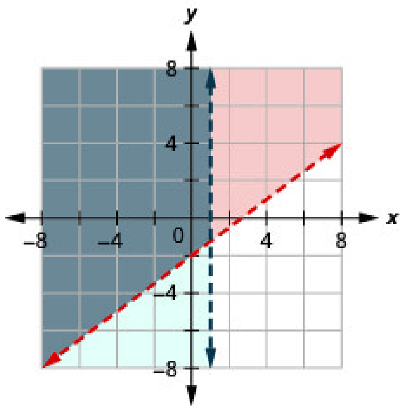
Exercise:

Problem: $\begin{cases} y \leq -\frac{1}{2}x + 3 \\ y < 1 \end{cases}$

Exercise:

Problem: $\begin{cases} 3x - 4y < 8 \\ x < 1 \end{cases}$

Solution:



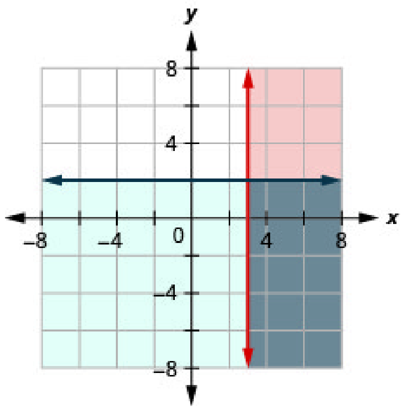
Exercise:

Problem: $\begin{cases} -3x + 5y > 10 \\ x > -1 \end{cases}$

Exercise:

Problem: $\begin{cases} x \geq 3 \\ y \leq 2 \end{cases}$

Solution:



Exercise:

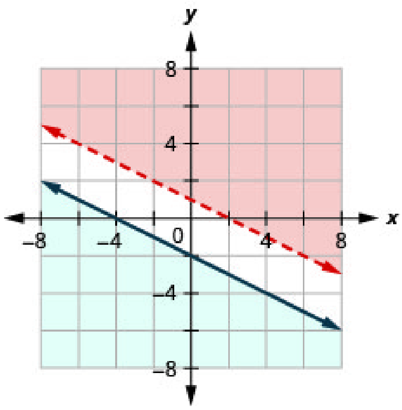
Problem: $\begin{cases} x \leq -1 \\ y \geq 3 \end{cases}$

Exercise:

Problem: $\begin{cases} 2x + 4y > 4 \\ y \leq -\frac{1}{2}x - 2 \end{cases}$

Solution:

No solution



Exercise:

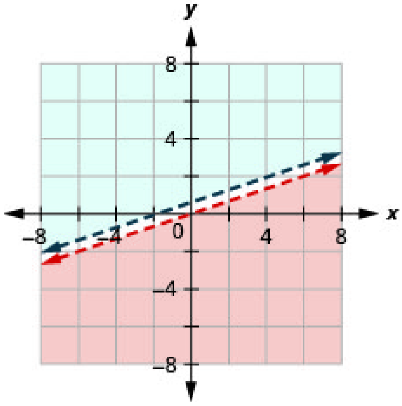
Problem: $\begin{cases} x - 3y \geq 6 \\ y > \frac{1}{3}x + 1 \end{cases}$

Exercise:

Problem: $\begin{cases} -2x + 6y < 0 \\ 6y > 2x + 4 \end{cases}$

Solution:

No solution



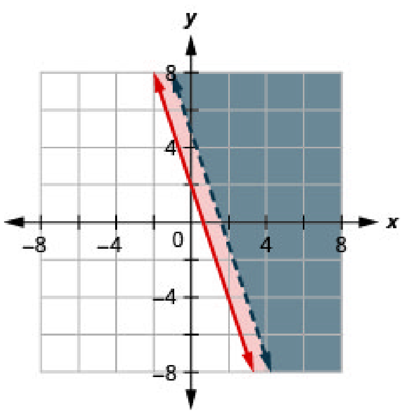
Exercise:

Problem: $\begin{cases} -3x+6y>12 \\ 4y \le 2x-4 \end{cases}$

Exercise:

Problem: $\begin{cases} y \ge -3x+2 \\ 3x+y>5 \end{cases}$

Solution:



Exercise:

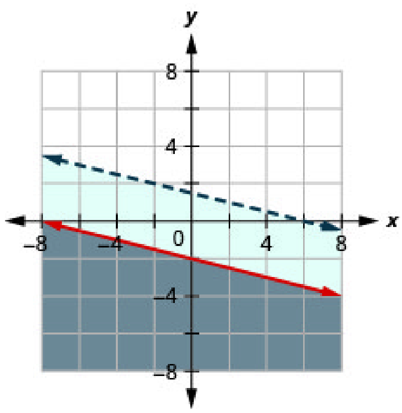
Problem: $\begin{cases} y \ge \frac{1}{2}x-1 \\ -2x+4y \ge 4 \end{cases}$

Exercise:

Problem: $\begin{cases} y \leq -\frac{1}{4}x - 2 \\ x + 4y < 6 \end{cases}$

Solution:

$x + 4y < 6$



Exercise:

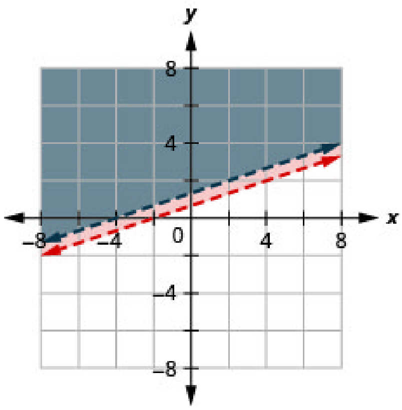
Problem: $\begin{cases} y \geq 3x - 1 \\ -3x + y > -4 \end{cases}$

Exercise:

Problem: $\begin{cases} 3y > x + 2 \\ -2x + 6y > 8 \end{cases}$

Solution:

$-2x + 6y > 8$



Exercise:

Problem: $\begin{cases} y < \frac{3}{4}x - 2 \\ -3x + 4y < 7 \end{cases}$

Solve Applications of Systems of Inequalities

In the following exercises, translate to a system of inequalities and solve.

Exercise:

Problem:

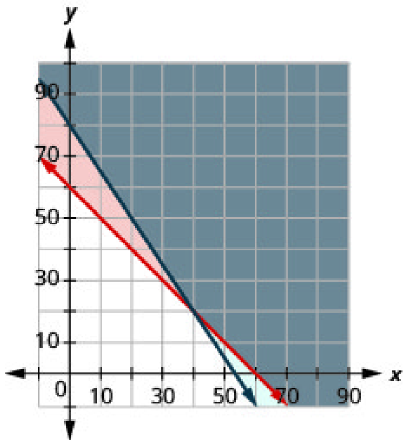
Caitlyn sells her drawings at the county fair. She wants to sell at least 60 drawings and has portraits and landscapes. She sells the portraits for $15 and the landscapes for $10. She needs to sell at least $800 worth of drawings in order to earn a profit.

ⓐ Write a system of inequalities to model this situation.
ⓑ Graph the system.
ⓒ Will she make a profit if she sells 20 portraits and 35 landscapes?
ⓓ Will she make a profit if she sells 50 portraits and 20 landscapes?

Solution:

ⓐ $\begin{cases} p + l \geq 60 \\ 15p + 10l \geq 800 \end{cases}$

ⓑ



ⓒ No
ⓓ Yes

Exercise:

Problem:

Jake does not want to spend more than \$50 on bags of fertilizer and peat moss for his garden. Fertilizer costs \$2 a bag and peat moss costs \$5 a bag. Jake’s van can hold at most 20 bags.

ⓐ Write a system of inequalities to model this situation.
ⓑ Graph the system.
ⓒ Can he buy 15 bags of fertilizer and 4 bags of peat moss?
ⓓ Can he buy 10 bags of fertilizer and 10 bags of peat moss?

Exercise:

Problem:

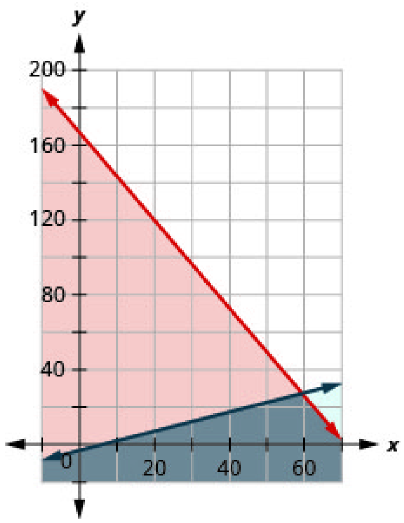
Reiko needs to mail her Christmas cards and packages and wants to keep her mailing costs to no more than \$500. The number of cards is at least 4 more than twice the number of packages. The cost of mailing a card (with pictures enclosed) is \$3 and for a package the cost is \$7.

ⓐ Write a system of inequalities to model this situation.
ⓑ Graph the system.
ⓒ Can she mail 60 cards and 26 packages?
ⓓ Can she mail 90 cards and 40 packages?

Solution:

ⓐ $\begin{cases} 7p + 3c \leq 500 \\ p \geq 2c + 4 \end{cases}$

ⓑ



ⓒ Yes
ⓓ No

Exercise:

Problem:

Juan is studying for his final exams in Chemistry and Algebra. He knows he only has 24 hours to study, and it will take him at least three times as long to study for Algebra than Chemistry.

ⓐ Write a system of inequalities to model this situation.
ⓑ Graph the system.
ⓒ Can he spend 4 hours on Chemistry and 20 hours on Algebra?
ⓓ Can he spend 6 hours on Chemistry and 18 hours on Algebra?

Exercise:

Problem:

Jocelyn is pregnant and needs to eat at least 500 more calories a day than usual. When buying groceries one day with a budget of $15 for the extra food, she buys bananas that have 90 calories each and chocolate granola bars that have 150 calories each. The bananas cost $0.35 each and the granola bars cost $2.50 each.

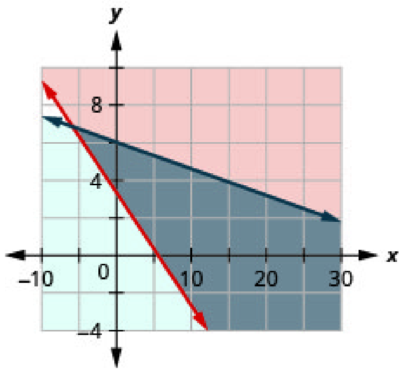
ⓐ Write a system of inequalities to model this situation.
ⓑ Graph the system.

ⓒ Could she buy 5 bananas and 6 granola bars?
ⓓ Could she buy 3 bananas and 4 granola bars?

Solution:

ⓐ $\begin{cases} 90b + 150g \geq 500 \\ 0.35b + 2.50g \leq 15 \end{cases}$

ⓑ



ⓒ No
ⓓ Yes

Exercise:

Problem:

Mark is attempting to build muscle mass and so he needs to eat at least an additional 80 grams of protein a day. A bottle of protein water costs $3.20 and a protein bar costs $1.75. The protein water supplies 27 grams of protein and the bar supplies 16 gram. If he has $ 10 dollars to spend

ⓐ Write a system of inequalities to model this situation.
ⓑ Graph the system.
ⓒ Could he buy 3 bottles of protein water and 1 protein bar?
ⓓ Could he buy no bottles of protein water and 5 protein bars?

Exercise:

Problem:

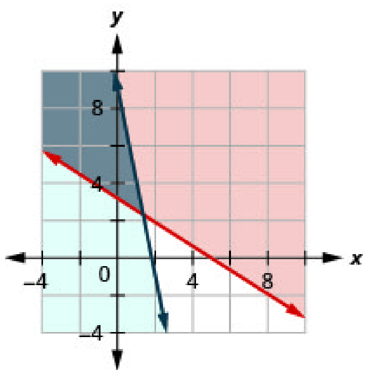
Jocelyn desires to increase both her protein consumption and caloric intake. She desires to have at least 35 more grams of protein each day and no more than an additional 200 calories daily. An ounce of cheddar cheese has 7 grams of protein and 110 calories. An ounce of parmesan cheese has 11 grams of protein and 22 calories.

ⓐ Write a system of inequalities to model this situation.
ⓑ Graph the system.
ⓒ Could she eat 1 ounce of cheddar cheese and 3 ounces of parmesan cheese?
ⓓ Could she eat 2 ounces of cheddar cheese and 1 ounce of parmesan cheese?

Solution:

ⓐ $\begin{cases} 7c + 11p \geq 35 \\ 110\text{c} + 22p \leq 200 \end{cases}$

ⓑ



ⓒ Yes
ⓓ No

Exercise:

Problem:

Mark is increasing his exercise routine by running and walking at least 4 miles each day. His goal is to burn a minimum of 1,500 calories from this exercise. Walking burns 270 calories/mile and running burns 650 calories.

ⓐ Write a system of inequalities to model this situation.
ⓑ Graph the system.
ⓒ Could he meet his goal by walking 3 miles and running 1 mile?
ⓓ Could he meet his goal by walking 2 miles and running 2 mile?

Everyday Math

Exercise:

Problem:

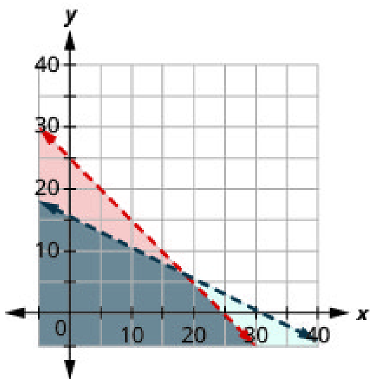
Tickets for an American Baseball League game for 3 adults and 3 children cost less than $75, while tickets for 2 adults and 4 children cost less than $62.

ⓐ Write a system of inequalities to model this problem.
ⓑ Graph the system.
ⓒ Could the tickets cost $20 for adults and $8 for children?
ⓓ Could the tickets cost $15 for adults and $5 for children?

Solution:

ⓐ $\begin{cases} 3a + 3c < 75 \\ 2\text{a} + 4c < 62 \end{cases}$

ⓑ



ⓒ No
ⓓ Yes

Exercise:

Problem:

Grandpa and Grandma are treating their family to the movies. Matinee tickets cost $4 per child and $4 per adult. Evening tickets cost $6 per child and $8 per adult. They plan on spending no more than $80 on the matinee tickets and no more than $100 on the evening tickets.

ⓐ Write a system of inequalities to model this situation.
ⓑ Graph the system.
ⓒ Could they take 9 children and 4 adults to both shows?
ⓓ Could they take 8 children and 5 adults to both shows?

Writing Exercises

Exercise:

Problem:

Graph the inequality $x - y \geq 3$. How do you know which side of the line $x - y = 3$ should be shaded?

Solution:

Answers will vary.

Exercise:

Problem: Graph the system $\begin{cases} x + 2y \leq 6 \\ y \geq -\frac{1}{2}x - 4 \end{cases}$. What does the solution mean?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
determine whether an ordered pair is a solution of a system of linear inequalities.			
solve a system of linear inequalities by graphing.			
solve applications of systems of inequalities.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

Chapter Review Exercises

Solve Systems of Equations by Graphing

Determine Whether an Ordered Pair is a Solution of a System of Equations.

In the following exercises, determine if the following points are solutions to the given system of equations.
Exercise:

Problem: $\begin{cases} x+3y=-9 \\ 2x-4y=12 \end{cases}$

ⓐ $(-3,-2)$ ⓑ $(0,-3)$

Solution:

ⓐ no ⓑ yes

Exercise:

Problem: $\begin{cases} x+y=8 \\ y=x-4 \end{cases}$

ⓐ $(6,2)$ ⓑ $(9,-1)$

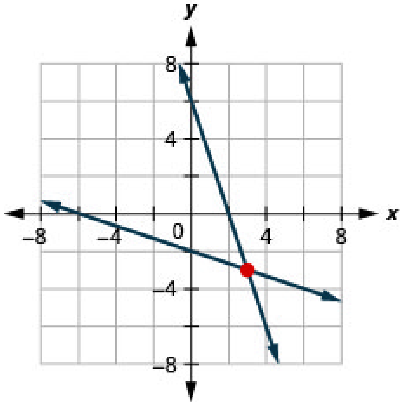
Solve a System of Linear Equations by Graphing

In the following exercises, solve the following systems of equations by graphing.
Exercise:

Problem: $\begin{cases} 3x+y=6 \\ x+3y=-6 \end{cases}$

Solution:

$(3,-3)$



Exercise:

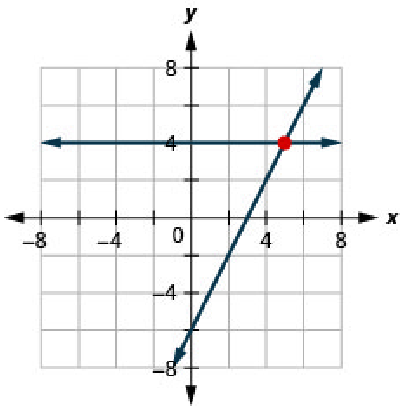
Problem: $\begin{cases} y = x - 2 \\ y = -2x - 2 \end{cases}$

Exercise:

Problem: $\begin{cases} 2x - y = 6 \\ y = 4 \end{cases}$

Solution:

$(5, 4)$



Exercise:

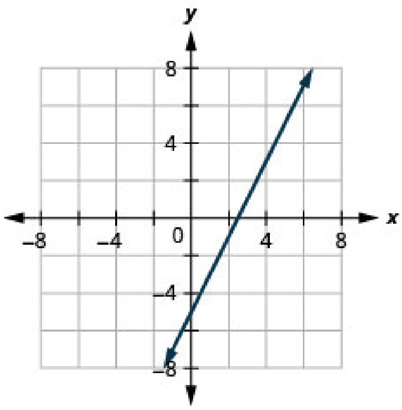
Problem: $\begin{cases} x + 4y = -1 \\ x = 3 \end{cases}$

Exercise:

Problem: $\begin{cases} 2x - y = 5 \\ 4x - 2y = 10 \end{cases}$

Solution:

coincident lines



Exercise:

Problem: $\begin{cases} -x+2y=4 \\ y=\frac{1}{2}x-3 \end{cases}$

Determine the Number of Solutions of a Linear System

In the following exercises, without graphing determine the number of solutions and then classify the system of equations.

Exercise:

Problem: $\begin{cases} y=\frac{2}{5}x+2 \\ -2x+5y=10 \end{cases}$

Solution:

infinitely many solutions, consistent system, dependent equations

Exercise:

Problem: $\begin{cases} 3x+2y=6 \\ y=-3x+4 \end{cases}$

Exercise:

Problem: $\begin{cases} 5x-4y=0 \\ y=\frac{5}{4}x-5 \end{cases}$

Solution:

no solutions, inconsistent system, independent equations

Exercise:

Problem: $\begin{cases} y = -\frac{3}{4}x + 1 \\ 6x + 8y = 8 \end{cases}$

Solve Applications of Systems of Equations by Graphing

Exercise:

Problem:

LaVelle is making a pitcher of caffe mocha. For each ounce of chocolate syrup, she uses five ounces of coffee. How many ounces of chocolate syrup and how many ounces of coffee does she need to make 48 ounces of caffe mocha?

Solution:

LaVelle needs 8 ounces of chocolate syrup and 40 ounces of coffee.

Exercise:

Problem:

Eli is making a party mix that contains pretzels and chex. For each cup of pretzels, he uses three cups of chex. How many cups of pretzels and how many cups of chex does he need to make 12 cups of party mix?

Solve Systems of Equations by Substitution

Solve a System of Equations by Substitution

In the following exercises, solve the systems of equations by substitution.

Exercise:

Problem: $\begin{cases} 3x - y = -5 \\ y = 2x + 4 \end{cases}$

Solution:

$(-1, 2)$

Exercise:

Problem: $\begin{cases} 3x - 2y = 2 \\ y = \frac{1}{2}x + 3 \end{cases}$

Exercise:

Problem: $\begin{cases} x - y = 0 \\ 2x + 5y = -14 \end{cases}$

Solution:

$(-2, -2)$

Exercise:

Problem: $\begin{cases} y = -2x + 7 \\ y = \frac{2}{3}x - 1 \end{cases}$

Exercise:

Problem: $\begin{cases} y = -5x \\ 5x + y = 6 \end{cases}$

Solution:

no solution

Exercise:

Problem: $\begin{cases} y = -\frac{1}{3}x + 2 \\ x + 3y = 6 \end{cases}$

Solve Applications of Systems of Equations by Substitution

In the following exercises, translate to a system of equations and solve.

Exercise:

Problem:

The sum of two number is 55. One number is 11 less than the other. Find the numbers.

Solution:

The numbers are 22 and 33.

Exercise:

Problem:

The perimeter of a rectangle is 128. The length is 16 more than the width. Find the length and width.

Exercise:

Problem:

The measure of one of the small angles of a right triangle is 2 less than 3 times the measure of the other small angle. Find the measure of both angles.

Solution:

The measures are 23 degrees and 67 degrees.

Exercise:

Problem:

Gabriela works for an insurance company that pays her a salary of $32,000 plus a commission of $100 for each policy she sells. She is considering changing jobs to a company that would pay a salary of $40,000 plus a commission of $80 for each policy sold. How many policies would Gabriela need to sell to make the total pay the same?

Solve Systems of Equations by Elimination

Solve a System of Equations by Elimination In the following exercises, solve the systems of equations by elimination.

Exercise:

Problem: $\begin{cases} x + y = 12 \\ x - y = -10 \end{cases}$

Solution:

$(1, 11)$

Exercise:

Problem: $\begin{cases} 4x+2y=2 \\ -4x-3y=-9 \end{cases}$

Exercise:

Problem: $\begin{cases} 3x-8y=20 \\ x+3y=1 \end{cases}$

Solution:

$(4,-1)$

Exercise:

Problem: $\begin{cases} 3x-2y=6 \\ 4x+3y=8 \end{cases}$

Exercise:

Problem: $\begin{cases} 9x+4y=2 \\ 5x+3y=5 \end{cases}$

Solution:

$(-2,5)$

Exercise:

Problem: $\begin{cases} -x+3y=8 \\ 2x-6y=-20 \end{cases}$

Solve Applications of Systems of Equations by Elimination

In the following exercises, translate to a system of equations and solve.

Exercise:

Problem:

The sum of two numbers is -90. Their difference is 16. Find the numbers.

Solution:

The numbers are -37 and -53.

Exercise:

Problem:

Omar stops at a donut shop every day on his way to work. Last week he had 8 donuts and 5 cappuccinos, which gave him a total of 3,000 calories. This week he had 6 donuts and 3 cappuccinos, which was a total of 2,160 calories. How many calories are in one donut? How many calories are in one cappuccino?

Choose the Most Convenient Method to Solve a System of Linear Equations

In the following exercises, decide whether it would be more convenient to solve the system of equations by substitution or elimination.

Exercise:

Problem: $\begin{cases} 6x - 5y = 27 \\ 3x + 10y = -24 \end{cases}$

Solution:

elimination

Exercise:

Problem: $\begin{cases} y = 3x - 9 \\ 4x - 5y = 23 \end{cases}$

Solve Applications with Systems of Equations

Translate to a System of Equations

In the following exercises, translate to a system of equations. Do not solve the system.

Exercise:

Problem:

The sum of two numbers is -32. One number is two less than twice the other. Find the numbers.

Solution:

$$\begin{cases} x + y = -32 \\ x = 2y - 2 \end{cases}$$

Exercise:

Problem:

Four times a number plus three times a second number is -9. Twice the first number plus the second number is three. Find the numbers.

Exercise:

Problem:

Last month Jim and Debbie earned \$7,200. Debbie earned \$1,600 more than Jim earned. How much did they each earn?

Solution:

$$\begin{cases} j + d = 7200 \\ d = j + 1600 \end{cases}$$

Exercise:

Problem:

Henri has \$24,000 invested in stocks and bonds. The amount in stocks is \$6,000 more than three times the amount in bonds. How much is each investment?

Solve Direct Translation Applications

In the following exercises, translate to a system of equations and solve.

Exercise:

Problem:

Pam is 3 years older than her sister, Jan. The sum of their ages is 99. Find their ages.

Solution:

Pam is 51 and Jan is 48.

Exercise:

Problem:

Mollie wants to plant 200 bulbs in her garden. She wantsall irises and tulips. She wants to plant three times as many tulips as irises. How many irises and how many tulips should she plant?

Solve Geometry Applications

In the following exercises, translate to a system of equations and solve.

Exercise:

Problem:

The difference of two supplementary angles is 58 degrees. Find the measures of the angles.

Solution:

The measures are 119 degrees and 61 degrees.

Exercise:

Problem:

Two angles are complementary. The measure of the larger angle is five more than four times the measure of the smaller angle. Find the measures of both angles.

Exercise:

Problem:

Becca is hanging a 28 foot floral garland on the two sides and top of a pergola to prepare for a wedding. The height is four feet less than the width. Find the height and width of the pergola.

Solution:

The pergola is 8 feet high and 12 feet wide.

Exercise:

Problem:

The perimeter of a city rectangular park is 1428 feet. The length is 78 feet more than twice the width. Find the length and width of the park.

Solve Uniform Motion Applications

In the following exercises, translate to a system of equations and solve.

Exercise:

Problem:

Sheila and Lenore were driving to their grandmother's house. Lenore left one hour after Sheila. Sheila drove at a rate of 45 mph, and Lenore drove at a rate of 60 mph. How long will it take for Lenore to catch up to Sheila?

Solution:

It will take Lenore 3 hours.

Exercise:

Problem:

Bob left home, riding his bike at a rate of 10 miles per hour to go to the lake. Cheryl, his wife, left 45 minutes $\left(\frac{3}{4}\right.$ hour) later, driving her car at a rate of 25 miles per hour. How long will it take Cheryl to catch up to Bob?

Exercise:

Problem:

Marcus can drive his boat 36 miles down the river in three hours but takes four hours to return upstream. Find the rate of the boat in still water and the rate of the current.

Solution:

The rate of the boat is 10.5 mph. The rate of the current is 1.5 mph.

Exercise:

Problem:

A passenger jet can fly 804 miles in 2 hours with a tailwind but only 776 miles in 2 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Solve Mixture Applications with Systems of Equations

Solve Mixture Applications

In the following exercises, translate to a system of equations and solve.

Exercise:

Problem:

Lynn paid a total of $2,780 for 261 tickets to the theater. Student tickets cost $10 and adult tickets cost $15. How many student tickets and how many adult tickets did Lynn buy?

Solution:

Lynn bought 227 student tickets and 34 adult tickets.

Exercise:

Problem:

Priam has dimes and pennies in a cup holder in his car. The total value of the coins is $4.21. The number of dimes is three less than four times the number of pennies. How many dimes and how many pennies are in the cup?

Exercise:

Problem:

Yumi wants to make 12 cups of party mix using candies and nuts. Her budget requires the party mix to cost her $1.29 per cup. The candies are $2.49 per cup and the nuts are $0.69 per cup. How many cups of candies and how many cups of nuts should she use?

Solution:

Yumi should use 4 cups of candies and 8 cups of nuts.

Exercise:

Problem:

A scientist needs 70 liters of a 40% solution of alcohol. He has a 30% and a 60% solution available. How many liters of the 30% and how many liters of the 60% solutions should he mix to make the 40% solution?

Solve Interest Applications

In the following exercises, translate to a system of equations and solve.

Exercise:

Problem:

Jack has $12,000 to invest and wants to earn 7.5% interest per year. He will put some of the money into a savings account that earns 4% per year and the rest into CD account that earns 9% per year. How much money should he put into each account?

Solution:

Jack should put $3600 into savings and $8400 into the CD.

Exercise:

Problem:

When she graduates college, Linda will owe $43,000 in student loans. The interest rate on the federal loans is 4.5% and the rate on the private bank loans is 2%. The total interest she owes for one year was $1585. What is the amount of each loan?

Graphing Systems of Linear Inequalities

Determine Whether an Ordered Pair is a Solution of a System of Linear Inequalities

In the following exercises, determine whether each ordered pair is a solution to the system.

Exercise:

Problem: $\begin{cases} 4x + y > 6 \\ 3x - y \leq 12 \end{cases}$

ⓐ $(2, -1)$ ⓑ $(3, -2)$

Solution:

ⓐ yes ⓑ yes

Exercise:

Problem: $\begin{cases} y > \frac{1}{3}x + 2 \\ x - \frac{1}{4}y \leq 10 \end{cases}$

ⓐ $(6, 5)$ ⓑ $(15, 8)$

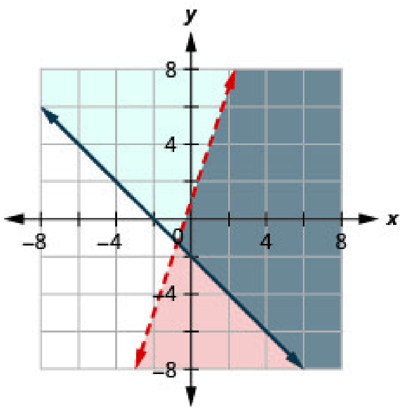
Solve a System of Linear Inequalities by Graphing

In the following exercises, solve each system by graphing.

Exercise:

Problem: $\begin{cases} y < 3x + 1 \\ y \geq -x - 2 \end{cases}$

Solution:



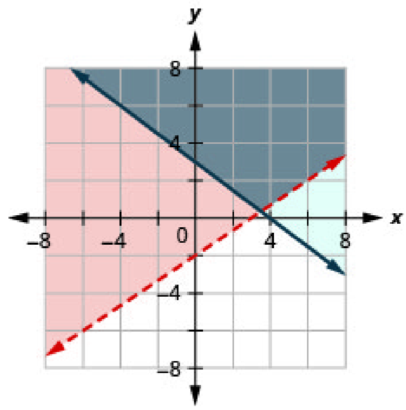
Exercise:

Problem: $\begin{cases} x - y > -1 \\ y < \frac{1}{3}x - 2 \end{cases}$

Exercise:

Problem: $\begin{cases} 2x - 3y < 6 \\ 3x + 4y \geq 12 \end{cases}$

Solution:



Exercise:

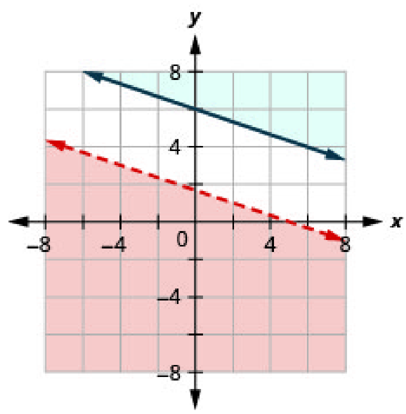
Problem: $\begin{cases} y \leq -\frac{3}{4}x + 1 \\ x \geq -5 \end{cases}$

Exercise:

Problem: $\begin{cases} x + 3y < 5 \\ y \geq -\frac{1}{3}x + 6 \end{cases}$

Solution:

No solution



Exercise:

Problem: $\begin{cases} y \geq 2x - 5 \\ -6x + 3y > -4 \end{cases}$

Solve Applications of Systems of Inequalities

In the following exercises, translate to a system of inequalities and solve.

Exercise:

Problem:

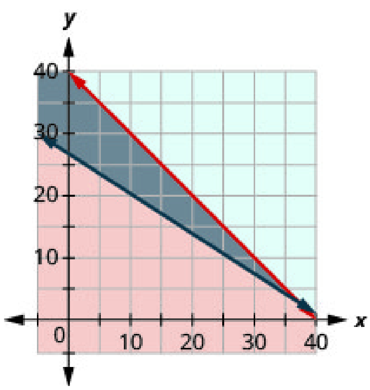
Roxana makes bracelets and necklaces and sells them at the farmers' market. She sells the bracelets for $12 each and the necklaces for $18 each. At the market next weekend she will have room to display no more than 40 pieces, and she needs to sell at least $500 worth in order to earn a profit.

ⓐ Write a system of inequalities to model this situation.
ⓑ Graph the system.
ⓒ Should she display 26 bracelets and 14 necklaces?
ⓓ Should she display 39 bracelets and 1 necklace?

Solution:

ⓐ $\begin{cases} b+n \leq 40 \\ 12b+18n \geq 500 \end{cases}$

ⓑ



ⓒ yes
ⓓ no

Exercise:

Problem:

Annie has a budget of $600 to purchase paperback books and hardcover books for her classroom. She wants the number of hardcover to be at least 5 more than three times the number of paperback books. Paperback books cost $4 each and hardcover books cost $15 each.

ⓐ Write a system of inequalities to model this situation.
ⓑ Graph the system.
ⓒ Can she buy 8 paperback books and 40 hardcover books?
ⓓ Can she buy 10 paperback books and 37 hardcover books?

Practice Test

Exercise:

Problem: $\begin{cases} x - 4y = -8 \\ 2x + 5y = 10 \end{cases}$

ⓐ $(0,2)$ ⓑ $(4,3)$

Solution:

ⓐ yes ⓑ no

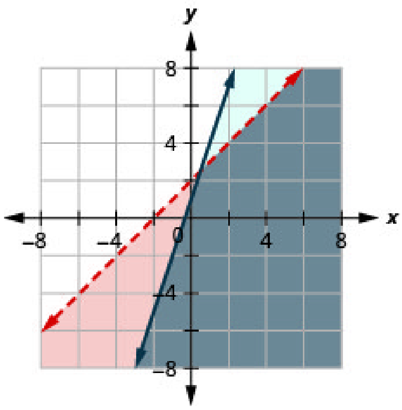
In the following exercises, solve the following systems by graphing.
Exercise:

Problem: $\begin{cases} x - y = 5 \\ x + 2y = -4 \end{cases}$

Exercise:

Problem: $\begin{cases} x - y > -2 \\ y \leq 3x + 1 \end{cases}$

Solution:



In the following exercises, solve each system of equations. Use either substitution or elimination.

Exercise:

Problem: $\begin{cases} 3x-2y=3 \\ y=2x-1 \end{cases}$

Exercise:

Problem: $\begin{cases} x+y=-3 \\ x-y=11 \end{cases}$

Solution:

$(4,-7)$

Exercise:

Problem: $\begin{cases} 4x-3y=7 \\ 5x-2y=0 \end{cases}$

Exercise:

Problem: $\begin{cases} y=-\frac{4}{5}x+1 \\ 8x+10y=10 \end{cases}$

Solution:

infinitely many solutions

Exercise:

Problem: $\begin{cases} 2x + 3y = 12 \\ -4x + 6y = -16 \end{cases}$

In the following exercises, translate to a system of equations and solve.

Exercise:

Problem:

The sum of two numbers is −24. One number is 104 less than the other. Find the numbers.

Solution:

The numbers are 40 and 64

Exercise:

Problem:

Ramon wants to plant cucumbers and tomatoes in his garden. He has room for 16 plants, and he wants to plant three times as many cucumbers as tomatoes. How many cucumbers and how many tomatoes should he plant?

Exercise:

Problem:

Two angles are complementary. The measure of the larger angle is six more than twice the measure of the smaller angle. Find the measures of both angles.

Solution:

The measures of the angles are 28 degrees and 62 degrees.

Exercise:

Problem:

On Monday, Lance ran for 30 minutes and swam for 20 minutes. His fitness app told him he had burned 610 calories. On Wednesday, the fitness app told him he burned 695 calories when he ran for 25 minutes and swam for 40 minutes. How many calories did he burn for one minute of running? How many calories did he burn for one minute of swimming?

Exercise:

Problem:

Kathy left home to walk to the mall, walking quickly at a rate of 4 miles per hour. Her sister Abby left home 15 minutes later and rode her bike to the mall at a rate of 10 miles per hour. How long will it take Abby to catch up to Kathy?

Solution:

It will take Kathy $\frac{1}{6}$ of an hour (or 10 minutes).

Exercise:

Problem:

It takes $5\frac{1}{2}$ hours for a jet to fly 2,475 miles with a headwind from San Jose, California to Lihue, Hawaii. The return flight from Lihue to San Jose with a tailwind, takes 5 hours. Find the speed of the jet in still air and the speed of the wind.

Exercise:

Problem:

Liz paid $160 for 28 tickets to take the Brownie troop to the science museum. Children’s tickets cost $5 and adult tickets cost $9. How many children’s tickets and how many adult tickets did Liz buy?

Solution:

Liz bought 23 children’s tickets and 5 adult tickets.

Exercise:

Problem:

A pharmacist needs 20 liters of a 2% saline solution. He has a 1% and a 5% solution available. How many liters of the 1% and how many liters of the 5% solutions should she mix to make the 2% solution?

Exercise:

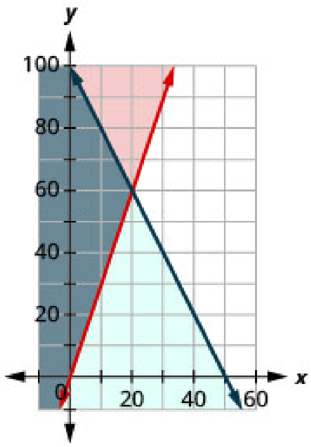
Problem: Translate to a system of inequalities and solve.

Andi wants to spend no more than $50 on Halloween treats. She wants to buy candy bars that cost $1 each and lollipops that cost $0.50 each, and she wants the number of lollipops to be at least three times the number of candy bars.

ⓐ Write a system of inequalities to model this situation.
ⓑ Graph the system.
ⓒ Can she buy 20 candy bars and 70 lollipops?
ⓓ Can she buy 15 candy bars and 65 lollipops?

Solution:

ⓐ $\begin{cases} C + 0.5L \leq 50 \\ L \geq 3C \end{cases}$
ⓑ



ⓒ No
ⓓ Yes

Glossary

system of linear inequalities
: Two or more linear inequalities grouped together form a system of linear inequalities.

Introduction
class="introduction"

Architects use polynomials to design curved shapes such as this suspension bridge, the Silver Jubilee bridge in Halton, England.

We have seen that the graphs of linear equations are straight lines. Graphs of other types of equations, called polynomial equations, are curves, like the outline of this suspension bridge. Architects use polynomials to design the shape of a bridge like this and to draw the blueprints for it. Engineers use polynomials to calculate the stress on the bridge's supports to ensure they

are strong enough for the intended load. In this chapter, you will explore operations with and properties of polynomials.

Add and Subtract Polynomials: ASE
By the end of this section, you will be able to:

- Identify polynomials, monomials, binomials, and trinomials
- Determine the degree of polynomials
- Add and subtract monomials
- Add and subtract polynomials
- Evaluate a polynomial for a given value

Identify Polynomials, Monomials, Binomials and Trinomials

You have learned that a *term* is a constant or the product of a constant and one or more variables. When it is of the form ax^m, where a is a constant and m is a whole number, it is called a monomial. Some examples of monomial are $8, -2x^2, 4y^3, \text{ and } 11z^7$.

Note:
Monomials
A **monomial** is a term of the form ax^m, where a is a constant and m is a positive whole number.

A monomial, or two or more monomials combined by addition or subtraction, is a polynomial. Some polynomials have special names, based on the number of terms. A monomial is a polynomial with exactly one term. A binomial has exactly two terms, and a trinomial has exactly three terms. There are no special names for polynomials with more than three terms.

Note:
Polynomials
polynomial—A monomial, or two or more monomials combined by addition or subtraction, is a polynomial.

- **monomial**—A polynomial with exactly one term is called a monomial.
- **binomial**—A polynomial with exactly two terms is called a binomial.
- **trinomial**—A polynomial with exactly three terms is called a trinomial.

Here are some examples of polynomials.

Polynomial	$b+1$	$4y^2-7y+2$	$4x^4+x^3+8x^2-9x+1$	
Monomial	14	$8y^2$	$-9x^3y^5$	$-$
Binomial	$a+7$	$4b-5$	y^2-16	$3a$
Trinomial	$x^2-7x+12$	$9y^2+2y-8$	$6m^4-m^3+8m$	z^4

Notice that every monomial, binomial, and trinomial is also a polynomial. They are just special members of the “family” of polynomials and so they have special names. We use the words *monomial*, *binomial*, and *trinomial* when referring to these special polynomials and just call all the rest *polynomials*.

Example:
Exercise:

Problem: Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial.

ⓐ $4y^2 - 8y - 6$
ⓑ $-5a^4b^2$
ⓒ $2x^5 - 5x^3 - 9x^2 + 3x + 4$
ⓓ $13 - 5m^3$
ⓔ q

Solution:
Solution

	Polynomial	Number of terms	Type
ⓐ	$4y^2 - 8y - 6$	3	Trinomial
ⓑ	$-5a^4b^2$	1	Monomial
ⓒ	$2x^5 - 5x^3 - 9x^2 + 3x + 4$	5	Polynomial
ⓓ	$13 - 5m^3$	2	Binomial
ⓔ	q	1	Monomial

Note:
Exercise:

Problem: Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial:

ⓐ $5b$ ⓑ $8y^3 - 7y^2 - y - 3$ ⓒ $-3x^2 - 5x + 9$ ⓓ $81 - 4a^2$ ⓔ $-5x^6$

Solution:

ⓐ monomial ⓑ polynomial ⓒ trinomial ⓓ binomial ⓔ monomial

Note:
Exercise:

Problem: Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial:

ⓐ $27z^3 - 8$ ⓑ $12m^3 - 5m^2 - 2m$ ⓒ $\frac{5}{6}$ ⓓ $8x^4 - 7x^2 - 6x - 5$ ⓔ $-n^4$

Solution:

ⓐ binomial ⓑ trinomial ⓒ monomial ⓓ polynomial ⓔ monomial

Determine the Degree of Polynomials

The degree of a polynomial and the degree of its terms are determined by the exponents of the variable.

A monomial that has no variable, just a constant, is a special case. The degree of a constant is 0—it has no variable.

Note:
Degree of a Polynomial
The **degree of a term** is the sum of the exponents of its variables.
The **degree of a constant** is 0.
The **degree of a polynomial** is the highest degree of all its terms.

Let’s see how this works by looking at several polynomials. We’ll take it step by step, starting with monomials, and then progressing to polynomials with more terms.

Monomial	14	$8y^2$	$-9x^3y^5$	$-13a$
Degree	0	2	8	1
Binomial	$a+7$	$4b^2-5b$	x^2y^2-16	$3n^3-9n^2$
Degree of each term	0 1	2 1	4 0	3 2
Degree of polynomial	1	2	4	3
Trinomial	$x^2-7x+12$	$9a^2+6ab+b^2$	$6m^4-m^3n^2+8mn^5$	z^4+3z^2-1
Degree of each term	2 1 0	2 2 2	4 5 6	4 2 0
Degree of polynomial	2	2	6	4
Polynomial	$b+1$	$4y^2-7y+2$	$4x^4+x^3+8x^2-9x+1$	
Degree of each term	1 0	2 1 0	4 3 2 1 0	
Degree of polynomial	1	2	4	

A polynomial is in **standard form** when the terms of a polynomial are written in descending order of degrees. Get in the habit of writing the term with the highest degree first.

Example:
Exercise:

Problem: Find the degree of the following polynomials.

ⓐ $10y$
ⓑ $4x^3-7x+5$
ⓒ -15

ⓓ $-8b^2+9b-2$
ⓔ $8xy^2+2y$

Solution:
Solution

ⓐ The exponent of y is one. $y=y^1$	$10y$ The degree is 1.
ⓑ The highest degree of all the terms is 3.	$4x^3-7x+5$ The degree is 3.
ⓒ The degree of a constant is 0.	-15 The degree is 0.
ⓓ The highest degree of all the terms is 2.	$-8b^2+9b-2$ The degree is 2.
ⓔ The highest degree of all the terms is 3.	$8xy^2+2y$ The degree is 3.

Note:
Exercise:

Problem: Find the degree of the following polynomials:

ⓐ $-15b$ ⓑ $10z^4+4z^2-5$ ⓒ $12c^5d^4+9c^3d^9-7$ ⓓ $3x^2y-4x$ ⓔ -9

Solution:

ⓐ 1 ⓑ 4 ⓒ 12 ⓓ 3 ⓔ 0

Note:
Exercise:

Problem: Find the degree of the following polynomials:

ⓐ 52 ⓑ a^4b-17a^4 ⓒ $5x+6y+2z$ ⓓ $3x^2-5x+7$ ⓔ $-a^3$

Solution:

ⓐ 0 ⓑ 5 ⓒ 1 ⓓ 2 ⓔ 3

Add and Subtract Monomials

You have learned how to simplify expressions by combining like terms. Remember, like terms must have the same variables with the same exponent. Since monomials are terms, adding and subtracting monomials is the same as combining like terms. If the monomials are like terms, we just combine them by adding or subtracting the coefficient.

Example:
Exercise:

Problem: Add: $25y^2 + 15y^2$.

Solution:
Solution

	$25y^2 + 15y^2$
Combine like terms.	$40y^2$

Note:
Exercise:

Problem: Add: $12q^2 + 9q^2$.

Solution:

$21q^2$

Note:
Exercise:

Problem: Add: $-15c^2 + 8c^2$.

Solution:

$-7c^2$

Example:
Exercise:

Problem: Subtract: $16p-(-7p)$.

Solution:
Solution

	$16p-(-7p)$
Combine like terms.	$23p$

Note:
Exercise:

Problem: Subtract: $8m-(-5m)$.

Solution:

$13m$

Note:
Exercise:

Problem: Subtract: $-15z^3-\left(-5z^3\right)$.

Solution:

$-10z^3$

Remember that like terms must have the same variables with the same exponents.

Example:
Exercise:

Problem: Simplify: $c^2+7d^2-6c^2$.

Solution:
Solution

	$c^2 + 7d^2 - 6c^2$
Combine like terms.	$-5c^2 + 7d^2$

Note:
Exercise:

Problem: Add: $8y^2 + 3z^2 - 3y^2$.

Solution:

$5y^2 + 3z^2$

Note:
Exercise:

Problem: Add: $3m^2 + n^2 - 7m^2$.

Solution:

$-4m^2 + n^2$

Example:
Exercise:

Problem: Simplify: $u^2v + 5u^2 - 3v^2$.

Solution:
Solution

	$u^2v + 5u^2 - 3v^2$
There are no like terms to combine.	$u^2v + 5u^2 - 3v^2$

Note:
Exercise:

Problem: Simplify: $m^2n^2 - 8m^2 + 4n^2$.

Solution:

There are no like terms to combine.

Note:
Exercise:

Problem: Simplify: $pq^2 - 6p - 5q^2$.

Solution:

There are no like terms to combine.

Add and Subtract Polynomials

We can think of adding and subtracting polynomials as just adding and subtracting a series of monomials. Look for the like terms—those with the same variables and the same exponent. The Commutative Property allows us to rearrange the terms to put like terms together.

Example:
Exercise:

Problem: Find the sum: $\left(5y^2 - 3y + 15\right) + \left(3y^2 - 4y - 11\right)$.

Solution:
Solution

Identify like terms.	$(5y^2 - 3y + 15) + (3y^2 - 4y - 11)$
Rearrange to get the like terms together.	$5y^2 + 3y^2 - 3y - 4y + 15 - 11$
Combine like terms.	$8y^2 - 7y + 4$

Note:
Exercise:

Problem: Find the sum: $\left(7x^2 - 4x + 5\right) + \left(x^2 - 7x + 3\right)$.

Solution:

$8x^2 - 11x + 1$

Note:
Exercise:

Problem: Find the sum: $\left(14y^2 + 6y - 4\right) + \left(3y^2 + 8y + 5\right)$.

Solution:

$17y^2 + 14y + 1$

Example:
Exercise:

Problem: Find the difference: $\left(9w^2 - 7w + 5\right) - \left(2w^2 - 4\right)$.

Solution:
Solution

	$(9w^2 - 7w + 5) - (2w^2 - 4)$
Distribute and identify like terms.	$\underline{\underline{9w^2}} - \underline{7w} + \underset{\sim}{5} - \underline{\underline{2w^2}} + \underset{\sim}{4}$
Rearrange the terms.	$\underline{\underline{9w^2 - 2w^2}} - \underline{7w} + \underset{\sim}{5} + \underset{\sim}{4}$
Combine like terms.	$7w^2 - 7w + 9$

Note:

Exercise:

Problem: Find the difference: $\left(8x^2+3x-19\right)-\left(7x^2-14\right)$.

Solution:

$15x^2+3x-5$

Note:
Exercise:

Problem: Find the difference: $\left(9b^2-5b-4\right)-\left(3b^2-5b-7\right)$.

Solution:

$6b^2+3$

Example:
Exercise:

Problem: Subtract: $\left(c^2-4c+7\right)$ from $\left(7c^2-5c+3\right)$.

Solution:
Solution

	Subtract (c^2-4c+7) from $(7c^2-5c+3)$.
	$(7c^2-5c+3)-(c^2-4c+7)$
Distribute and identify like terms.	$7c^2-5c+3-c^2+4c-7$
Rearrange the terms.	$7c^2-c^2-5c+4c+3-7$
Combine like terms.	$6c^2-c-4$

Note:
Exercise:

Problem: Subtract: $\left(5z^2 - 6z - 2\right)$ from $\left(7z^2 + 6z - 4\right)$.

Solution:

$2z^2 + 12z - 2$

Note:
Exercise:

Problem: Subtract: $\left(x^2 - 5x - 8\right)$ from $\left(6x^2 + 9x - 1\right)$.

Solution:

$5x^2 + 14x + 7$

Example:
Exercise:

Problem: Find the sum: $\left(u^2 - 6uv + 5v^2\right) + \left(3u^2 + 2uv\right)$.

Solution:
Solution

	$\left(u^2 - 6uv + 5v^2\right) + \left(3u^2 + 2uv\right)$
Distribute.	$u^2 - 6uv + 5v^2 + 3u^2 + 2uv$
Rearrange the terms, to put like terms together.	$u^2 + 3u^2 - 6uv + 2uv + 5v^2$
Combine like terms.	$4u^2 - 4uv + 5v^2$

Note:
Exercise:

Problem: Find the sum: $\left(3x^2 - 4xy + 5y^2\right) + \left(2x^2 - xy\right)$.

Solution:

$5x^2 - 5xy + 5y^2$

Note:
Exercise:

Problem: Find the sum: $\left(2x^2 - 3xy - 2y^2\right) + \left(5x^2 - 3xy\right)$.

Solution:

$7x^2 - 6xy - 2y^2$

Example:
Exercise:

Problem: Find the difference: $\left(p^2 + q^2\right) - \left(p^2 + 10pq - 2q^2\right)$.

Solution:
Solution

	$\left(p^2 + q^2\right) - \left(p^2 + 10pq - 2q^2\right)$
Distribute.	$p^2 + q^2 - p^2 - 10pq + 2q^2$
Rearrange the terms, to put like terms together.	$p^2 - p^2 - 10pq + q^2 + 2q^2$
Combine like terms.	$-10pq^2 + 3q^2$

Note:
Exercise:

Problem: Find the difference: $\left(a^2 + b^2\right) - \left(a^2 + 5ab - 6b^2\right)$.

Solution:

$-5ab - 5b^2$

Note:
Exercise:

Problem: Find the difference: $\left(m^2+n^2\right)-\left(m^2-7mn-3n^2\right)$.

Solution:

$4n^2+7mn$

Example:
Exercise:

Problem: Simplify: $\left(a^3-a^2b\right)-\left(ab^2+b^3\right)+\left(a^2b+ab^2\right)$.

Solution:
Solution

	$\left(a^3-a^2b\right)-\left(ab^2+b^3\right)+\left(a^2b+ab^2\right)$
Distribute.	$a^3-a^2b-ab^2-b^3+a^2b+ab^2$
Rearrange the terms, to put like terms together.	$a^3-a^2b+a^2b-ab^2+ab^2-b^3$
Combine like terms.	a^3-b^3

Note:
Exercise:

Problem: Simplify: $\left(x^3-x^2y\right)-\left(xy^2+y^3\right)+\left(x^2y+xy^2\right)$.

Solution:

x^3-y^3

Note:
Exercise:

Problem: Simplify: $\left(p^3-p^2q\right)+\left(pq^2+q^3\right)-\left(p^2q+pq^2\right)$.

Solution:

$p^3-2p^2q+q^3$

Evaluate a Polynomial for a Given Value

We have already learned how to evaluate expressions. Since polynomials are expressions, we'll follow the same procedures to evaluate a polynomial. We will substitute the given value for the variable and then simplify using the order of operations.

Example:
Exercise:

Problem: Evaluate $5x^2 - 8x + 4$ when

ⓐ $x = 4$
ⓑ $x = -2$
ⓒ $x = 0$

Solution:
Solution

ⓐ $x = 4$	
	$5x^2 - 8x + 4$
Substitute 4 for x.	$5(4)^2 - 8(4) + 4$
Simplify the exponents.	$5 \cdot 16 - 8(4) + 4$
Multiply.	$80 - 32 + 4$
Simplify.	52

ⓑ $x = -2$	
	$5x^2 - 8x + 4$

Substitute -2 for x.	$5(-2)^2 - 8(-2) + 4$
Simplify the exponents.	$5 \cdot 4 - 8(-2) + 4$
Multiply.	$20 + 16 + 4$
Simplify.	40

ⓒ $x = 0$	
	$5x^2 - 8x + 4$
Substitute 0 for x.	$5(0)^2 - 8(0) + 4$
Simplify the exponents.	$5 \cdot 0 - 8(0) + 4$
Multiply.	$0 + 0 + 4$
Simplify.	4

Note:
Exercise:

Problem: Evaluate: $3x^2 + 2x - 15$ when

ⓐ $x = 3$
ⓑ $x = -5$
ⓒ $x = 0$

Solution:

ⓐ 18 ⓑ 50 ⓒ -15

Note:
Exercise:

Problem: Evaluate: $5z^2 - z - 4$ when

ⓐ $z = -2$
ⓑ $z = 0$
ⓒ $z = 2$

Solution:

ⓐ 18 ⓑ -4 ⓒ 14

Example:
Exercise:

Problem:

The polynomial $-16t^2 + 250$ gives the height of a ball t seconds after it is dropped from a 250 foot tall building. Find the height after $t = 2$ seconds. The polynomial ignores wind resistance and assumes that the ball has not already hit the ground.

Solution:
Solution

	$-16t^2 + 250$
Substitute $t = 2$.	$-16(2)^2 + 250$
Simplify.	$-16 \cdot 4 + 250$
Simplify.	$-64 + 250$
Simplify.	186
	After 2 seconds the height of the ball is 186 feet.

Note:
Exercise:

Problem:

The polynomial $-16t^2 + 250$ gives the height of a ball t seconds after it is dropped from a 250-foot tall building. Find the height after $t = 0$ seconds.

Solution:

250

Note:
Exercise:

Problem:

The polynomial $-16t^2 + 250$ gives the height of a ball t seconds after it is dropped from a 250-foot tall building. Find the height after $t = 3$ seconds.

Solution:

106

Example:
Exercise:

Problem:

The polynomial $6x^2 + 15xy$ gives the cost, in dollars, of producing a rectangular container whose top and bottom are squares with side x feet and sides of height y feet. Find the cost of producing a box with $x = 4$ feet and $y = 6$ feet.

Solution:
Solution

	$6x^2 + 15xy$
Substitute $x = 4$, $y = 6$.	$6(4)^2 + 15(4)(6)$
Simplify.	$6 \cdot 16 + 15(4)(6)$
Simplify.	$96 + 360$
Simplify.	456
	The cost of producing the box is $456.

Note:
Exercise:

Problem:

The polynomial $6x^2 + 15xy$ gives the cost, in dollars, of producing a rectangular container whose top and bottom are squares with side x feet and sides of height y feet. Find the cost of producing a box with $x = 6$ feet and $y = 4$ feet.

Solution:

$576

Note:
Exercise:

Problem:

The polynomial $6x^2 + 15xy$ gives the cost, in dollars, of producing a rectangular container whose top and bottom are squares with side x feet and sides of height y feet. Find the cost of producing a box with $x = 5$ feet and $y = 8$ feet.

Solution:

$750

Note:
Access these online resources for additional instruction and practice with adding and subtracting polynomials.

- Add and Subtract Polynomials 1
- Add and Subtract Polynomials 2
- Add and Subtract Polynomial 3
- Add and Subtract Polynomial 4

Key Concepts

- **Monomials**
  - A monomial is a term of the form ax^m, where a is a constant and m is a whole number
- **Polynomials**
  - **polynomial**—A monomial, or two or more monomials combined by addition or subtraction is a polynomial.
  - **monomial**—A polynomial with exactly one term is called a monomial.
  - **binomial**—A polynomial with exactly two terms is called a binomial.
  - **trinomial**—A polynomial with exactly three terms is called a trinomial.

- **Degree of a Polynomial**
    - The **degree of a term** is the sum of the exponents of its variables.
    - The **degree of a constant** is 0.
    - The **degree of a polynomial** is the highest degree of all its terms.

Practice Makes Perfect

Identify Polynomials, Monomials, Binomials, and Trinomials

In the following exercises, determine if each of the following polynomials is a monomial, binomial, trinomial, or other polynomial.

Exercise:

ⓐ $81b^5 - 24b^3 + 1$
ⓑ $5c^3 + 11c^2 - c - 8$
ⓒ $\frac{14}{15}y + \frac{1}{7}$
ⓓ 5
Problem: ⓔ $4y + 17$

Solution:

ⓐ trinomial ⓑ polynomial ⓒ binomial ⓓ monomial ⓔ binomial

Exercise:

ⓐ $x^2 - y^2$
ⓑ $-13c^4$
ⓒ $x^2 + 5x - 7$
ⓓ $x^2y^2 - 2xy + 8$
Problem: ⓔ 19

Exercise:

ⓐ $8 - 3x$
ⓑ $z^2 - 5z - 6$
ⓒ $y^3 - 8y^2 + 2y - 16$
ⓓ $81b^5 - 24b^3 + 1$
Problem: ⓔ -18

Solution:

ⓐ binomial ⓑ trinomial ⓒ polynomial ⓓ trinomial ⓔ monomial

Exercise:

ⓐ $11y^2$
ⓑ -73
ⓒ $6x^2 - 3xy + 4x - 2y + y^2$
ⓓ $4y + 17$
Problem: ⓔ $5c^3 + 11c^2 - c - 8$

Determine the Degree of Polynomials

In the following exercises, determine the degree of each polynomial.

Exercise:

Problem:
ⓐ $6a^2 + 12a + 14$
ⓑ $18xy^2z$
ⓒ $5x + 2$
ⓓ $y^3 - 8y^2 + 2y - 16$
ⓔ -24

Solution:

ⓐ 2 ⓑ 4 ⓒ 1 ⓓ 3 ⓔ 0

Exercise:

Problem:
ⓐ $9y^3 - 10y^2 + 2y - 6$
ⓑ $-12p^4$
ⓒ $a^2 + 9a + 18$
ⓓ $20x^2y^2 - 10a^2b^2 + 30$
ⓔ 17

Exercise:

Problem:
ⓐ $14 - 29x$
ⓑ $z^2 - 5z - 6$
ⓒ $y^3 - 8y^2 + 2y - 16$
ⓓ $23ab^2 - 14$
ⓔ -3

Solution:

ⓐ 1 ⓑ 2 ⓒ 3 ⓓ 3 ⓔ 0

Exercise:

Problem:
ⓐ $62y^2$
ⓑ 15
ⓒ $6x^2 - 3xy + 4x - 2y + y^2$
ⓓ $10 - 9x$
ⓔ $m^4 + 4m^3 + 6m^2 + 4m + 1$

Add and Subtract Monomials

In the following exercises, add or subtract the monomials.

Exercise:

Problem: $7\text{x}^2 + 5x^2$

Solution:

$12x^2$

Exercise:

Problem: $4\text{y}^3 + 6y^3$

Exercise:

Problem: $-12w + 18w$

Solution:

$6w$

Exercise:

Problem: $-3m + 9m$

Exercise:

Problem: $4\text{a} - 9a$

Solution:

$-5a$

Exercise:

Problem: $-y - 5y$

Exercise:

Problem: $28x - (-12x)$

Solution:

$40x$

Exercise:

Problem: $13z - (-4z)$

Exercise:

Problem: $-5b - 17b$

Solution:

$-22b$

Exercise:

Problem: $-10x - 35x$

Exercise:

Problem: $12a + 5b - 22a$

Solution:

$-10\text{a} + 5b$

Exercise:

Problem: $14\text{x} - 3y - 13x$

Exercise:

Problem: $2a^2 + b^2 - 6a^2$

Solution:

$-4a^2 + b^2$

Exercise:

Problem: $5u^2 + 4v^2 - 6u^2$

Exercise:

Problem: $xy^2 - 5x - 5y^2$

Solution:

$xy^2 - 5x - 5y^2$

Exercise:

Problem: $pq^2 - 4p - 3q^2$

Exercise:

Problem: $a^2b - 4a - 5ab^2$

Solution:

$a^2b - 4a - 5ab^2$

Exercise:

Problem: $x^2y - 3x + 7xy^2$

Exercise:

Problem: $12\text{a} + 8b$

Solution:

$12\text{a} + 8b$

Exercise:

Problem: $19\text{y} + 5z$

Exercise:

Problem: Add: $4a$, $-3b$, $-8a$

Solution:

$-4a-3b$

Exercise:

Problem: Add: $4\text{x}, 3y, -3x$

Exercise:

Problem: Subtract $5x^6 \text{from} - 12x^6$.

Solution:

$-17x^6$

Exercise:

Problem: Subtract $2p^4 \text{from} - 7p^4$.

Add and Subtract Polynomials

In the following exercises, add or subtract the polynomials.

Exercise:

Problem: $(5y^2+12y+4)+(6y^2-8y+7)$

Solution:

$11y^2+4y+11$

Exercise:

Problem: $(4y^2+10y+3)+(8y^2-6y+5)$

Exercise:

Problem: $(x^2+6x+8)+(-4x^2+11x-9)$

Solution:

$-3x^2+17x-1$

Exercise:

Problem: $(y^2+9y+4)+(-2y^2-5y-1)$

Exercise:

Problem: $(8x^2-5x+2)+(3x^2+3)$

Solution:

$11x^2-5x+5$

Exercise:

Problem: $(7x^2-9x+2)+(6x^2-4)$

Exercise:

Problem: $(5a^2 + 8) + (a^2 - 4a - 9)$

Solution:

$6a^2 - 4a - 1$

Exercise:

Problem: $(p^2 - 6p - 18) + (2p^2 + 11)$

Exercise:

Problem: $(4m^2 - 6m - 3) - (2m^2 + m - 7)$

Solution:

$2m^2 - 7m + 4$

Exercise:

Problem: $(3b^2 - 4b + 1) - (5b^2 - b - 2)$

Exercise:

Problem: $(a^2 + 8a + 5) - (a^2 - 3a + 2)$

Solution:

$5a + 3$

Exercise:

Problem: $(b^2 - 7b + 5) - (b^2 - 2b + 9)$

Exercise:

Problem: $(12s^2 - 15s) - (s - 9)$

Solution:

$12s^2 - 14s + 9$

Exercise:

Problem: $(10r^2 - 20r) - (r - 8)$

Exercise:

Problem: Subtract $\left(9x^2 + 2\right)$ from $\left(12x^2 - x + 6\right)$.

Solution:

$3x^2 - x + 4$

Exercise:

Problem: Subtract $\left(5y^2 - y + 12\right)$ from $\left(10y^2 - 8y - 20\right)$.

Exercise:

Problem: Subtract $\left(7w^2 - 4w + 2\right)$ from $\left(8w^2 - w + 6\right)$.

Solution:

$w^2 + 3w + 4$

Exercise:

Problem: Subtract $\left(5x^2 - x + 12\right)$ from $\left(9x^2 - 6x - 20\right)$.

Exercise:

Problem: Find the sum of $\left(2p^3 - 8\right)$ and $\left(p^2 + 9p + 18\right)$.

Solution:

$2p^3 + p^2 + 9p + 10$

Exercise:

Problem: Find the sum of $\left(q^2 + 4q + 13\right)$ and $\left(7q^3 - 3\right)$.

Exercise:

Problem: Find the sum of $\left(8a^3 - 8a\right)$ and $\left(a^2 + 6a + 12\right)$.

Solution:

$8a^3 + a^2 - 2a + 12$

Exercise:

Problem: Find the sum of $\left(b^2 + 5b + 13\right)$ and $\left(4b^3 - 6\right)$.

Exercise:

Problem: Find the difference of $\left(w^2 + w - 42\right)$ and $\left(w^2 - 10w + 24\right)$.

Solution:

$11w - 64$

Exercise:

Problem: Find the difference of $\left(z^2 - 3z - 18\right)$ and $\left(z^2 + 5z - 20\right)$.

Exercise:

Problem: Find the difference of $\left(c^2+4c-33\right)$ and $\left(c^2-8c+12\right)$.

Solution:

$12c-45$

Exercise:

Problem: Find the difference of $\left(t^2-5t-15\right)$ and $\left(t^2+4t-17\right)$.

Exercise:

Problem: $\left(7x^2-2xy+6y^2\right)+\left(3x^2-5xy\right)$

Solution:

$10x^2-7xy+6y^2$

Exercise:

Problem: $\left(-5x^2-4xy-3y^2\right)+\left(2x^2-7xy\right)$

Exercise:

Problem: $\left(7m^2+mn-8n^2\right)+\left(3m^2+2mn\right)$

Solution:

$10m^2+3mn-8n^2$

Exercise:

Problem: $\left(2r^2-3rs-2s^2\right)+\left(5r^2-3rs\right)$

Exercise:

Problem: $\left(a^2-b^2\right)-\left(a^2+3ab-4b^2\right)$

Solution:

$-3ab+3b^2$

Exercise:

Problem: $\left(m^2+2n^2\right)-\left(m^2-8mn-n^2\right)$

Exercise:

Problem: $\left(u^2-v^2\right)-\left(u^2-4uv-3v^2\right)$

Solution:

$4uv + 2v^2$

Exercise:

Problem: $(j^2 - k^2) - (j^2 - 8jk - 5k^2)$

Exercise:

Problem: $(p^3 - 3p^2q) + (2pq^2 + 4q^3) - (3p^2q + pq^2)$

Solution:

$p^3 - 6p^2q + pq^2 + 4q^3$

Exercise:

Problem: $(a^3 - 2a^2b) + (ab^2 + b^3) - (3a^2b + 4ab^2)$

Exercise:

Problem: $(x^3 - x^2y) - (4xy^2 - y^3) + (3x^2y - xy^2)$

Solution:

$x^3 + 2x^2y - 5xy^2 + y^3$

Exercise:

Problem: $(x^3 - 2x^2y) - (xy^2 - 3y^3) - (x^2y - 4xy^2)$

Evaluate a Polynomial for a Given Value

In the following exercises, evaluate each polynomial for the given value.

Exercise:

Problem: Evaluate $8y^2 - 3y + 2$ when:

ⓐ $y = 5$
ⓑ $y = -2$
ⓒ $y = 0$

Solution:

ⓐ 187 ⓑ 46 ⓒ 2

Exercise:

Problem: Evaluate $5y^2 - y - 7$ when:

ⓐ $y = -4$
ⓑ $y = 1$
ⓒ $y = 0$

Exercise:

Problem: Evaluate $4 - 36x$ when:

ⓐ $x = 3$
ⓑ $x = 0$
ⓒ $x = -1$

Solution:

ⓐ −104 ⓑ 4 ⓒ 40

Exercise:

Problem: Evaluate $16 - 36x^2$ when:

ⓐ $x = -1$
ⓑ $x = 0$
ⓒ $x = 2$

Exercise:

Problem:

A painter drops a brush from a platform 75 feet high. The polynomial $-16t^2 + 75$ gives the height of the brush t seconds after it was dropped. Find the height after $t = 2$ seconds.

Solution:

11

Exercise:

Problem:

A girl drops a ball off a cliff into the ocean. The polynomial $-16t^2 + 250$ gives the height of a ball t seconds after it is dropped from a 250-foot tall cliff. Find the height after $t = 2$ seconds.

Exercise:

Problem:

A manufacturer of stereo sound speakers has found that the revenue received from selling the speakers at a cost of p dollars each is given by the polynomial $-4p^2 + 420p$. Find the revenue received when $p = 60$ dollars.

Solution:

$10,800

Exercise:

Problem:

A manufacturer of the latest basketball shoes has found that the revenue received from selling the shoes at a cost of p dollars each is given by the polynomial $-4p^2 + 420p$. Find the revenue received when $p = 90$ dollars.

Everyday Math

Exercise:

Problem:

Fuel Efficiency The fuel efficiency (in miles per gallon) of a car going at a speed of x miles per hour is given by the polynomial $-\frac{1}{150}x^2 + \frac{1}{3}x$. Find the fuel efficiency when $x = 30\ \text{mph}$.

Solution:

4

Exercise:

Problem:

Stopping Distance The number of feet it takes for a car traveling at x miles per hour to stop on dry, level concrete is given by the polynomial $0.06x^2 + 1.1x$. Find the stopping distance when $x = 40\ \text{mph}$.

Exercise:

Problem:

Rental Cost The cost to rent a rug cleaner for d days is given by the polynomial $5.50d + 25$. Find the cost to rent the cleaner for 6 days.

Solution:

$58

Exercise:

Problem:

Height of Projectile The height (in feet) of an object projected upward is given by the polynomial $-16t^2 + 60t + 90$ where t represents time in seconds. Find the height after $t = 2.5$ seconds.

Exercise:

Problem:

Temperature Conversion The temperature in degrees Fahrenheit is given by the polynomial $\frac{9}{5}c + 32$ where c represents the temperature in degrees Celsius. Find the temperature in degrees Fahrenheit when $c = 65^\circ$.

Solution:

149

Writing Exercises

Exercise:

Problem: Using your own words, explain the difference between a monomial, a binomial, and a trinomial.

Exercise:

Problem:

Using your own words, explain the difference between a polynomial with five terms and a polynomial with a degree of 5.

Solution:

Answers will vary.

Exercise:

Problem: Ariana thinks the sum $6y^2 + 5y^4$ is $11y^6$. What is wrong with her reasoning?

Exercise:

Problem: Jonathan thinks that $\frac{1}{3}$ and $\frac{1}{x}$ are both monomials. What is wrong with his reasoning?

Solution:

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
identify polynomials, monomials, binomials, and trinomials.			
determine the degree of polynomials.			
add and subtract monomials.			
add and subtract polynomials.			
evaluate a polynomial for a given value.			

ⓑ If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

Glossary

binomial
: A binomial is a polynomial with exactly two terms.

degree of a constant
: The degree of any constant is 0.

degree of a polynomial
: The degree of a polynomial is the highest degree of all its terms.

degree of a term
: The degree of a term is the exponent of its variable.

monomial

A monomial is a term of the form ax^m, where a is a constant and m is a whole number; a monomial has exactly one term.

polynomial

A polynomial is a monomial, or two or more monomials combined by addition or subtraction.

standard form

A polynomial is in standard form when the terms of a polynomial are written in descending order of degrees.

trinomial

A trinomial is a polynomial with exactly three terms.

Use Multiplication Properties of Exponents: ASE
By the end of this section, you will be able to:

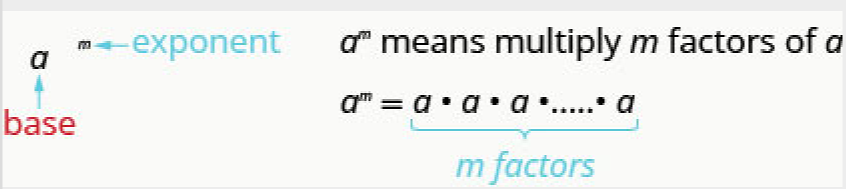
- Simplify expressions with exponents
- Simplify expressions using the Product Property for Exponents
- Simplify expressions using the Power Property for Exponents
- Simplify expressions using the Product to a Power Property
- Simplify expressions by applying several properties
- Multiply monomials

Simplify Expressions with Exponents

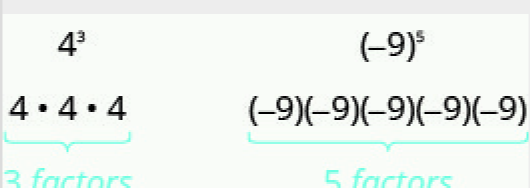
Remember that an exponent indicates repeated multiplication of the same quantity. For example, 2^4 means to multiply 2 by itself 4 times, so 2^4 means $2 \cdot 2 \cdot 2 \cdot 2$.

Let’s review the vocabulary for expressions with exponents.

Note:
Exponential Notation



This is read a to the m^{th} power.
In the expression a^m, the *exponent* m tells us how many times we use the *base* a as a factor.



Before we begin working with variable expressions containing exponents, let's simplify a few expressions involving only numbers.

Example:
Exercise:

Problem: Simplify: ⓐ 4^3 ⓑ 7^1 ⓒ $\left(\frac{5}{6}\right)^2$ ⓓ $(0.63)^2$.

Solution:
Solution

ⓐ	4^3
Multiply three factors of 4.	$4 \cdot 4 \cdot 4$
Simplify.	64
ⓑ	7^1
Multiply one factor of 7.	7
ⓒ	$\left(\frac{5}{6}\right)^2$
Multiply two factors.	$\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)$
Simplify.	$\frac{25}{36}$
ⓓ	$(0.63)^2$

Multiply two factors.	$(0.63)(0.63)$
Simplify.	0.3969

Note:
Exercise:

Problem: Simplify: ⓐ 6^3 ⓑ 15^1 ⓒ $\left(\frac{3}{7}\right)^2$ ⓓ $(0.43)^2$.

Solution:

ⓐ 216 ⓑ 15 ⓒ $\frac{9}{49}$ ⓓ 0.1849

Note:
Exercise:

Problem: Simplify: ⓐ 2^5 ⓑ 21^1 ⓒ $\left(\frac{2}{5}\right)^3$ ⓓ $(0.218)^2$.

Solution:

ⓐ 32 ⓑ 21 ⓒ $\frac{8}{125}$ ⓓ 0.047524

Example:
Exercise:

Problem: Simplify: ⓐ $(-5)^4$ ⓑ -5^4.

Solution:
Solution

ⓐ	$(-5)^4$
Multiply four factors of -5.	$(-5)(-5)(-5)(-5)$
Simplify.	625
ⓑ	-5^4
Multiply four factors of 5.	$-(5 \cdot 5 \cdot 5 \cdot 5)$
Simplify.	-625

Note:
Exercise:

Problem: Simplify: ⓐ $(-3)^4$ ⓑ -3^4.

Solution:

ⓐ 81 ⓑ -81

Note:

Exercise:

Problem: Simplify: ⓐ $(-13)^2$ ⓑ -13^2.

Solution:

ⓐ 169 ⓑ -169

Notice the similarities and differences in [link]ⓐ and [link]ⓑ! Why are the answers different? As we follow the order of operations in part ⓐ the parentheses tell us to raise the (-5) to the 4th power. In part ⓑ we raise just the 5 to the 4th power and then take the opposite.

Simplify Expressions Using the Product Property for Exponents

You have seen that when you combine like terms by adding and subtracting, you need to have the same base with the same exponent. But when you multiply and divide, the exponents may be different, and sometimes the bases may be different, too.

We'll derive the properties of exponents by looking for patterns in several examples.

First, we will look at an example that leads to the Product Property.

	$x^2 \cdot x^3$

What does this mean? How many factors altogether?	$\underbrace{\underbrace{x \cdot x}_{\text{2 factors}} \cdot \underbrace{x \cdot x \cdot x}_{\text{3 factors}}}_{\text{5 factors}}$
So, we have	x^5
Notice that 5 is the sum of the exponents, 2 and 3.	$x^2 \cdot x^3$ is x^{2+3}, or x^5

We write:
Equation:

$$x^2 \cdot x^3$$
$$x^{2+3}$$
$$x^5$$

The base stayed the same and we added the exponents. This leads to the **Product Property for Exponents**.

Note:
Product Property for Exponents
If a is a real number, and m and n are counting numbers, then
Equation:

$$a^m \cdot a^n = a^{m+n}$$

To multiply with like bases, add the exponents.

An example with numbers helps to verify this property.
Equation:

$$\begin{aligned} 2^2 \cdot 2^3 &\stackrel{?}{=} 2^{2+3} \\ 4 \cdot 8 &\stackrel{?}{=} 2^5 \\ 32 &= 32 \checkmark \end{aligned}$$

Example:
Exercise:

Problem: Simplify: $y^5 \cdot y^6$.

Solution:
Solution

	$y^5 \cdot y^6$
Use the product property, $a^m \cdot a^n = a^{m+n}$.	y^{5+6}
Simplify.	y^{11}

Note:
Exercise:

Problem: Simplify: $b^9 \cdot b^8$.

Solution:

b^{17}

Note:
Exercise:

Problem: Simplify: $x^{12} \cdot x^4$.

Solution:

x^{16}

Example:
Exercise:

Problem: Simplify: ⓐ $2^5 \cdot 2^9$ ⓑ $3 \cdot 3^4$.

Solution:
Solution

ⓐ

	$2^5 \cdot 2^9$
Use the product property, $a^m \cdot a^n = a^{m+n}$.	2^{5+9}
Simplify.	2^{14}

ⓑ

	$3 \cdot 3^4$
Use the product property, $a^m \cdot a^n = a^{m+n}$.	3^{1+4}
Simplify.	3^5

Note:
Exercise:

Problem: Simplify: ⓐ $5 \cdot 5^5$ ⓑ $4^9 \cdot 4^9$.

Solution:

ⓐ 5^6 ⓑ 4^{18}

Note:
Exercise:

Problem: Simplify: ⓐ $7^6 \cdot 7^8$ ⓑ $10 \cdot 10^{10}$.

Solution:

ⓐ 7^{14} ⓑ 10^{11}

Example:
Exercise:

Problem: Simplify: ⓐ $a^7 \cdot a$ ⓑ $x^{27} \cdot x^{13}$.

Solution:
Solution

ⓐ

	$a^7 \cdot a$

Rewrite, a = a^1.	$a^7 \cdot a^1$
Use the product property, $a^m \cdot a^n = a^{m+n}$.	a^{7+1}
Simplify.	a^8

ⓑ

	$x^{27} \cdot x^{13}$
Notice, the bases are the same, so add the exponents.	x^{27+13}
Simplify.	x^{40}

Note:
Exercise:

Problem: Simplify: ⓐ $p^5 \cdot p$ ⓑ $y^{14} \cdot y^{29}$.

Solution:

ⓐ p^6 ⓑ y^{43}

Note:
Exercise:

Problem: Simplify: ⓐ $z \cdot z^7$ ⓑ $b^{15} \cdot b^{34}$.

Solution:

ⓐ z^8 ⓑ b^{49}

We can extend the Product Property for Exponents to more than two factors.

Example:
Exercise:

Problem: Simplify: $d^4 \cdot d^5 \cdot d^2$.

Solution:
Solution

	$d^4 \cdot d^5 \cdot d^2$

Add the exponents, since bases are the same.	d^{4+5+2}
Simplify.	d^{11}

Note:
Exercise:

Problem: Simplify: $x^6 \cdot x^4 \cdot x^8$.

Solution:

x^{18}

Note:
Exercise:

Problem: Simplify: $b^5 \cdot b^9 \cdot b^5$.

Solution:

b^{19}

Simplify Expressions Using the Power Property for Exponents

Now let’s look at an exponential expression that contains a power raised to a power. See if you can discover a general property.

	$(x^2)^3$
What does this mean? How many factors altogether?	$\underbrace{\underbrace{x \cdot x}_{2\ factors} \cdot \underbrace{x \cdot x}_{2\ factors} \cdot \underbrace{x \cdot x}_{2\ factors}}_{6\ factors}$ ($x^2 \cdot x^2 \cdot x^2$)
So we have	x^6
Notice that 6 is the product of the exponents, 2 and 3.	$(x^2)^3$ is $x^{2\cdot3}$ or x^6

We write:
Equation:

$$\left(x^2\right)^3$$
$$x^{2\cdot 3}$$
$$x^6$$

We multiplied the exponents. This leads to the **Power Property for Exponents.**

Note:
Power Property for Exponents
If a is a real number, and m and n are whole numbers, then
Equation:

$$\left(a^m\right)^n = a^{m\cdot n}$$

To raise a power to a power, multiply the exponents.

An example with numbers helps to verify this property.

Equation:

$$\begin{aligned}\left(3^2\right)^3 &\stackrel{?}{=} 3^{2\cdot 3}\\ (9)^3 &\stackrel{?}{=} 3^6\\ 729 &= 729\ \checkmark\end{aligned}$$

Example:

Exercise:

Problem: Simplify: ⓐ $\left(y^5\right)^9$ ⓑ $\left(4^4\right)^7$.

Solution:

Solution

ⓐ

	$(y^5)^9$
Use the power property, $(a^m)^n = a^{m\cdot n}$.	$y^{5\cdot 9}$

Simplify.	y^{45}

ⓑ

	$(4^4)^7$
Use the power property.	$4^{4 \cdot 7}$
Simplify.	4^{28}

Note:
Exercise:

Problem: Simplify: ⓐ $(b^7)^5$ ⓑ $(5^4)^3$.

Solution:

ⓐ b^{35} ⓑ 5^{12}

Note:
Exercise:

Problem: Simplify: ⓐ $(z^6)^9$ ⓑ $(3^7)^7$.

Solution:

ⓐ z^{54} ⓑ 3^{49}

Simplify Expressions Using the Product to a Power Property

We will now look at an expression containing a product that is raised to a power. Can you find this pattern?

	$(2x)^3$
What does this mean?	$2x \cdot 2x \cdot 2x$
We group the like factors together.	$2 \cdot 2 \cdot 2 \cdot x \cdot x \cdot x$
How many factors of 2 and of x?	$2^3 \cdot x^3$

Notice that each factor was raised to the power and $(2x)^3$ is $2^3 \cdot x^3$.

We write:	$(2x)^3$
	$2^3 \cdot x^3$

The exponent applies to each of the factors. This leads to the **Product to a Power Property for Exponents.** It is also called the **Distributive Property of Exponentiation over Multiplication**. Notice that it has the same pattern as the Distributive Property of Multiplication over Addition which is normally just called the Distributive Property.

Note:
Product to a Power Property for Exponents
If a and b are real numbers and m is a whole number, then
Equation:

$$(ab)^m = a^m b^m$$

To raise a product to a power, raise each factor to that power.

An example with numbers helps to verify this property:
Equation:

$$\begin{aligned} (2 \cdot 3)^2 &\stackrel{?}{=} 2^2 \cdot 3^2 \\ 6^2 &\stackrel{?}{=} 4 \cdot 9 \\ 36 &= 36 \checkmark \end{aligned}$$

Example:
Exercise:

Problem: Simplify: ⓐ $(-9d)^2$ ⓑ $(3mn)^3$.

Solution:
Solution

ⓐ

	$(-9d)^2$
Use Power of a Product Property, $(ab)^m = a^m b^m$.	$(-9)^2 d^2$
Simplify.	$81d^2$

ⓑ

	$(3mn)^3$
Use Power of a Product Property, $(ab)^m = a^m b^m$.	$(3)^3 m^3 n^3$

Simplify.	$27m^3n^3$

Note:
Exercise:

Problem: Simplify: ⓐ $(-12y)^2$ ⓑ $(2wx)^5$.

Solution:

ⓐ $144y^2$ ⓑ $32w^5x^5$

Note:
Exercise:

Problem: Simplify: ⓐ $(5wx)^3$ ⓑ $(-3y)^3$.

Solution:

ⓐ $125w^3x^3$ ⓑ $-27y^3$

Simplify Expressions by Applying Several Properties

We now have three properties for multiplying expressions with exponents. Let's summarize them and then we'll do some examples that use more than one of the properties.

Note:
Properties of Exponents
If a and b are real numbers, and m and n are whole numbers, then

Product Property	$a^m \cdot a^n \ = \ a^{m+n}$
Power Property	$(a^m)^n \ = \ a^{m \cdot n}$
Product to a Power	$(ab)^m \ = \ a^m b^m$

All exponent properties hold true for any real numbers m and n. Right now, we only use whole number exponents.

Example:
Exercise:

Problem: Simplify: ⓐ $(y^3)^6(y^5)^4$ ⓑ $(-6x^4y^5)^2$.

Solution:
Solution

ⓐ

	$(y^3)^6(y^5)^4$
Use the Power Property.	$y^{18} \cdot y^{20}$
Add the exponents.	y^{38}
ⓑ	$\left(-6x^4y^5\right)^2$
Use the Product to a Power Property.	$(-6)^2(x^4)^2(y^5)^2$
Use the Power Property.	$(-6)^2$
Simplify.	$36x^8y^{10}$

Note:
Exercise:

Problem: Simplify: ⓐ $(a^4)^5(a^7)^4$ ⓑ $\left(-2c^4d^2\right)^3$.

Solution:

ⓐ a^{48} ⓑ $-8c^{12}d^6$

Note:
Exercise:

Problem: Simplify: ⓐ $\left(-3x^6y^7\right)^4$ ⓑ $(q^4)^5(q^3)^3$.

Solution:

ⓐ $81x^{24}y^{28}$ ⓑ q^{29}

Example:
Exercise:

Problem: Simplify: ⓐ $(5m)^2(3m^3)$ ⓑ $(3x^2y)^4(2xy^2)^3$.

Solution:
Solution

ⓐ	$(5m)^2(3m^3)$
Raise $5m$ to the second power.	$5^2m^2 \cdot 3m^3$
Simplify.	$25m^2 \cdot 3m^3$
Use the Commutative Property.	$25 \cdot 3 \cdot m^2 \cdot m^3$
Multiply the constants and add the exponents.	$75m^5$
ⓑ	$(3x^2y)^4(2xy^2)^3$
Use the Product to a Power Property.	$(3^4x^8y^4)(2^3x^3y^6)$
Simplify.	$(81x^8y^4)(8x^3y^6)$
Use the Commutative Property.	$81 \cdot 8 \cdot x^8 \cdot x^3 \cdot y^4 \cdot y^6$

Multiply the constants and add the exponents.	$648x^{11}y^{10}$

Note:
Exercise:

Problem: Simplify: ⓐ $(5n)^2(3n^{10})$ ⓑ $(c^4d^2)^5(3cd^5)^4$.

Solution:

ⓐ $75n^{12}$ ⓑ $81c^{24}d^{30}$

Note:
Exercise:

Problem: Simplify: ⓐ $(a^3b^2)^6(4ab^3)^4$ ⓑ $(2x)^3(5x^7)$.

Solution:

ⓐ $256a^{22}b^{24}$ ⓑ $40x^{10}$

Multiply Monomials

Since a monomial is an algebraic expression, we can use the properties of exponents to multiply monomials.

Example:
Exercise:

Problem: Multiply: $\left(3x^2\right)\left(-4x^3\right)$.

Solution:
Solution

	$\left(3x^2\right)\left(-4x^3\right)$
Use the Commutative Property to rearrange the terms.	$3 \cdot (-4) \cdot x^2 \cdot x^3$
Multiply.	$-12x^5$

Note:
Exercise:

Problem: Multiply: $\left(5y^7\right)\left(-7y^4\right)$.

Solution:

$-35y^{11}$

Note:

Exercise:

Problem: Multiply: $\left(-6b^4\right)\left(-9b^5\right)$.

Solution:

$54b^9$

Example:
Exercise:

Problem: Multiply: $\left(\frac{5}{6}x^3y\right)\left(12xy^2\right)$.

Solution:
Solution

	$\left(\frac{5}{6}x^3y\right)\left(12xy^2\right)$
Use the Commutative Property to rearrange the terms.	$\frac{5}{6}\cdot 12\cdot x^3\cdot x\cdot y\cdot y^2$
Multiply.	$10x^4y^3$

Note:
Exercise:

Problem: Multiply: $\left(\frac{2}{5}a^4b^3\right)\left(15ab^3\right)$.

Solution:

$6a^5b^6$

Note:
Exercise:

Problem: Multiply: $\left(\frac{2}{3}r^5s\right)\left(12r^6s^7\right)$.

Solution:

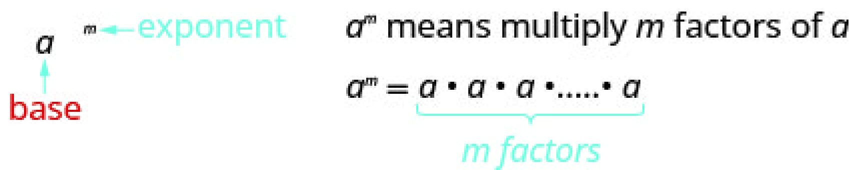
$8r^{11}s^8$

Note:
Access these online resources for additional instruction and practice with using multiplication properties of exponents:

- Multiplication Properties of Exponents

Key Concepts

- **Exponential Notation**



- **Properties of Exponents**
  - If a, b are real numbers and m, n are whole numbers, then

$$\begin{array}{lrcl} \textbf{Product Property} & a^m \cdot a^n & = & a^{m+n} \\ \textbf{Power Property} & (a^m)^n & = & a^{m \cdot n} \\ \textbf{Product to a Power} & (ab)^m & = & a^m b^m \end{array}$$

Practice Makes Perfect

Simplify Expressions with Exponents

In the following exercises, simplify each expression with exponents.

Exercise:

Problem:
ⓐ 3^5
ⓑ 9^1
ⓒ $\left(\frac{1}{3}\right)^2$
ⓓ $(0.2)^4$

Exercise:

Problem:
ⓐ 10^4
ⓑ 17^1
ⓒ $\left(\frac{2}{9}\right)^2$
ⓓ $(0.5)^3$

Solution:

ⓐ 10,000 ⓑ 17 ⓒ $\frac{4}{81}$ ⓓ 0.125

Exercise:

Problem: ⓐ 2^6
ⓑ 14^1
ⓒ $\left(\frac{2}{5}\right)^3$
ⓓ $(0.7)^2$

Exercise:

Problem: ⓐ 8^3
ⓑ 8^1
ⓒ $\left(\frac{3}{4}\right)^3$
ⓓ $(0.4)^3$

Solution:

ⓐ 512 ⓑ 8 ⓒ $\frac{27}{64}$
ⓓ 0.064

Exercise:

Problem: ⓐ $(-6)^4$
ⓑ -6^4

Exercise:

Problem: ⓐ $(-2)^6$
ⓑ -2^6

Solution:

ⓐ 64 ⓑ -64

Exercise:

Problem: ⓐ $-\left(\frac{1}{4}\right)^4$
ⓑ $\left(-\frac{1}{4}\right)^4$

Exercise:

Problem: ⓐ $-\left(\frac{2}{3}\right)^2$
ⓑ $\left(-\frac{2}{3}\right)^2$

Solution:

ⓐ $-\frac{4}{9}$
ⓑ $\frac{4}{9}$

Exercise:

Problem: ⓐ -0.5^2
ⓑ $(-0.5)^2$

Exercise:

Problem: ⓐ -0.1^4
ⓑ $(-0.1)^4$

Solution:

ⓐ -0.001 ⓑ 0.001

Simplify Expressions Using the Product Property for Exponents

In the following exercises, simplify each expression using the Product Property for Exponents.

Exercise:

Problem: $d^3 \cdot d^6$

Exercise:

Problem: $x^4 \cdot x^2$

Solution:

x^6

Exercise:

Problem: $n^{19} \cdot n^{12}$

Exercise:

Problem: $q^{27} \cdot q^{15}$

Solution:

q^{42}

Exercise:

Problem: ⓐ $4^5 \cdot 4^9$ ⓑ $8^9 \cdot 8$

Exercise:

Problem: ⓐ $3^{10} \cdot 3^6$ ⓑ $5 \cdot 5^4$

Solution:

ⓐ 3^{16} ⓑ 5^5

Exercise:

Problem: ⓐ $y \cdot y^3$ ⓑ $z^{25} \cdot z^8$

Exercise:

Problem: ⓐ $w^5 \cdot w$ ⓑ $u^{41} \cdot u^{53}$

Solution:

ⓐ w^6 ⓑ u^{94}

Exercise:

Problem: $w \cdot w^2 \cdot w^3$

Exercise:

Problem: $y \cdot y^3 \cdot y^5$

Solution:

y^9

Exercise:

Problem: $a^4 \cdot a^3 \cdot a^9$

Exercise:

Problem: $c^5 \cdot c^{11} \cdot c^2$

Solution:

c^{18}

Exercise:

Problem: $m^x \cdot m^3$

Exercise:

Problem: $n^y \cdot n^2$

Solution:

n^{y+2}

Exercise:

Problem: $y^a \cdot y^b$

Exercise:

Problem: $x^p \cdot x^q$

Solution:

x^{p+q}

Simplify Expressions Using the Power Property for Exponents

In the following exercises, simplify each expression using the Power Property for Exponents.

Exercise:

Problem: ⓐ $\left(m^4\right)^2$ ⓑ $\left(10^3\right)^6$

Exercise:

Problem: ⓐ $\left(b^2\right)^7$ ⓑ $\left(3^8\right)^2$

Solution:

ⓐ b^{14} ⓑ 3^{16}

Exercise:

Problem: ⓐ $\left(y^3\right)^x$ ⓑ $(5^x)^y$

Exercise:

Problem: ⓐ $\left(x^2\right)^y$ ⓑ $(7^a)^b$

Solution:

ⓐ x^{2y} ⓑ 7^{ab}

Simplify Expressions Using the Product to a Power Property

In the following exercises, simplify each expression using the Product to a Power Property.

Exercise:

Problem: ⓐ $(6a)^2$ ⓑ $(3xy)^2$

Exercise:

Problem: ⓐ $(5x)^2$ ⓑ $(4ab)^2$

Solution:

ⓐ $25x^2$ ⓑ $16a^2b^2$

Exercise:

Problem: ⓐ $(-4m)^3$ ⓑ $(5ab)^3$

Exercise:

Problem: ⓐ $(-7n)^3$ ⓑ $(3xyz)^4$

Solution:

ⓐ $-343n^3$ ⓑ $81x^4y^4z^4$

Simplify Expressions by Applying Several Properties

In the following exercises, simplify each expression.
Exercise:

Problem: ⓐ $\left(y^2\right)^4 \cdot \left(y^3\right)^2$ ⓑ $(10a^2b)^3$

Exercise:

Problem: ⓐ $\left(w^4\right)^3 \cdot \left(w^5\right)^2$ ⓑ $(2xy^4)^5$

Solution:

ⓐ w^{22} ⓑ $32x^5y^{20}$

Exercise:

Problem: ⓐ $(-2r^3s^2)^4$ ⓑ $\left(m^5\right)^3 \cdot \left(m^9\right)^4$

Exercise:

Problem: ⓐ $(-10q^2p^4)^3$ ⓑ $\left(n^3\right)^{10} \cdot \left(n^5\right)^2$

Solution:

ⓐ $-1000q^6p^{12}$ ⓑ n^{40}

Exercise:

Problem: ⓐ $(3x)^2(5x)$
ⓑ $(5t^2)^3(3t)^2$

Exercise:

Problem: ⓐ $(2y)^3(6y)$
ⓑ $(10k^4)^3(5k^6)^2$

Solution:

ⓐ $48y^4$ ⓑ $25{,}000k^{24}$

Exercise:

Problem: ⓐ $(5a)^2(2a)^3$
ⓑ $(\frac{1}{2}y^2)^3(\frac{2}{3}y)^2$

Exercise:

Problem: ⓐ $(4b)^2(3b)^3$
ⓑ $(\frac{1}{2}j^2)^5(\frac{2}{5}j^3)^2$

Solution:

ⓐ $432b^5$ ⓑ $\frac{1}{200}j^{16}$

Exercise:

Problem: ⓐ $\left(\frac{2}{5}x^2y\right)^3$ ⓑ $\left(\frac{8}{9}xy^4\right)^2$

Exercise:

Problem: ⓐ $(2r^2)^3(4r)^2$ ⓑ $(3x^3)^3(x^5)^4$

Solution:

ⓐ $128r^8$ ⓑ $\frac{1}{200}j^{16}$

Exercise:

Problem: ⓐ $(m^2n)^2(2mn^5)^4$ ⓑ $(3pq^4)^2(6p^6q)^2$

Multiply Monomials

In the following exercises, multiply the monomials.

Exercise:

Problem: $\left(6y^7\right)\left(-3y^4\right)$

Solution:

$-18y^{11}$

Exercise:

Problem: $\left(-10x^5\right)\left(-3x^3\right)$

Exercise:

Problem: $(-8u^6)(-9u)$

Solution:

$72u^7$

Exercise:

Problem: $(-6c^4)(-12c)$

Exercise:

Problem: $(\frac{1}{5}f^8)(20f^3)$

Solution:

$4f^{11}$

Exercise:

Problem: $(\frac{1}{4}d^5)(36d^2)$

Exercise:

Problem: $(4a^3b)(9a^2b^6)$

Solution:

$36a^5b^7$

Exercise:

Problem: $(6m^4n^3)(7mn^5)$

Exercise:

Problem: $\left(\frac{4}{7}rs^2\right)\left(14rs^3\right)$

Solution:

$8r^2s^5$

Exercise:

Problem: $\left(\frac{5}{8}x^3y\right)\left(24x^5y\right)$

Exercise:

Problem: $\left(\frac{2}{3}x^2y\right)\left(\frac{3}{4}xy^2\right)$

Solution:

$\frac{1}{2}x^3y^3$

Exercise:

Problem: $\left(\frac{3}{5}m^3n^2\right)\left(\frac{5}{9}m^2n^3\right)$

Mixed Practice

In the following exercises, simplify each expression.
Exercise:

Problem: $\left(x^2\right)^4 \cdot \left(x^3\right)^2$

Solution:

x^{14}

Exercise:

Problem: $\left(y^4\right)^3 \cdot \left(y^5\right)^2$

Exercise:

Problem: $\left(a^2\right)^6 \cdot \left(a^3\right)^8$

Solution:

a^{36}

Exercise:

Problem: $\left(b^7\right)^5 \cdot \left(b^2\right)^6$

Exercise:

Problem: $(2m^6)^3$

Solution:

$8m^{18}$

Exercise:

Problem: $(3y^2)^4$

Exercise:

Problem: $(10x^2y)^3$

Solution:

$1000x^6y^3$

Exercise:

Problem: $(2mn^4)^5$

Exercise:

Problem: $(-2a^3b^2)^4$

Solution:

$16a^{12}b^8$

Exercise:

Problem: $(-10u^2v^4)^3$

Exercise:

Problem: $\left(\frac{2}{3}x^2y\right)^3$

Solution:

$\frac{8}{27}x^6y^3$

Exercise:

Problem: $\left(\frac{7}{9}pq^4\right)^2$

Exercise:

Problem: $(8a^3)^2(2a)^4$

Solution:

$1024a^{10}$

Exercise:

Problem: $(5r^2)^3(3r)^2$

Exercise:

Problem: $(10p^4)^3(5p^6)^2$

Solution:

$25000p^{24}$

Exercise:

Problem: $(4x^3)^3(2x^5)^4$

Exercise:

Problem: $\left(\frac{1}{2}x^2y^3\right)^4\left(4x^5y^3\right)^2$

Solution:

$x^{18}y^{18}$

Exercise:

Problem: $\left(\frac{1}{3}m^3n^2\right)^4\left(9m^8n^3\right)^2$

Exercise:

Problem: $(3m^2n)^2(2mn^5)^4$

Solution:

$144m^8n^{22}$

Exercise:

Problem: $(2pq^4)^3(5p^6q)^2$

Everyday Math

Exercise:

Problem:

Email Kate emails a flyer to ten of her friends and tells them to forward it to ten of their friends, who forward it to ten of their friends, and so on. The number of people who receive the email on the second round is 10^2, on the third round is 10^3, as shown in the table below. How many people will receive the email on the sixth round? Simplify the expression to show the number of people who receive the email.

Round	Number of people
1	10
2	10^2
3	10^3
…	…
6	?

Solution:

1,000,000

Exercise:

Problem:

Salary Jamal’s boss gives him a 3% raise every year on his birthday. This means that each year, Jamal’s salary is 1.03 times his last year’s salary. If his original salary was $35,000, his salary after 1 year was $\$35{,}000\,(1.03)$, after 2 years was $\$35{,}000(1.03)^2$, after 3 years was $\$35{,}000(1.03)^3$, as shown in the table below. What will Jamal’s salary be after 10 years? Simplify the expression, to show Jamal’s salary in dollars.

Year	**Salary**
1	$\$35{,}000\,(1.03)$
2	$\$35{,}000(1.03)^2$
3	$\$35{,}000(1.03)^3$
…	…
10	?

Exercise:

Problem:

Clearance A department store is clearing out merchandise in order to make room for new inventory. The plan is to mark down items by 30% each week. This means that each week the cost of an item is 70% of the previous week's cost. If the original cost of a sofa was \$1,000, the cost for the first week would be $\$1,000\,(0.70)$ and the cost of the item during the second week would be $\$1,000(0.70)^2$. Complete the table shown below. What will be the cost of the sofa during the fifth week? Simplify the expression, to show the cost in dollars.

Week	**Cost**
1	$\$1,000\,(0.70)$
2	$\$1,000(0.70)^2$
3	
…	…
8	?

Solution:

\$168.07

Exercise:

Problem:

Depreciation Once a new car is driven away from the dealer, it begins to lose value. Each year, a car loses 10% of its value. This means that each year the value of a car is 90% of the previous year's value. If a new car was purchased for $20,000, the value at the end of the first year would be $\$20{,}000\,(0.90)$ and the value of the car after the end of the second year would be $\$20{,}000(0.90)^2$. Complete the table shown below. What will be the value of the car at the end of the eighth year? Simplify the expression, to show the value in dollars.

Week	**Cost**
1	$\$20{,}000\,(0.90)$
2	$\$20{,}000(0.90)^2$
3	
4	…
5	?

Writing Exercises

Exercise:

Problem:

Use the Product Property for Exponents to explain why $x \cdot x = x^2$.

Solution:

Answers will vary.

Exercise:

Problem: Explain why $-5^3 = (-5)^3$ but $-5^4 \neq (-5)^4$.

Exercise:

Problem: Jorge thinks $\left(\frac{1}{2}\right)^2$ is 1. What is wrong with his reasoning?

Solution:

Answers will vary.

Exercise:

Problem: Explain why $x^3 \cdot x^5$ is x^8, and not x^{15}.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
simplify expressions with exponents.			
simplify expressions using the Product Property for Exponents.			
simplify expressions using the Power Property for Exponents.			
simplify expressions using the Product to a Power Property.			
simplify expressions by applying several properties.			
multiply monomials.			

ⓑ After reviewing this checklist, what will you do to become confident for all goals?

Multiply Polynomials: ASE

By the end of this section, you will be able to:

- Multiply a polynomial by a monomial
- Multiply a binomial by a binomial
- Multiply a trinomial by a binomial

Multiply a Polynomial by a Monomial

We have used the Distributive Property to simplify expressions like $2(x-3)$. You multiplied both terms in the parentheses, x and 3, by 2, to get $2x-6$. With this chapter’s new vocabulary, you can say you were multiplying a binomial, $x-3$, by a monomial, 2.

Multiplying a binomial by a monomial is nothing new for you! Here’s an example:

Example:
Exercise:

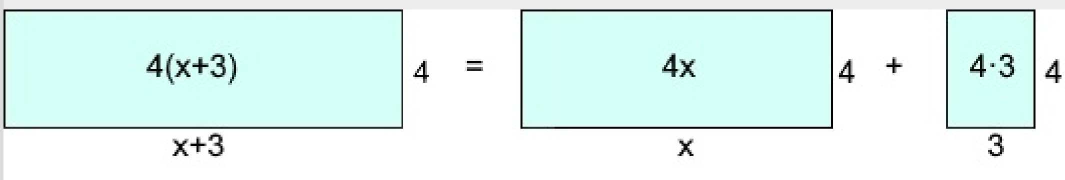
Problem: Multiply: $4\left(x+3\right)$.

Solution:
Solution

	$4(x+3)$
Distribute.	$4 \cdot x + 4 \cdot 3$

Simplify.	$4x + 12$

Area Model of Multiplication and the Distributive Property
We can visualize multiplication as the area of a rectangle where the length and width of the rectangle are the two factors being multiplied. In the above example, either the length or the width is 4 and the other is x + 3. We can visualize the distributive property as breaking this rectangle into two smaller rectangles. One will be 4 by x and the other 4 by 3. We know that the area of the 4 by 3 rectangle is 12, but the x by x rectangle can not be simplified beyond 4x.



Note:
Exercise:

Problem: Multiply: $5\left(x+7\right)$.

Solution:

$5x+35$

Note:
Exercise:

Problem: Multiply: $3\left(y+13\right)$.

Solution:

$3y+39$

Example:
Exercise:

Problem: Multiply: $y\left(y-2\right)$.

Solution:
Solution

	$y(y-2)$
Distribute.	$y\cdot y-y\cdot 2$
Simplify.	y^2-2y

Note:
Exercise:

Problem: Multiply: $x\left(x-7\right)$.

Solution:

x^2-7x

Note:
Exercise:

Problem: Multiply: $d\left(d-11\right)$.

Solution:

d^2-11d

Example:
Exercise:

Problem: Multiply: $7x(2x+y)$.

Solution:
Solution

	$7x(2x+y)$
Distribute.	$7x \cdot 2x + 7x \cdot y$
Simplify.	$14x^2 + 7xy$

Note:
Exercise:

Problem: Multiply: $5x(x+4y)$.

Solution:

$5x^2 + 20xy$

Note:
Exercise:

Problem: Multiply: $2p(6p+r)$.

Solution:

$12p^2 + 2pr$

Example:
Exercise:

Problem: Multiply: $-2y\left(4y^2+3y-5\right)$.

Solution:
Solution

	$-2y(4y^2+3y-5)$
Distribute.	$-2y \cdot 4y^2+(-2y) \cdot 3y-(-2y) \cdot 5$
Simplify.	$-8y^3-6y^2+10y$

Note:
Exercise:

Problem: Multiply: $-3y\left(5y^2+8y-7\right)$.

Solution:

$-15y^3-24y^2+21y$

Note:
Exercise:

Problem: Multiply: $4x^2(2x^2 - 3x + 5)$.

Solution:

$8x^4 - 24x^3 + 20x^2$

Example:
Exercise:

Problem: Multiply: $2x^3(x^2 - 8x + 1)$.

Solution:
Solution

	$2x^3(x^2 - 8x + 1)$
Distribute.	$2x^3 \cdot x^2 + (2x^3) \cdot (-8x) + (2x^3) \cdot 1$
Simplify.	$2x^5 - 16x^4 + 2x^3$

Note:
Exercise:

Problem: Multiply: $4x\left(3x^2 - 5x + 3\right)$.

Solution:

$12x^3 - 20x^2 + 12x$

Note:
Exercise:

Problem: Multiply: $-6a^3(3a^2 - 2a + 6)$.

Solution:

$-18a^5 + 12a^4 - 36a^3$

Example:
Exercise:

Problem: Multiply: $(x + 3)p$.

Solution:
Solution

The monomial is the second factor.	$(x + 3)p$
Distribute.	$x \cdot p + 3 \cdot p$
Simplify.	$xp + 3p$

Note:
Exercise:

Problem: Multiply: $(x + 8)p$.

Solution:

$xp + 8p$

Note:
Exercise:

Problem: Multiply: $(a + 4)p$.

Solution:

$ap + 4p$

Multiply a Binomial by a Binomial

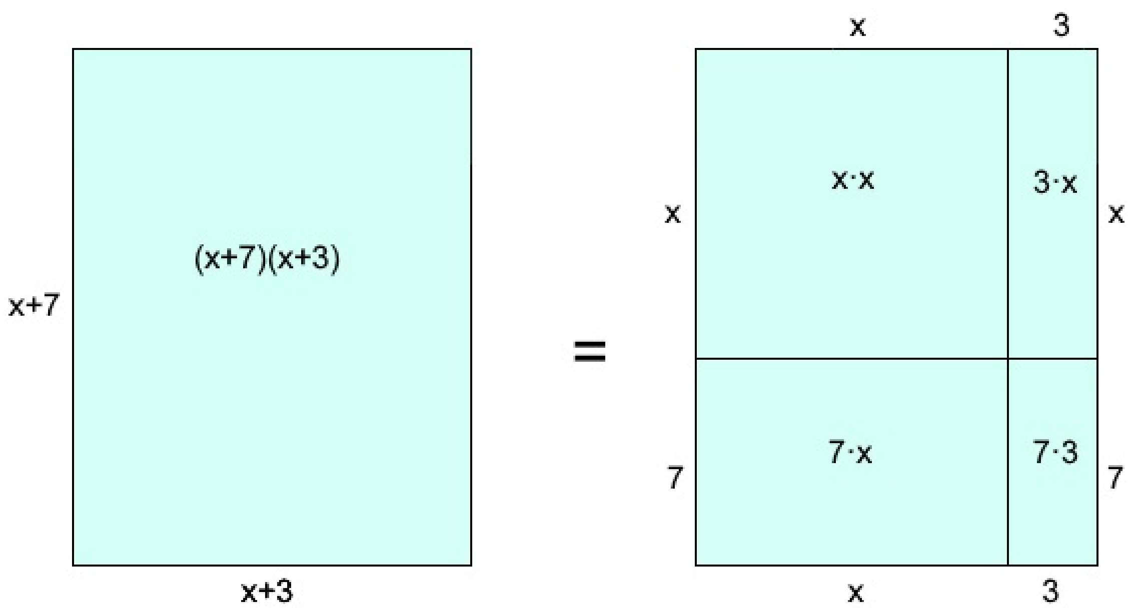
Just like there are different ways to represent multiplication of numbers, there are several methods that can be used to multiply a binomial times a binomial. We will start by using the Distributive Property.

Multiply a Binomial by a Binomial Using the Distributive Property

Look at [link], where we multiplied a binomial by a monomial.

	$(x+3)p$
We distributed the p to get:	$xp+3p$
What if we have $(x+7)$ instead of p?	$(x+3)(x+7)$
Distribute $(x+7)$.	$x(x+7)+3(x+7)$
Distribute again.	$x^2+7x+3x+21$
Combine like terms.	$x^2+10x+21$

Notice that before combining like terms, you had four terms. You multiplied the two terms of the first binomial by the two terms of the second binomial—four multiplications. You can see this if you look at the multiplication using the area model.



Example:
Exercise:

Problem: Multiply: $(y+5)(y+8)$.

Solution:
Solution

	$(y+5)(y+8)$
Distribute (y + 8).	$y(y+8)+5(y+8)$
Distribute again	$y^2+8y+5y+40$
Combine like terms.	$y^2+13y+40$

Note:
Exercise:

Problem: Multiply: $(x+8)(x+9)$.

Solution:

$x^2+17x+72$

Note:
Exercise:

Problem: Multiply: $(5x+9)(4x+3)$.

Solution:

$20x^2+51x+27$

Example:
Exercise:

Problem: Multiply: $(2y+5)(3y+4)$.

Solution:
Solution

	$(2y+5)(3y+4)$
Distribute (3*y* + 4).	$2y(3y+4)+5(3y+4)$
Distribute again	$6y^2+8y+15y+20$
Combine like terms.	$6y^2+23y+20$

Note:
Exercise:

Problem: Multiply: $(3b+5)(4b+6)$.

Solution:

$12b^2+38b+30$

Note:
Exercise:

Problem: Multiply: $(a+10)(a+7)$.

Solution:

$a^2+17a+70$

Example:
Exercise:

Problem: Multiply: $(4y+3)(2y-5)$.

Solution:
Solution

	$(4y+3)(2y-5)$

Distribute.	$4y(2y-5)+3(2y-5)$
Distribute again.	$8y^2-20y+6y-15$
Combine like terms.	$8y^2-14y-15$

Note:
Exercise:

Problem: Multiply: $(5y+2)(6y-3)$.

Solution:

$30y^2-3y-6$

Note:
Exercise:

Problem: Multiply: $(3c+4)(5c-2)$.

Solution:

$15c^2+14c-8$

Example:
Exercise:

Problem: Multiply: $(x+2)(x-y)$.

Solution:
Solution

	$(x-2)(x-y)$
Distribute.	$x(x-y)-2(x-y)$
Distribute again.	$x^2-xy-2x+2y$
There are no like terms to combine.	

Note:
Exercise:

Problem: Multiply: $(a+7)(a-b)$.

Solution:

$a^2 - ab + 7a - 7b$

Note:
Exercise:

Problem: Multiply: $(x + 5)(x - y)$.

Solution:

$x^2 - xy + 5x - 5y$

Multiply a Binomial by a Binomial Using the FOIL Method

Remember that when you multiply a binomial by a binomial you get four terms. Sometimes you can combine like terms to get a trinomial, but sometimes, like in [link], there are no like terms to combine.

Let's look at the last example again and pay particular attention to how we got the four terms.
Equation:

$$(x - 2)(x - y)$$
$$x^2 - xy - 2x + 2y$$

Where did the first term, x^2, come from?

It is the product of x and x, the *first* terms in $(x – 2)$ and $(x – y)$.

$(x – 2)(x – y)$

First

The next term, $–xy$, is the product of x and $–y$, the two *outer* terms.

$(x – 2)(x – y)$

Outer

The third term, $–2x$, is the product of $–2$ and x, the two *inner* terms.

$(x – 2)(x – y)$

Inner

And the last term, $+2y$, came from multiplying the two *last* terms, $–2$ and $–y$.

$(x – 2)(x – y)$

Last

We abbreviate “First, Outer, Inner, Last” as FOIL. The letters stand for ‘**F**irst, **O**uter, **I**nner, **L**ast’. The word FOIL is easy to remember and ensures we find all four products.

Equation:

$$(x-2)(x-y)$$
$$\underset{F}{x^2} - \underset{O}{xy} - \underset{I}{2x} + \underset{L}{2y}$$

Let’s look at $(x+3)\,(x+7)$.

Distributive Property	FOIL

$(x+3)(x+7)$	$(x+3)(x+7)$
$x(x+7)+3(x+7)$	
$\underset{F}{x^2} + \underset{O}{7x} + \underset{I}{3x} + \underset{L}{21}$	$\underset{F}{x^2} + \underset{O}{7x} + \underset{I}{3x} + \underset{L}{21}$
$x^2+10x+21$	$x^2+10x+21$

Notice how the terms in third line fit the FOIL pattern.

Now we will do an example where we use the FOIL pattern to multiply two binomials.

Example:
How to Multiply a Binomial by a Binomial using the FOIL Method
Exercise:

Problem: Multiply using the FOIL method: $(x+5)\,(x+9)$.

Solution:
Solution

Step 1. Multiply the *First* terms.	$(x+5)(x+9)$ $(x+5)(x+9)$	$\underset{F}{x^2} + \underset{O}{\_\_} + \underset{I}{\_\_} + \underset{L}{\_\_}$

Step 2. Multiply the *Outer* terms.	$(x+5)(x+9)$	$\underset{F}{x^2} + \underset{O}{9x} + \underset{I}{\_} + \underset{L}{\_}$
Step 3. Multiply the *Inner* terms.	$(x+5)(x+9)$	$\underset{F}{x^2} + \underset{O}{9x} + \underset{I}{5x} + \underset{L}{\_}$
Step 4. Multiply the *Last* terms.	$(x+5)(x+9)$	$\underset{F}{x^2} + \underset{O}{9x} + \underset{I}{5x} + \underset{L}{45}$
Step 5. Combine like terms, when possible.		$x^2 + 14x + 45$

Note:
Exercise:

Problem: Multiply using the FOIL method: $(x+6)(x+8)$.

Solution:

$x^2 + 14x + 48$

Note:
Exercise:

Problem: Multiply using the FOIL method: $(y+17)(y+3)$.

Solution:

$y^2 + 20y + 51$

We summarize the steps of the FOIL method below. The FOIL method only applies to multiplying binomials, not other polynomials!

Note:
Multiply two binomials using the FOIL method

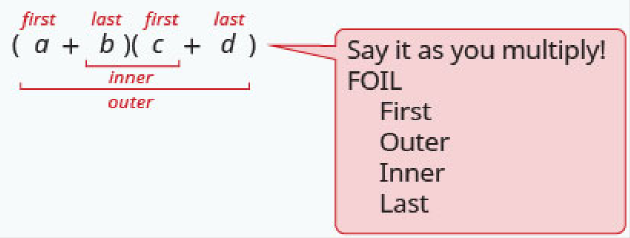
Step 1. Multiply the *First* terms.

Step 2. Multiply the *Outer* terms.

Step 3. Multiply the *Inner* terms.

Step 4. Multiply the *Last* terms.

Step 5. Combine like terms, when possible.



When you multiply by the FOIL method, drawing the lines will help your brain focus on the pattern and make it easier to apply.

Example:
Exercise:

Problem: Multiply: $(y-7)(y+4)$.

Solution:
Solution

		$(y-7)(y+4)$
Multiply the *First* terms.	$(y-7)(y+4)$	$\underset{F}{y^2} + \underset{O}{\_\_} + \underset{I}{\_\_} + \underset{L}{\_\_}$
Multiply the *Outer* terms.	$(y-7)(y+4)$	$\underset{F}{y^2} + \underset{O}{4y} + \underset{I}{\_\_} + \underset{L}{\_\_}$
Multiply the *Inner* terms.	$(y-7)(y+4)$	$\underset{F}{y^2} + \underset{O}{4y} \underset{I}{- 7y} + \underset{L}{\_\_}$
Multiply the *Last* terms.	$(y-7)(y+4)$	$\underset{F}{y^2} + \underset{O}{4y} \underset{I}{- 7y} \underset{L}{-28}$
Combine like terms.		$y^2 - 3y - 28$

Note:
Exercise:

Problem: Multiply: $(x-7)(x+5)$.

Solution:

$x^2 - 2x - 35$

Note:
Exercise:

Problem: Multiply: $(b-3)(b+6)$.

Solution:

$b^2 + 3b - 18$

Example:
Exercise:

Problem: Multiply: $(4x + 3)\,(2x - 5)$.

Solution:
Solution

	$(4x - 3)(2x - 5)$
	$(4x + 3)(2x - 5)$
Multiply the *First* terms, $4x \cdot 2x$.	$8x^2 + \underset{O}{\_\_} + \underset{I}{\_\_} + \underset{L}{\_\_}$, with $8x^2$ marked F
Multiply the *Outer* terms, $4x \cdot (-5)$.	$\underset{F}{8x^2} \underset{O}{- 20x} + \underset{I}{\_\_} + \underset{L}{\_\_}$
Multiply the *Inner* terms, $3 \cdot 2x$.	$\underset{F}{8x^2} \underset{O}{- 2x} \underset{I}{+ 6x} + \underset{L}{\_\_}$
Multiply the *Last* terms, $3 \cdot (-5)$.	$\underset{F}{8x^2} \underset{O}{- 20x} \underset{I}{+ 6x} \underset{L}{-15}$
Combine like terms.	$8x^2 - 14x - 15$

Note:
Exercise:

Problem: Multiply: $(3x+7)(5x-2)$.

Solution:

$15x^2+29x-14$

Note:
Exercise:

Problem: Multiply: $(4y+5)(4y-10)$.

Solution:

$16y^2-20y-50$

The final products in the last four examples were trinomials because we could combine the two middle terms. This is not always the case.

Example:
Exercise:

Problem: Multiply: $(3x-y)(2x-5)$.

Solution:
Solution

	$(3x - y)(2x - 5)$
	$(3x - y)(2x - 5)$
Multiply the *First*.	$\underset{F}{6x^2} + \underset{O}{\_\_} + \underset{I}{\_\_} + \underset{L}{\_\_}$
Multiply the *Outer*.	$\underset{F}{6x^2} \underset{O}{- 15x} + \underset{I}{\_\_} + \underset{L}{\_\_}$
Multiply the *Inner*.	$\underset{F}{6x^2} \underset{O}{- 15x} \underset{I}{- 2xy} + \underset{L}{\_\_}$
Multiply the *Last*.	$\underset{F}{6x^2} \underset{O}{- 15x} \underset{I}{- 2xy} \underset{L}{+ 5y}$
Combine like terms—there are none.	$6x^2 - 15x - 2xy + 5y$

Note:
Exercise:

Problem: Multiply: $(10c - d)\,(c - 6)$.

Solution:

$10c^2 - 60c - cd + 6d$

Note:
Exercise:

Problem: Multiply: $(7x - y)(2x - 5)$.

Solution:

$14x^2 - 35x - 2xy + 10y$

Be careful of the exponents in the next example.

Example:
Exercise:

Problem: Multiply: $(n^2 + 4)(n - 1)$.

Solution:
Solution

	$(n^2 + 4)(n - 1)$

	$(n^2+4)(n-1)$
Multiply the *First*.	$\underset{F}{n^3} + \underset{O}{\_\_} + \underset{I}{\_\_} + \underset{L}{\_\_}$
Multiply the *Outer*.	$\underset{F}{n^3} \underset{O}{- n^2} + \underset{I}{\_\_} + \underset{L}{\_\_}$
Multiply the *Inner*.	$\underset{F}{n^3} \underset{O}{- n^2} \underset{I}{+ 4n} + \underset{L}{\_\_}$
Multiply the *Last*.	$\underset{F}{n^3} \underset{O}{- n^2} \underset{I}{+ 4n} \underset{L}{- 4}$
Combine like terms—there are none.	$n^3 - n^2 + 4n - 4$

Note:
Exercise:

Problem: Multiply: $\left(x^2+6\right)(x-8)$.

Solution:

$x^3 - 8x^2 + 6x - 48$

Note:
Exercise:

Problem: Multiply: $\left(y^2+7\right)(y-9)$.

Solution:

$y^3 - 9y^2 + 7y - 63$

Example:
Exercise:

Problem: Multiply: $(3pq+5)(6pq-11)$.

Solution:
Solution

	$(3pq+5)(6pq-11)$	
Multiply the *First*.	$\underset{F}{18p^2q^2} + \underset{O}{\_\_} + \underset{I}{\_\_} + \underset{L}{\_\_}$	$(3pq+5)(6pq-11)$
Multiply the *Outer*.		

	$\underset{F}{18p^2q^2} \underset{O}{- 33pq} + \underset{I}{\_} + \underset{L}{\_}$	
Multiply the *Inner*.	$\underset{F}{18p^2q^2} \underset{O}{- 33pq} \underset{I}{+ 30pq} + \underset{L}{\_}$	
Multiply the *Last*.	$\underset{F}{18p^2q^2} \underset{O}{- 33pq} \underset{I}{+ 30pq} \underset{L}{- 55}$	
Combine like terms—there are none.	$18p^2q^2 - 3pq - 55$	

Note:
Exercise:

Problem: Multiply: $(2ab+5)(4ab-4)$.

Solution:

$8a^2b^2 + 12ab - 20$

Note:
Exercise:

Problem: Multiply: $(2xy+3)(4xy-5)$.

Solution:

$8x^2y^2 + 2xy - 15$

Multiply a Binomial by a Binomial Using the Vertical Method

The FOIL method is usually the quickest method for multiplying two binomials, but it *only* works for binomials. You can use the Distributive Property to find the product of any two polynomials. Another method that works for all polynomials is the Vertical Method. It is very much like the method you use to multiply whole numbers. Look carefully at this example of multiplying two-digit numbers.

23	
×46	
138 partial product	Start by multiplying 23 by 6 to get 138.
92 partial product	Next, multiply 23 by 4, lining up the partial product in the correct columns.
1058 product	Last you add the partial products.

Now we'll apply this same method to multiply two binomials.

Example:
Exercise:

Problem: Multiply using the Vertical Method: $(3y-1)(2y-6)$.

Solution:
Solution

It does not matter which binomial goes on the top.

$$\begin{array}{lrl} \text{Multiply } 3y-1 \text{ by } -6. & 3y-1 & \\ \text{Multiply } 3y-1 \text{ by 2y.} & \underline{\times\ 2y-6} & \\ & -18y+6 & \text{partial product} \\ & \underline{6y^2-2y} & \text{partial product} \\ \text{Add like terms.} & 6y^2-20y+6 & \text{product} \end{array}$$

Notice the partial products are the same as the terms in the FOIL method.

$$\begin{array}{r} (3y-1)(2y-6) \\ 6y^2-2y-18y+6 \\ \hline 6y^2-20y+6 \end{array} \qquad \begin{array}{r} 3y-1 \\ \times\ 2y-6 \\ \hline -18y+6 \\ 6y^2-2y \quad\;\; \\ \hline 6y^2-20x+6 \end{array}$$

Note:
Exercise:

Problem: Multiply using the Vertical Method: $(5m-7)(3m-6)$.

Solution:

$15m^2-51m+42$

Note:
Exercise:

Problem: Multiply using the Vertical Method: $(6b-5)(7b-3)$.

Solution:

$42b^2-53b+15$

We have now used three methods for multiplying binomials. Be sure to practice each method, and try to decide which one you prefer. The methods are listed here all together, to help you remember them.

Note:
Multiplying Two Binomials
To multiply binomials, use the:

- Distributive Property
- FOIL Method
- Vertical Method

Remember, FOIL only works when multiplying two binomials.

Multiply a Trinomial by a Binomial

We have multiplied monomials by monomials, monomials by polynomials, and binomials by binomials. Now we're ready to multiply a trinomial by a binomial. Remember, FOIL will not work in this case, but we can use either the Distributive Property or the Vertical Method. We first look at an example using the Distributive Property.

Example:
Exercise:

Problem:

Multiply using the Distributive Property: $(b+3)(2b^2-5b+8)$.

Solution:
Solution

	$(b+3)(2b^2-5b+8)$
Distribute.	$b(2b^2-5b+8)+3(2b^2-5b+8)$
Multiply.	$2b^3-5b^2+8b+6b^2-15b+24$
Combine like terms.	$2b^3+b^2-7b+24$

Note:
Exercise:

Problem:

Multiply using the Distributive Property: $(y-3)(y^2-5y+2)$.

Solution:

$y^3-8y^2+17y-6$

Note:
Exercise:

Problem:

Multiply using the Distributive Property: $(x+4)(2x^2-3x+5)$.

Solution:

$2x^3+5x^2-7x+20$

Now let’s do this same multiplication using the Vertical Method.

Example:
Exercise:

Problem: Multiply using the Vertical Method: $(b+3)(2b^2-5b+8)$.

Solution:
Solution

It is easier to put the polynomial with fewer terms on the bottom because we get fewer partial products this way.

Multiply $(2b^2 - 5b + 8)$ by 3.	$\begin{array}{r} 2b^2-5b+8 \\ \times \quad b+3 \\ \hline 6b^2-15b+24 \end{array}$

	$2b^3 - 5b^2 + 8b$
Multiply $(2b^2 - 5b + 8)$ by b.	$2b^3 + b^2 - 7b + 24$
Add like terms.	

Note:
Exercise:

Problem: Multiply using the Vertical Method: $(y-3)(y^2-5y+2)$.

Solution:

$y^3 - 8y^2 + 17y - 6$

Note:
Exercise:

Problem:

Multiply using the Vertical Method: $(x+4)(2x^2-3x+5)$.

Solution:

$2x^3 + 5x^2 - 7x + 20$

We have now seen two methods you can use to multiply a trinomial by a binomial. After you practice each method, you’ll probably find you prefer one way over the other. We list both methods are listed here, for easy reference.

Note:
Multiplying a Trinomial by a Binomial
To multiply a trinomial by a binomial, use the:

- Distributive Property
- Vertical Method

Note:
Access these online resources for additional instruction and practice with multiplying polynomials:

- Multiplying Exponents 1
- Multiplying Exponents 2
- Multiplying Exponents 3

Key Concepts

- **FOIL Method for Multiplying Two Binomials**—To multiply two binomials:

  Multiply the**First**terms.
  Multiply the**Outer**terms.
  Multiply the**Inner**terms.
  Multiply the**Last**terms.

- **Multiplying Two Binomials**—To multiply binomials, use the:

- Distributive Property ([link])
- FOIL Method ([link])
- Vertical Method ([link])

- **Multiplying a Trinomial by a Binomial**—To multiply a trinomial by a binomial, use the:
  - Distributive Property ([link])
  - Vertical Method ([link])

Practice Makes Perfect

Multiply a Polynomial by a Monomial

In the following exercises, multiply.

Exercise:

Problem: $4(w+10)$

Solution:

$4w+40$

Exercise:

Problem: $6(b+8)$

Exercise:

Problem: $-3(a+7)$

Solution:

$-3a-21$

Exercise:

Problem: $-5(p+9)$

Exercise:

Problem: $2(x-7)$

Solution:

$2x-14$

Exercise:

Problem: $7(y-4)$

Exercise:

Problem: $-3(k-4)$

Solution:

$-3k+12$

Exercise:

Problem: $-8(j-5)$

Exercise:

Problem: $q(q+5)$

Solution:

q^2+5q

Exercise:

Problem: $k(k+7)$

Exercise:

Problem: $-b(b+9)$

Solution:

$-b^2-9b$

Exercise:

Problem: $-y(y+3)$

Exercise:

Problem: $-x(x-10)$

Solution:

$-x^2+10x$

Exercise:

Problem: $-p(p-15)$

Exercise:

Problem: $6r(4r+s)$

Solution:

$24r^2+6rs$

Exercise:

Problem: $5c(9c+d)$

Exercise:

Problem: $12x(x-10)$

Solution:

$12x^2 - 120x$

Exercise:

Problem: $9m(m - 11)$

Exercise:

Problem: $-9a(3a + 5)$

Solution:

$-27a^2 - 45a$

Exercise:

Problem: $-4p(2p + 7)$

Exercise:

Problem: $3(p^2 + 10p + 25)$

Solution:

$3p^2 + 30p + 75$

Exercise:

Problem: $6(y^2 + 8y + 16)$

Exercise:

Problem: $-8x(x^2 + 2x - 15)$

Solution:

$-8x^3 - 16x^2 + 120x$

Exercise:

Problem: $-5t(t^2 + 3t - 18)$

Exercise:

Problem: $5q^3\left(q^3 - 2q + 6\right)$

Solution:

$5q^6 - 10q^4 + 30q^3$

Exercise:

Problem: $4x^3\left(x^4 - 3x + 7\right)$

Exercise:

Problem: $-8y(y^2 + 2y - 15)$

Solution:

$-8y^3 - 16y^2 + 120y$

Exercise:

Problem: $-5m(m^2 + 3m - 18)$

Exercise:

Problem: $5q^3(q^2 - 2q + 6)$

Solution:

$5q^5 - 10q^4 + 30q^3$

Exercise:

Problem: $9r^3(r^2 - 3r + 5)$

Exercise:

Problem: $-4z^2(3z^2 + 12z - 1)$

Solution:

$-12z^4 - 48z^3 + 4z^2$

Exercise:

Problem: $-3x^2(7x^2 + 10x - 1)$

Exercise:

Problem: $(2m - 9)m$

Solution:

$2m^2 - 9m$

Exercise:

Problem: $(8j - 1)j$

Exercise:

Problem: $(w - 6) \cdot 8$

Solution:

$8w - 48$

Exercise:

Problem: $(k - 4) \cdot 5$

Exercise:

Problem: $4(x+10)$

Solution:

$4x+40$

Exercise:

Problem: $6(y+8)$

Exercise:

Problem: $15(r-24)$

Solution:

$15r-360$

Exercise:

Problem: $12(v-30)$

Exercise:

Problem: $-3(m+11)$

Solution:

$-3m-33$

Exercise:

Problem: $-4(p+15)$

Exercise:

Problem: $-8(z-5)$

Solution:

$-8z+40$

Exercise:

Problem: $-3(x-9)$

Exercise:

Problem: $u(u+5)$

Solution:

u^2+5u

Exercise:

Problem: $q(q+7)$

Exercise:

Problem: $n(n^2-3n)$

Solution:

n^3-3n^2

Exercise:

Problem: $s(s^2-6s)$

Exercise:

Problem: $6x(4x+y)$

Solution:

$24x^2 + 6xy$

Exercise:

Problem: $5a(9a + b)$

Exercise:

Problem: $5p(11p - 5q)$

Solution:

$55p^2 - 25pq$

Exercise:

Problem: $12u(3u - 4v)$

Exercise:

Problem: $3(v^2 + 10v + 25)$

Solution:

$3v^2 + 30v + 75$

Exercise:

Problem: $6(x^2 + 8x + 16)$

Exercise:

Problem: $2n(4n^2 - 4n + 1)$

Solution:

$8n^3 - 8n^2 + 2n$

Exercise:

Problem: $3r(2r^2 - 6r + 2)$

Exercise:

Problem: $-8y(y^2 + 2y - 15)$

Solution:

$-8y^3 - 16y^2 + 120y$

Exercise:

Problem: $-5m(m^2 + 3m - 18)$

Exercise:

Problem: $5q^3(q^2 - 2q + 6)$

Solution:

$5q^5 - 10q^4 + 30q^3$

Exercise:

Problem: $9r^3(r^2 - 3r + 5)$

Exercise:

Problem: $-4z^2(3z^2 + 12z - 1)$

Solution:

$-12z^4 - 48z^3 + 4z^2$

Exercise:

Problem: $-3x^2(7x^2 + 10x - 1)$

Exercise:

Problem: $(2y-9)y$

Solution:

$18y^2-9y$

Exercise:

Problem: $(8b-1)b$

Multiply a Binomial by a Binomial

In the following exercises, multiply the following binomials using: ⓐ the Distributive Property ⓑ the FOIL method ⓒ the Vertical Method.

Exercise:

Problem: $(w+5)(w+7)$

Solution:

$w^2+12w+35$

Exercise:

Problem: $(y+9)(y+3)$

Exercise:

Problem: $(p+11)(p-4)$

Solution:

$p^2+7p-44$

Exercise:

Problem: $(q+4)(q-8)$

In the following exercises, multiply the binomials. Use any method.

Exercise:

Problem: $(x+8)(x+3)$

Solution:

$x^2+11x+24$

Exercise:

Problem: $(y+7)(y+4)$

Exercise:

Problem: $(y-6)(y-2)$

Solution:

$y^2-8y+12$

Exercise:

Problem: $(x-7)(x-2)$

Exercise:

Problem: $(w-4)(w+7)$

Solution:

$w^2+3w-28$

Exercise:

Problem: $(q-5)(q+8)$

Exercise:

Problem: $(p+12)(p-5)$

Solution:

$p^2+7p-60$

Exercise:

Problem: $(m+11)(m-4)$

Exercise:

Problem: $(6p+5)(p+1)$

Solution:

$6p^2+11p+5$

Exercise:

Problem: $(7m+1)(m+3)$

Exercise:

Problem: $(2t-9)(10t+1)$

Solution:

$20t^2-88t-9$

Exercise:

Problem: $(3r-8)(11r+1)$

Exercise:

Problem: $(5x - y)\,(3x - 6)$

Solution:

$15x^2 - 3xy - 30x + 6y$

Exercise:

Problem: $(10a - b)\,(3a - 4)$

Exercise:

Problem: $(a + b)(2a + 3b)$

Solution:

$2a^2 + 5ab + 3b^2$

Exercise:

Problem: $(r + s)(3r + 2s)$

Exercise:

Problem: $(4z - y)(z - 6)$

Solution:

$4z^2 - 24z - zy + 6y$

Exercise:

Problem: $(5x - y)(x - 4)$

Exercise:

Problem: $(x^2 + 3)(x + 2)$

Solution:

$x^3 + 2x^2 + 3x + 6$

Exercise:

Problem: $(y^2 - 4)(y + 3)$

Exercise:

Problem: $(x^2 + 8)(x^2 - 5)$

Solution:

$x^4 + 3x^2 - 40$

Exercise:

Problem: $(y^2 - 7)(y^2 - 4)$

Exercise:

Problem: $(5ab - 1)(2ab + 3)$

Solution:

$10a^2b^2 + 13ab - 3$

Exercise:

Problem: $(2xy + 3)(3xy + 2)$

Exercise:

Problem: $(6pq - 3)(4pq - 5)$

Solution:

$24p^2q^2 - 42pq + 15$

Exercise:

Problem: $(3rs - 7)(3rs - 4)$

Multiply a Trinomial by a Binomial

In the following exercises, multiply using ⓐ the Distributive Property ⓑ the Vertical Method.

Exercise:

Problem: $(x + 5)(x^2 + 4x + 3)$

Solution:

$x^3 + 9x^2 + 23x + 15$

Exercise:

Problem: $(u + 4)(u^2 + 3u + 2)$

Exercise:

Problem: $(y + 8)(4y^2 + y - 7)$

Solution:

$4y^3 + 33y^2 + y - 56$

Exercise:

Problem: $(a + 10)(3a^2 + a - 5)$

In the following exercises, multiply. Use either method.

Exercise:

Problem: $(w-7)(w^2-9w+10)$

Solution:

$w^3-16w^2+73w-70$

Exercise:

Problem: $(p-4)(p^2-6p+9)$

Exercise:

Problem: $(3q+1)(q^2-4q-5)$

Solution:

$3q^3-11q^2-19q-5$

Exercise:

Problem: $(6r+1)(r^2-7r-9)$

Mixed Practice

Exercise:

Problem: $(10y-6)+(4y-7)$

Solution:

$14y-13$

Exercise:

Problem: $(15p-4)+(3p-5)$

Exercise:

Problem: $\left(x^2 - 4x - 34\right) - \left(x^2 + 7x - 6\right)$

Solution:

$-11x - 28$

Exercise:

Problem: $\left(j^2 - 8j - 27\right) - \left(j^2 + 2j - 12\right)$

Exercise:

Problem: $5q(3q^2 - 6q + 11)$

Solution:

$15q^3 - 30q^2 + 55q$

Exercise:

Problem: $8t(2t^2 - 5t + 6)$

Exercise:

Problem: $(s - 7)(s + 9)$

Solution:

$s^2 + 2s - 63$

Exercise:

Problem: $(x - 5)(x + 13)$

Exercise:

Problem: $\left(y^2 - 2y\right)(y + 1)$

Solution:

$y^3 - y^2 - 2y$

Exercise:

Problem: $\left(a^2 - 3a\right)(4a + 5)$

Exercise:

Problem: $(3n - 4)\left(n^2 + n - 7\right)$

Solution:

$3n^3 - n^2 - 25n + 28$

Exercise:

Problem: $(6k - 1)\left(k^2 + 2k - 4\right)$

Exercise:

Problem: $(7p + 10)(7p - 10)$

Solution:

$49p^2 - 100$

Exercise:

Problem: $(3y + 8)(3y - 8)$

Exercise:

Problem: $(4m^2 - 3m - 7)m^2$

Solution:

$4m^4 - 3m^3 - 7m^2$

Exercise:

Problem: $(15c^2 - 4c + 5)c^4$

Exercise:

Problem: $(5a + 7b)\,(5a + 7b)$

Solution:

$25a^2 + 70ab + 49b^2$

Exercise:

Problem: $(3x - 11y)\,(3x - 11y)$

Exercise:

Problem: $(4y + 12z)\,(4y - 12z)$

Solution:

$16y^2 - 144z^2$

Everyday Math

Exercise:

Problem:

Mental math You can use binomial multiplication to multiply numbers without a calculator. Say you need to multiply 13 times 15. Think of 13 as $10 + 3$ and 15 as $10 + 5$.

ⓐ Multiply $(10 + 3)(10 + 5)$ by the FOIL method.

ⓑ Multiply $13 \cdot 15$ without using a calculator.
ⓒ Which way is easier for you? Why?

Exercise:

Problem:

Mental math You can use binomial multiplication to multiply numbers without a calculator. Say you need to multiply 18 times 17. Think of 18 as $20 - 2$ and 17 as $20 - 3$.

ⓐ Multiply $(20 - 2)(20 - 3)$ by the FOIL method.
ⓑ Multiply $18 \cdot 17$ without using a calculator.
ⓒ Which way is easier for you? Why?

Solution:

ⓐ 306 ⓑ 306 ⓒ Answers will vary.

Writing Exercises

Exercise:

Problem:

Which method do you prefer to use when multiplying two binomials: the Distributive Property, the FOIL method, or the Vertical Method? Why?

Exercise:

Problem:

Which method do you prefer to use when multiplying a trinomial by a binomial: the Distributive Property or the Vertical Method? Why?

Solution:

Answers will vary.

Exercise:

Problem: Multiply the following:

$(x+2)(x-2)$
$(y+7)(y-7)$
$(w+5)(w-5)$

Explain the pattern that you see in your answers.

Exercise:

Problem: Multiply the following:

$(m-3)(m+3)$
$(n-10)(n+10)$
$(p-8)(p+8)$

Explain the pattern that you see in your answers.

Solution:

Answers may vary.

Exercise:

Problem: Multiply the following:

$(p+3)(p+3)$
$(q+6)(q+6)$
$(r+1)(r+1)$

Explain the pattern that you see in your answers.

Exercise:

Problem: Multiply the following:

$(x - 4)(x - 4)$
$(y - 1)(y - 1)$
$(z - 7)(z - 7)$

Explain the pattern that you see in your answers.

Solution:

Answers may vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
multiply a polynomial by a monomial.			
multiply a binomial by a binomial.			
multiply a trinomial by a binomial.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Special Products: ASE
By the end of this section, you will be able to:

- Square a binomial using the Binomial Squares Pattern
- Multiply conjugates using the Product of Conjugates Pattern
- Recognize and use the appropriate special product pattern

Square a Binomial Using the Binomial Squares Pattern

Mathematicians like to look for patterns that will make their work easier. A good example of this is squaring binomials. While you can always get the product by writing the binomial twice and using the methods of the last section, there is less work to do if you learn to use a pattern.

Let's start by looking at $(x+9)^2$.	
What does this mean?	$(x+9)^2$
It means to multiply $(x+9)$ by itself.	$(x+9)(x+9)$
Then, using FOIL, we get:	$x^2+9x+9x+81$
Combining like terms gives:	$x^2+18x+81$

Here's another one:	$(y-7)^2$
Multiply $(y-7)$ by itself.	$(y-7)(y-7)$
Using FOIL, we get:	$y^2-7y-7y+49$
And combining like terms:	$y^2-14y+49$

And one more:	$(2x+3)^2$
Multiply.	$(2x+3)(2x+3)$
Use FOIL:	$4x^2+6x+6x+9$
Combine like terms.	$4x^2+12x+9$

Look at these results. Do you see any patterns?

What about the number of terms? In each example we squared a binomial and the result was a trinomial.

Equation:

$$(a+b)^2 = \_\_\_\_ + \_\_\_\_ + \_\_\_\_$$

Now look at the ***first term*** in each result. Where did it come from?

$(x+9)^2$	$(y-7)^2$	$(2x+3)^2$
$(x+9)(x+9)$	$(y-7)(y-7)$	$(2x+3)(2x+3)$
$x^2+9x+9x+81$	$y^2-7y-7y+49$	$4x^2+6x+6x+9$
$x^2+18x+81$	$y^2-14y+49$	$4x^2+12x+9$

The first term is the product of the first terms of each binomial. Since the binomials are identical, it is just the square of the first term!

Equation:

$$(a+b)^2 = a^2 + \_\_\_\_ + \_\_\_\_$$

To get the ***first term*** of the product, ***square the first term***.

Where did the ***last term*** come from? Look at the examples and find the pattern.

The last term is the product of the last terms, which is the square of the last term.

Equation:

$$(a+b)^2 = \_\_\_\_ + \_\_\_\_ + b^2$$

*To get the **last term** of the product, **square the last term**.*

Finally, look at the ***middle term***. Notice it came from adding the “outer” and the “inner” terms—which are both the same! So the middle term is double the product of the two terms of the binomial.

Equation:

$$(a+b)^2 = \_\_\_\_ + 2ab + \_\_\_\_$$
$$(a-b)^2 = \_\_\_\_ - 2ab + \_\_\_\_$$

*To get the **middle term** of the product, **multiply the terms and double their product**.*

Putting it all together:

Note:
Binomial Squares Pattern
If a and b are real numbers,
Equation:

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$

$$\underbrace{(a+b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} + \underbrace{2ab}_{2(\text{product of terms})} + \underbrace{b^2}_{(\text{last term})^2}$$

To square a binomial:

- square the first term
- square the last term
- double their product

A number example helps verify the pattern.

	$(10+4)^2$
Square the first term.	$10^2 + \_\_\_ +$

Square the last term.	$10^2 + \_\_\_ + 4^2$
Double their product.	$10^2 + 2 \cdot 10 \cdot 4 + 4^2$
Simplify.	$100 + 80 + 16$
Simplify.	196

To multiply $(10+4)^2$ usually you'd follow the Order of Operations.
Equation:

$$(10+4)^2$$
$$(14)^2$$
$$196$$

The pattern works!

Example:
Exercise:

Problem: Multiply: $(x+5)^2$.

Solution:
Solution

	$\left(\overset{a}{x}+\overset{b}{5}\right)^2$
Square the first term.	$\overset{a^2}{x^2}+\overset{2ab}{\_\_\_\_}+\overset{b^2}{\_\_\_\_}$
Square the last term.	$\overset{a^2}{x^2}+\overset{2ab}{\_\_\_\_}+\overset{b^2}{5^2}$

Double the product.	$\color{red}{a^2 + 2 \cdot a \cdot b + b^2}$ $x^2 + 2 \cdot x \cdot 5 + 5^2$
Simplify.	$x^2 + 10x + 25$

Note:
Exercise:

Problem: Multiply: $(x+9)^2$.

Solution:

$x^2 + 18x + 81$

Note:
Exercise:

Problem: Multiply: $(y+11)^2$.

Solution:

$y^2 + 22y + 121$

Example:
Exercise:

Problem: Multiply: $(y-3)^2$.

Solution:
Solution

	$\overset{a\ -\ b}{(y-3)}^2$
Square the first term.	$\overset{a^2\ -\ 2ab\ +\ b^2}{y^2 - \_\_\_ + \_\_\_}$
Square the last term.	$\overset{a^2\ -\ 2ab\ +\ b^2}{y^2 - \_\_\_ + 3^2}$
Double the product.	$\overset{a^2\ -\ 2\cdot a\cdot b\ +\ b^2}{y^2 - 2\cdot y\cdot 3 + 3^2}$
Simplify.	$y^2 - 6y + 9$

Note:
Exercise:

Problem: Multiply: $(x-9)^2$.

Solution:

$x^2 - 18x + 81$

Note:
Exercise:

Problem: Multiply: $(p-13)^2$.

Solution:

$p^2 - 26p + 169$

Example:
Exercise:

Problem: Multiply: $(4x+6)^2$.

Solution:
Solution

	$\overset{a\ \ +\ \ b}{(4x+6)^2}$
Use the pattern.	$\overset{a^2\ +\ 2\cdot a\cdot b\ +\ b^2}{(4x)^2+2\cdot 4x\cdot 6+6^2}$
Simplify.	$16x^2+48x+36$

Note:
Exercise:

Problem: Multiply: $(6x+3)^2$.

Solution:

$36x^2+36x+9$

Note:
Exercise:

Problem: Multiply: $(4x+9)^2$.

Solution:

$16x^2 + 72x + 81$

Example:
Exercise:

Problem: Multiply: $(2x - 3y)^2$.

Solution:
Solution

	$(\overset{a}{2x} \overset{-}{-} \overset{b}{3y})^2$
Use the pattern.	$\overset{a^2}{(2x)^2} - \overset{2 \cdot a \cdot b}{2 \cdot 2x \cdot 3y} + \overset{b^2}{(3y)^2}$
Simplify.	$4x^2 - 12xy + 9y^2$

Note:
Exercise:

Problem: Multiply: $(2c - d)^2$.

Solution:

$4c^2 - 4cd + d^2$

Note:
Exercise:

Problem: Multiply: $(4x-5y)^2$.

Solution:

$16x^2-40xy+25y^2$

Example:
Exercise:

Problem: Multiply: $\left(4u^3+1\right)^2$.

Solution:
Solution

	$\left(\overset{a}{4u^3}+\overset{b}{1}\right)^2$
Use the pattern.	$\overset{a^2}{(4u^3)^2}+2\cdot\overset{a}{4u^3}\cdot\overset{b}{1}+\overset{b^2}{(1)^2}$
Simplify.	$16u^6+8u^3+1$

Note:
Exercise:

Problem: Multiply: $\left(2x^2+1\right)^2$.

Solution:

$4x^4+4x^2+1$

Note:
Exercise:

Problem: Multiply: $\left(3y^3+2\right)^2$.

Solution:

$9y^6+12y^3+4$

Multiply Conjugates Using the Product of Conjugates Pattern

We just saw a pattern for squaring binomials that we can use to make multiplying some binomials easier. Similarly, there is a pattern for another product of binomials. But before we get to it, we need to introduce some vocabulary.

What do you notice about these pairs of binomials?
Equation:

$$(x-9)\,(x+9) \qquad (y-8)\,(y+8) \qquad (2x-5)\,(2x+5)$$

Look at the first term of each binomial in each pair.

$$(x-9)(x+9) \qquad (y-8)(y+8) \qquad (2x-5)(2x+5)$$

Notice the first terms are the same in each pair.

Look at the last terms of each binomial in each pair.

$$(x-9)(x+9) \qquad (y-8)(y+8) \qquad (2x-5)(2x+5)$$

Notice the last terms are the same in each pair.

Notice how each pair has one sum and one difference.

$$\left(\underset{\text{Difference}}{\underset{\uparrow}{x-9}}\right)\left(\underset{\text{Sum}}{\underset{\uparrow}{x+9}}\right) \qquad \left(\underset{\text{Difference}}{\underset{\uparrow}{y-8}}\right)\left(\underset{\text{Sum}}{\underset{\uparrow}{y+8}}\right) \qquad \left(\underset{\text{Difference}}{\underset{\uparrow}{2x-5}}\right)\left(\underset{\text{Sum}}{\underset{\uparrow}{2x+5}}\right)$$

A pair of binomials that each have the same first term and the same last term, but one is a sum and one is a difference has a special name. It is called a *conjugate pair* and is of the form $(a-b),(a+b)$.

Note:
Conjugate Pair
A **conjugate pair** is two binomials of the form
Equation:

$$(a-b),(a+b).$$

The pair of binomials each have the same first term and the same last term, but one binomial is a sum and the other is a difference.

There is a nice pattern for finding the product of conjugates. You could, of course, simply FOIL to get the product, but using the pattern makes your work easier.

Let’s look for the pattern by using FOIL to multiply some conjugate pairs.
Equation:

$$\begin{array}{ccc} (x-9)\,(x+9) & (y-8)\,(y+8) & (2x-5)\,(2x+5) \\ x^2+9x-9x-81 & y^2+8y-8y-64 & 4x^2+10x-10x-25 \\ x^2-81 & y^2-64 & 4x^2-25 \end{array}$$

$$\begin{array}{ccc} (x+9)(x-9) & (y-8)(y+8) & (2x-5)(2x+5) \\ x^2-9x+9x-81 & y^2+8y-8y-64 & 4x^2+10x-10x-25 \\ x^2-81 & y^2-64 & 4x^2-25 \end{array}$$

Each **first term** is the product of the first terms of the binomials, and since they are identical it is the square of the first term.
Equation:

$$(a+b)(a-b)=a^2-\_\_\_\_$$
$$\text{To get the } \textbf{first term, square the first term}.$$

The **last term** came from multiplying the last terms, the square of the last term.
Equation:

$$(a+b)(a-b)=a^2-b^2$$
$$\text{To get the } \textbf{last term, square the last term}.$$

What do you observe about the products?

The product of the two binomials is also a binomial! Most of the products resulting from FOIL have been trinomials.

Why is there no middle term? Notice the two middle terms you get from FOIL combine to 0 in every case, the result of one addition and one subtraction.

The product of conjugates is always of the form $a^2 - b^2$. This is called a difference of squares.

This leads to the pattern:

Note:
Product of Conjugates Pattern
If a and b are real numbers,

$$(a-b)(a+b) = a^2 - b^2$$

$$\underbrace{(a-b)(a+b)}_{\text{conjugates}} = \underbrace{a^2 \overset{\text{difference}}{-} b^2}_{\text{squares}}$$

The product is called a difference of squares.
To multiply conjugates, square the first term, square the last term, and write the product as a difference of squares.

Let’s test this pattern with a numerical example.

	$(10-2)(10+2)$
It is the product of conjudgates, so the result will be the difference of two squares.	____ $-$ ____
Square the first term.	$10^2 -$ ____
Square the last term.	$10^2 - 2^2$
Simplify.	$100 - 4$
Simplify.	96
What do you get using the order of operations?	

	$(10-2)(10+2)$ $(8)(12)$ 96

Notice, the result is the same!

Example:
Exercise:

Problem: Multiply: $(x-8)\,(x+8)$.

Solution:
Solution

First, recognize this as a product of conjugates. The binomials have the same first terms, and the same last terms, and one binomial is a sum and the other is a difference.

It fits the pattern.	$\begin{array}{c} (a - b)(a + b) \\ (x - 8)(x + 8) \end{array}$
Square the first term, *x*.	$\begin{array}{c} a^2 - b^2 \\ x^2 - \_\_ \end{array}$
Square the last term, 8.	$\begin{array}{c} a^2 - b^2 \\ x^2 - 8^2 \end{array}$
The product is a difference of squares.	$\begin{array}{c} a^2 - b^2 \\ x^2 - 64 \end{array}$

Note:

Exercise:

Problem: Multiply: $(x-5)\,(x+5)$.

Solution:

x^2-25

Note:
Exercise:

Problem: Multiply: $(w-3)\,(w+3)$.

Solution:

w^2-9

Example:
Exercise:

Problem: Multiply: $(2x+5)\,(2x-5)$.

Solution:
Solution

Are the binomials conjugates?

It is the product of conjugates.	$(\overset{a}{2x}\overset{+}{+}\overset{b}{5})\,(\overset{a}{2x}\overset{-}{-}\overset{b}{5})$
Square the first term, $2x$.	$\overset{a^2}{(2x)^2}\overset{-}{-}\overset{b^2}{\_\_}$
Square the last term, 5.	

	$\overset{a^2 \;-\; b^2}{(2x)^2 - 5^2}$
Simplify. The product is a difference of squares.	$\overset{a^2 \;-\; b^2}{4x^2 - 25}$

Note:
Exercise:

Problem: Multiply: $\left(6x+5\right)\left(6x-5\right)$.

Solution:

$36x^2-25$

Note:
Exercise:

Problem: Multiply: $\left(2x+7\right)\left(2x-7\right)$.

Solution:

$4x^2-49$

The binomials in the next example may look backwards – the variable is in the second term. But the two binomials are still conjugates, so we use the same pattern to multiply them.

Example:
Exercise:

Problem: Find the product: $\left(3+5x\right)\left(3-5x\right)$.

Solution:
Solution

It is the product of conjugates.	$\overset{a\ -\ b}{(3+5x)}\overset{a\ +\ b}{(3-5x)}$
Use the pattern.	$\overset{a^2\ -\ b^2}{3^2-(5x)^2}$
Simplify.	$9-25x^2$

Note:
Exercise:

Problem: Multiply: $(7+4x)(7-4x)$.

Solution:

$49-16x^2$

Note:
Exercise:

Problem: Multiply: $(9-2y)(9+2y)$.

Solution:

$81-4y^2$

Now we'll multiply conjugates that have two variables.

Example:
Exercise:

Problem: Find the product: $(5m-9n)(5m+9n)$.

Solution:
Solution

This fits the pattern.	$\overset{a}{(5m} \overset{-}{-} \overset{b}{9n)}\overset{a}{(5m} \overset{+}{+} \overset{b}{9n)}$
Use the pattern.	$\overset{a^2}{(5m)^2} \overset{-}{-} \overset{b^2}{(9n)^2}$
Simplify.	$25m^2 - 81n^2$

Note:
Exercise:

Problem: Find the product: $(4p - 7q)\,(4p + 7q)$.

Solution:

$16p^2 - 49q^2$

Note:
Exercise:

Problem: Find the product: $(3x - y)\,(3x + y)$.

Solution:

$9x^2 - y^2$

Example:
Exercise:

Problem: Find the product: $(cd-8)\,(cd+8)$.

Solution:
Solution

This fits the pattern.	$\left(\overset{a}{cd}-\overset{b}{8}\right)\left(\overset{a}{cd}+\overset{b}{8}\right)$
Use the pattern.	$\overset{a^2}{(cd)^2}-\overset{b^2}{(8)^2}$
Simplify.	c^2d^2-64

Note:
Exercise:

Problem: Find the product: $(xy-6)(xy+6)$.

Solution:

x^2y^2-36

Note:
Exercise:

Problem: Find the product: $(ab-9)(ab+9)$.

Solution:

$a^2b^2 - 81$

Example:
Exercise:

Problem: Find the product: $\left(6u^2 - 11v^5\right)\left(6u^2 + 11v^5\right)$.

Solution:
Solution

This fits the pattern.	$\left(\overset{a}{6u^2} - \overset{b}{11v^5}\right)\left(\overset{a}{6u^2} + \overset{b}{11v^5}\right)$
Use the pattern.	$\overset{a^2}{(6u^2)^2} - \overset{b^2}{(11v^5)^2}$
Simplify.	$36u^4 - 121v^{10}$

Note:
Exercise:

Problem: Find the product: $\left(3x^2 - 4y^3\right)\left(3x^2 + 4y^3\right)$.

Solution:

$9x^4 - 16y^6$

Note:
Exercise:

Problem: Find the product: $\left(2m^2 - 5n^3\right)\left(2m^2 + 5n^3\right)$.

Solution:

$4m^4 - 25n^6$

Recognize and Use the Appropriate Special Product Pattern

We just developed special product patterns for Binomial Squares and for the Product of Conjugates. The products look similar, so it is important to recognize when it is appropriate to use each of these patterns and to notice how they differ. Look at the two patterns together and note their similarities and differences.

Note:
Comparing the Special Product Patterns

Binomial Squares	**Product of Conjugates**
$(a+b)^2 = a^2 + 2ab + b^2$	$(a-b)(a+b) = a^2 - b^2$
$(a-b)^2 = a^2 - 2ab + b^2$	
- Squaring a binomial	- Multiplying conjugates
- Product is a **trinomial**	- Product is a **binomial**
- Inner and outer terms with FOIL are **the same.**	- Inner and outer terms with FOIL are **opposites.**
- Middle term is **double the product** of the terms.	- There is **no** middle term.

Example:

Exercise:

Problem: Choose the appropriate pattern and use it to find the product:

ⓐ $(2x-3)(2x+3)$ ⓑ $(5x-8)^2$ ⓒ $(6m+7)^2$ ⓓ $(5x-6)(6x+5)$

Solution:
Solution

ⓐ $(2x-3)(2x+3)$ These are conjugates. They have the same first numbers, and the same last numbers, and one binomial is a sum and the other is a difference. It fits the Product of Conjugates pattern.

This fits the pattern.	$\overset{a\ -\ b}{(2x-3)}\overset{a\ +\ b}{(2x+3)}$
Use the pattern.	$\overset{a^2\ -\ b^2}{(2x)^2-3^2}$
Simplify.	$4x^2-9$

ⓑ $(8x-5)^2$ We are asked to square a binomial. It fits the **binomial squares** pattern.

	$\overset{a\ -\ b}{(8x-5)}^2$
Use the pattern.	$\overset{a^2\ -\ 2ab\ +\ b^2}{(8x)^2-2\cdot 8x\cdot 5+5^2}$
Simplify.	$64x^2-80x+25$

ⓒ $(6m+7)^2$ Again, we will square a binomial so we use the **binomial squares** pattern.

	$\left(\overset{a}{6m}+\overset{b}{7}\right)^2$
Use the pattern.	$\overset{a^2}{(6m)^2} + \overset{2ab}{2\cdot 6m\cdot 7} + \overset{b^2}{7^2}$
Simplify.	$36m^2+84m+49$

ⓓ $(5x-6)(6x+5)$ This product does not fit the patterns, so we will use FOIL.

	$(5x-6)(6x+5)$
Use FOIL.	$30x^2+25x-36x-30$
Simplify.	$30x^2-11x-30$

Note:
Exercise:

Problem: Choose the appropriate pattern and use it to find the product:

ⓐ $(9b-2)(2b+9)$ ⓑ $(9p-4)^2$ ⓒ $(7y+1)^2$ ⓓ $(4r-3)(4r+3)$

Solution:

ⓐ FOIL; $18b^2 + 77b - 18$ ⓑ Binomial Squares; $81p^2 - 72p + 16$ ⓒ Binomial Squares; $49y^2 + 14y + 1$ ⓓ Product of Conjugates; $16r^2 - 9$

Note:
Exercise:

Problem: Choose the appropriate pattern and use it to find the product:

ⓐ $(6x + 7)^2$ ⓑ $(3x - 4)\,(3x + 4)$ ⓒ $(2x - 5)\,(5x - 2)$ ⓓ $(6n - 1)^2$

Solution:

ⓐ Binomial Squares; $36x^2 + 84x + 49$ ⓑ Product of Conjugates; $9x^2 - 16$ ⓒ FOIL; $10x^2 - 29x + 10$ ⓓ Binomial Squares; $36n^2 - 12n + 1$

Note:
Access these online resources for additional instruction and practice with special products:

- [Special Products]

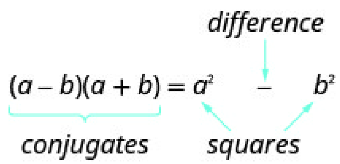
Key Concepts

- **Binomial Squares Pattern**
  - If a, b are real numbers,

$$\underbrace{(a+b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} + \underbrace{2ab}_{2(\text{product of terms})} + \underbrace{b^2}_{(\text{last term})^2}$$

  - $(a + b)^2 = a^2 + 2ab + b^2$
  - $(a - b)^2 = a^2 - 2ab + b^2$
  - To square a binomial: square the first term, square the last term, double their product.

- **Product of Conjugates Pattern**
  - If a, b are real numbers,



- $\left(a-b\right)\left(a+b\right)={a}^{2}-{b}^{2}$
- The product is called a difference of squares.

- **To multiply conjugates:**
  - **square the first term square the last term** write it as a difference of squares

Practice Makes Perfect

Square a Binomial Using the Binomial Squares Pattern

In the following exercises, square each binomial using the Binomial Squares Pattern.
Exercise:

Problem: ${\left(w+4\right)}^{2}$

Exercise:

Problem: ${\left(q+12\right)}^{2}$

Solution:

${q}^{2}+24q+144$

Exercise:

Problem: ${\left(y+\frac{1}{4}\right)}^{2}$

Exercise:

Problem: ${\left(x+\frac{2}{3}\right)}^{2}$

Solution:

${x}^{2}+\frac{4}{3}x+\frac{4}{9}$

Exercise:

Problem: ${\left(b-7\right)}^{2}$

Exercise:

Problem: $(y-6)^2$

Solution:

$y^2-12y+36$

Exercise:

Problem: $(m-15)^2$

Exercise:

Problem: $(p-13)^2$

Solution:

$p^2-26p+169$

Exercise:

Problem: $(3d+1)^2$

Exercise:

Problem: $(4a+10)^2$

Solution:

$16a^2+80a+100$

Exercise:

Problem: $(2q+\frac{1}{3})^2$

Exercise:

Problem: $\left(3z+\frac{1}{5}\right)^2$

Solution:

$9z^2+\frac{6}{5}z+\frac{1}{25}$

Exercise:

Problem: $(3x-y)^2$

Exercise:

Problem: $(2y-3z)^2$

Solution:

$4y^2-12yz+9z^2$

Exercise:

Problem: $\left(\frac{1}{5}x-\frac{1}{7}y\right)^2$

Exercise:

Problem: $\left(\frac{1}{8}x-\frac{1}{9}y\right)^2$

Solution:

$\frac{1}{64}x^2-\frac{1}{36}xy+\frac{1}{81}y^2$

Exercise:

Problem: $(3x^2+2)^2$

Exercise:

Problem: $(5u^2+9)^2$

Solution:

$25u^4+90u^2+81$

Exercise:

Problem: $(4y^3-2)^2$

Exercise:

Problem: $(8p^3-3)^2$

Solution:

$64p^6-48p^3+9$

Multiply Conjugates Using the Product of Conjugates Pattern

In the following exercises, multiply each pair of conjugates using the Product of Conjugates Pattern.

Exercise:

Problem: $(m-7)(m+7)$

Exercise:

Problem: $(c-5)(c+5)$

Solution:

c^2-25

Exercise:

Problem: $\left(x+\frac{3}{4}\right)\left(x-\frac{3}{4}\right)$

Exercise:

Problem: $\left(b+\frac{6}{7}\right)\left(b-\frac{6}{7}\right)$

Solution:

$b^2-\frac{36}{49}$

Exercise:

Problem: $(5k+6)(5k-6)$

Exercise:

Problem: $(8j+4)(8j-4)$

Solution:

$64j^2-16$

Exercise:

Problem: $(11k+4)(11k-4)$

Exercise:

Problem: $(9c+5)(9c-5)$

Solution:

$81c^2 - 25$

Exercise:

Problem: $(11 - b)(11 + b)$

Exercise:

Problem: $(13 - q)(13 + q)$

Solution:

$169 - q^2$

Exercise:

Problem: $(5 - 3x)(5 + 3x)$

Exercise:

Problem: $(4 - 6y)(4 + 6y)$

Solution:

$16 - 36y^2$

Exercise:

Problem: $(9c - 2d)(9c + 2d)$

Exercise:

Problem: $(7w + 10x)(7w - 10x)$

Solution:

$49w^2 - 100x^2$

Exercise:

Problem: $\left(m + \frac{2}{3}n\right)\left(m - \frac{2}{3}n\right)$

Exercise:

Problem: $\left(p + \frac{4}{5}q\right)\left(p - \frac{4}{5}q\right)$

Solution:

$p^2 - \frac{16}{25}q^2$

Exercise:

Problem: $(ab-4)(ab+4)$

Exercise:

Problem: $(xy-9)(xy+9)$

Solution:

$x^2y^2 - 81$

Exercise:

Problem: $(uv - \frac{3}{5})(uv + \frac{3}{5})$

Exercise:

Problem: $(rs - \frac{2}{7})(rs + \frac{2}{7})$

Solution:

$r^2s^2 - \frac{4}{49}$

Exercise:

Problem: $(2x^2 - 3y^4)(2x^2 + 3y^4)$

Exercise:

Problem: $(6m^3 - 4n^5)(6m^3 + 4n^5)$

Solution:

$36m^6 - 16n^{10}$

Exercise:

Problem: $(12p^3 - 11q^2)(12p^3 + 11q^2)$

Exercise:

Problem: $(15m^2 - 8n^4)(15m^2 + 8n^4)$

Solution:

$225m^4 - 64n^8$

Recognize and Use the Appropriate Special Product Pattern

In the following exercises, find each product.

Exercise:

Problem:

ⓐ $(p-3)(p+3)$
ⓑ $(t-9)^2$
ⓒ $(m+n)^2$
ⓓ $(2x+y)(x-2y)$

Exercise:

Problem:

ⓐ $(2r+12)^2$
ⓑ $(3p+8)(3p-8)$
ⓒ $(7a+b)(a-7b)$
ⓓ $(k-6)^2$

Solution:

ⓐ $4r^2+48r+144$ ⓑ $9p^2-64$ ⓒ $7a^2-48ab-7b^2$ ⓓ $k^2-12k+36$

Exercise:

Problem:

ⓐ $(a^5-7b)^2$
ⓑ $(x^2+8y)(8x-y^2)$
ⓒ $(r^6+s^6)(r^6-s^6)$
ⓓ $(y^4+2z)^2$

Exercise:

Problem:

ⓐ $(x^5+y^5)(x^5-y^5)$
ⓑ $(m^3-8n)^2$
ⓒ $(9p+8q)^2$
ⓓ $(r^2-s^3)(r^3+s^2)$

Solution:

ⓐ $x^{10} - y^{10}$ ⓑ $m^6 - 16m^3n + 64n^2$ ⓒ $81p^2 + 144pq + 64q^2$ ⓓ $r^5 + r^2s^2 - r^3s^3 - s^5$

Everyday Math

Exercise:

Problem:

Mental math You can use the product of conjugates pattern to multiply numbers without a calculator. Say you need to multiply 47 times 53. Think of 47 as $50 - 3$ and 53 as $50 + 3$.

ⓐ Multiply $(50 - 3)(50 + 3)$ by using the product of conjugates pattern, $(a - b)(a + b) = a^2 - b^2$.
ⓑ Multiply $47 \cdot 53$ without using a calculator.
ⓒ Which way is easier for you? Why?

Exercise:

Problem:

Mental math You can use the binomial squares pattern to multiply numbers without a calculator. Say you need to square 65. Think of 65 as $60 + 5$.

ⓐ Multiply $(60 + 5)^2$ by using the binomial squares pattern, $(a + b)^2 = a^2 + 2ab + b^2$.
ⓑ Square 65 without using a calculator.
ⓒ Which way is easier for you? Why?

Solution:

ⓐ 4,225 ⓑ 4,225 ⓒ Answers will vary.

Writing Exercises

Exercise:

Problem: How do you decide which pattern to use?

Exercise:

Problem: Why does $(a + b)^2$ result in a trinomial, but $(a - b)(a + b)$ result in a binomial?

Solution:

Answers will vary.

Exercise:

Problem: Marta did the following work on her homework paper:

Equation:

$$\begin{array}{c}(3-y)^2\\ 3^2-y^2\\ 9-y^2\end{array}$$

Explain what is wrong with Marta's work.

Exercise:

Problem:

Use the order of operations to show that $(3+5)^2$ is 64, and then use that numerical example to explain why $(a+b)^2 \neq a^2+b^2$.

Solution:

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
square a binomial using the binomial squares pattern.			
multiply conjugates using the product of conjugates pattern.			
recognize and use the appropriate special product pattern.			

ⓑ On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Glossary

conjugate pair

A conjugate pair is two binomials of the form $(a - b), (a + b)$; the pair of binomials each have the same first term and the same last term, but one binomial is a sum and the other is a difference.

Divide Monomials: ASE
By the end of this section, you will be able to:

- Simplify expressions using the Quotient Property for Exponents
- Simplify expressions with zero exponents
- Simplify expressions using the quotient to a Power Property
- Simplify expressions by applying several properties
- Divide monomials

Simplify Expressions Using the Quotient Property for Exponents

Earlier in this chapter, we developed the properties of exponents for multiplication. We summarize these properties below.

Note:
Summary of Exponent Properties for Multiplication
If a and b are real numbers, and m and n are whole numbers, then

Product Property	$a^m \cdot a^n = a^{m+n}$
Power Property	$(a^m)^n = a^{m \cdot n}$
Product to a Power	$(ab)^m = a^m b^m$

Now we will look at the exponent properties for division. A quick memory refresher may help before we get started. You have learned to simplify fractions by dividing out common factors from the numerator and denominator using the Equivalent Fractions Property. This property will also help you work with algebraic fractions—which are also quotients.

Note:

Equivalent Fractions Property
If $a, b,$ and c are whole numbers where $b \neq 0, c \neq 0,$
Equation:

$$\text{then} \quad \frac{a}{b} = \frac{a \cdot c}{b \cdot c} \quad \text{and} \quad \frac{a \cdot c}{b \cdot c} = \frac{a}{b}$$

As before, we'll try to discover a property by looking at some examples.

Consider	$\frac{x^5}{x^2}$	and	$\frac{x^2}{x^3}$
What do they mean?	$\frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x}$		$\frac{x \cdot x}{x \cdot x \cdot x}$
Use the Equivalent Fractions Property.	$\frac{\cancel{x} \cdot \cancel{x} \cdot x \cdot x \cdot x}{\cancel{x} \cdot \cancel{x}}$		$\frac{\cancel{x} \cdot \cancel{x} \cdot 1}{\cancel{x} \cdot \cancel{x} \cdot x}$
Simplify.	x^3		$\frac{1}{x}$

Notice, in each case the bases were the same and we subtracted exponents.

When the larger exponent was in the numerator, we were left with factors in the numerator.

When the larger exponent was in the denominator, we were left with factors in the denominator—notice the numerator of 1.

We write:
Equation:

$$\begin{array}{cc} \frac{x^5}{x^2} & \frac{x^2}{x^3} \\ x^{5-2} & \frac{1}{x^{3-2}} \\ x^3 & \frac{1}{x} \end{array}$$

This leads to the *Quotient Property for Exponents*.

Note:
Quotient Property for Exponents
If a is a real number, $a \neq 0$, and m and n are whole numbers, then
Equation:

$$\frac{a^m}{a^n} = a^{m-n}, m > n \text{ and } \frac{a^m}{a^n} = \frac{1}{a^{n-m}}, n > m$$

A couple of examples with numbers may help to verify this property.
Equation:

$$\begin{array}{rclcrcl} \frac{3^4}{3^2} & = & 3^{4-2} & \qquad & \frac{5^2}{5^3} & = & \frac{1}{5^{3-2}} \\ \frac{81}{9} & = & 3^2 & & \frac{25}{125} & = & \frac{1}{5^1} \\ 9 & = & 9\checkmark & & \frac{1}{5} & = & \frac{1}{5}\checkmark \end{array}$$

Example:
Exercise:

Problem: Simplify: ⓐ $\frac{x^9}{x^7}$ ⓑ $\frac{3^{10}}{3^2}$.

Solution:
Solution

To simplify an expression with a quotient, we need to first compare the exponents in the numerator and denominator.

ⓐ

Since 9 > 7, there are more factors of x in the numerator.	$\frac{x^9}{x^7}$
Use the Quotient Property, $\frac{a^m}{a^n} = a^{m-n}$.	x^{9-7}
Simplify.	x^2

ⓑ

Since 10 > 2, there are more factors of x in the numerator.	$\frac{3^{10}}{3^2}$
Use the Quotient Property, $\frac{a^m}{a^n} = a^{m-n}$.	3^{10-2}
Simplify.	3^8

Notice that when the larger exponent is in the numerator, we are left with factors in the numerator.

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{x^{15}}{x^{10}}$ ⓑ $\frac{6^{14}}{6^5}$.

Solution:

ⓐ x^5 ⓑ 6^9

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{y^{43}}{y^{37}}$ ⓑ $\frac{10^{15}}{10^7}$.

Solution:

ⓐ y^6 ⓑ 10^8

Example:
Exercise:

Problem: Simplify: ⓐ $\frac{b^8}{b^{12}}$ ⓑ $\frac{7^3}{7^5}$.

Solution:
Solution

To simplify an expression with a quotient, we need to first compare the exponents in the numerator and denominator.

ⓐ

Since 12 > 8, there are more factors of *b* in the denominator.	$\frac{b^8}{b^{12}}$
Use the Quotient Property, $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$.	

	$\frac{1}{b^{12-8}}$
Simplify.	$\frac{1}{b^4}$

ⓑ

Since 5 > 3, there are more factors of 3 in the denominator.	$\frac{7^3}{7^5}$
Use the Quotient Property, $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$.	$\frac{1}{7^{5-3}}$
Simplify.	$\frac{1}{7^2}$
Simplify.	$\frac{1}{49}$

Notice that when the larger exponent is in the denominator, we are left with factors in the denominator.

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{x^{18}}{x^{22}}$ ⓑ $\frac{12^{15}}{12^{30}}$.

Solution:

ⓐ $\frac{1}{x^4}$ ⓑ $\frac{1}{12^{15}}$

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{m^7}{m^{15}}$ ⓑ $\frac{9^8}{9^{19}}$.

Solution:

ⓐ $\frac{1}{m^8}$ ⓑ $\frac{1}{9^{11}}$

Notice the difference in the two previous examples:

- If we start with more factors in the numerator, we will end up with factors in the numerator.
- If we start with more factors in the denominator, we will end up with factors in the denominator.

The first step in simplifying an expression using the Quotient Property for Exponents is to determine whether the exponent is larger in the numerator or the denominator.

Example:
Exercise:

Problem: Simplify: ⓐ $\frac{a^5}{a^9}$ ⓑ $\frac{x^{11}}{x^7}$.

Solution:

Solution

ⓐ Is the exponent of a larger in the numerator or denominator? Since 9 > 5, there are more $a\prime$s in the denominator and so we will end up with factors in the denominator.

	$\frac{a^5}{a^9}$
Use the Quotient Property, $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$.	$\frac{1}{a^{9-5}}$
Simplify.	$\frac{1}{a^4}$

ⓑ Notice there are more factors of x in the numerator, since 11 > 7. So we will end up with factors in the numerator.

	$\frac{x^{11}}{x^7}$
Use the Quotient Property, $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$.	x^{11-7}
Simplify.	x^4

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{b^{19}}{b^{11}}$ ⓑ $\frac{z^5}{z^{11}}$.

Solution:

ⓐ b^8 ⓑ $\frac{1}{z^6}$

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{p^9}{p^{17}}$ ⓑ $\frac{w^{13}}{w^9}$.

Solution:

ⓐ $\frac{1}{p^8}$ ⓑ w^4

Simplify Expressions with an Exponent of Zero

A special case of the Quotient Property is when the exponents of the numerator and denominator are equal, such as an expression like $\frac{a^m}{a^m}$. From your earlier work with fractions, you know that:

Equation:

$$\frac{2}{2} = 1 \qquad \frac{17}{17} = 1 \qquad \frac{-43}{-43} = 1$$

In words, a number divided by itself is 1. So, $\frac{x}{x} = 1$, for any x $(x \neq 0)$, since any number divided by itself is 1.

The Quotient Property for Exponents shows us how to simplify $\frac{a^m}{a^n}$ when $m > n$ and when $n < m$ by subtracting exponents. What if $m = n$?

Consider $\frac{8}{8}$, which we know is 1.

	$\frac{8}{8} = 1$
Write 8 as 2^3.	$\frac{2^3}{2^3} = 1$
Subtract exponents.	$2^{3-3} = 1$
Simplify.	$2^0 = 1$

Now we will simplify $\frac{a^m}{a^m}$ in two ways to lead us to the definition of the zero exponent. In general, for $a \neq 0$:

$\frac{a^m}{a^m}$	$\frac{a^m}{a^m}$
a^{m-m}	$\dfrac{\overbrace{\cancel{a} \cdot \cancel{a} \cdot \ldots \cdot \cancel{a}}^{m \text{ factors}}}{\underbrace{\cancel{a} \cdot \cancel{a} \cdot \ldots \cdot \cancel{a}}_{m \text{ factors}}}$
a^0	1

We see $\frac{a^m}{a^m}$ simplifies to a^0 and to 1. So $a^0 = 1$.

Note:
Zero Exponent
If a is a non-zero number, then $a^0 = 1$.
Any nonzero number raised to the zero power is 1.

In this text, we assume any variable that we raise to the zero power is not zero.

Example:
Exercise:

Problem: Simplify: ⓐ 9^0 ⓑ n^0.

Solution:
Solution

The definition says any non-zero number raised to the zero power is 1.

ⓐ Use the definition of the zero exponent.	9^0 1
ⓑ Use the definition of the zero exponent.	n^0 1

Note:
Exercise:

Problem: Simplify: ⓐ 15^0 ⓑ m^0.

Solution:

ⓐ 1 ⓑ 1

Note:
Exercise:

Problem: Simplify: ⓐ k^0 ⓑ 29^0.

Solution:

ⓐ 1 ⓑ 1

Now that we have defined the zero exponent, we can expand all the Properties of Exponents to include whole number exponents.

What about raising an expression to the zero power? Let's look at $(2x)^0$. We can use the product to a power rule to rewrite this expression.

	$(2x)^0$
Use the product to a power rule.	2^0x^0
Use the zero exponent property.	$1 \cdot 1$
Simplify.	1

This tells us that any nonzero expression raised to the zero power is one.

Example:
Exercise:

Problem: Simplify: ⓐ $(5b)^0$ ⓑ $\left(-4a^2b\right)^0$.

Solution:
Solution

ⓐ	$(5b)^0$
Use the definition of the zero exponent.	1
ⓑ	$\left(-4a^2b\right)^0$
Use the definition of the zero exponent.	1

Note:
Exercise:

Problem: Simplify: ⓐ $(11z)^0$ ⓑ $\left(-11pq^3\right)^0$.

Solution:

ⓐ 1 ⓑ 1

Note:
Exercise:

Problem: Simplify: ⓐ $(-6d)^0$ ⓑ $\left(-8m^2n^3\right)^0$.

Solution:

ⓐ 1 ⓑ 1

Simplify Expressions Using the Quotient to a Power Property

Now we will look at an example that will lead us to the Quotient to a Power Property.

	$\left(\frac{x}{y}\right)^3$
This means:	$\frac{x}{y} \cdot \frac{x}{y} \cdot \frac{x}{y}$
Multiply the fractions.	$\frac{x \cdot x \cdot x}{y \cdot y \cdot y}$
Write with exponents.	$\frac{x^3}{y^3}$

Notice that the exponent applies to both the numerator and the denominator.

We write:	$\left(\frac{x}{y}\right)^3$
	$\frac{x^3}{y^3}$

This leads to the *Quotient to a Power Property for Exponents*.

Note:
Quotient to a Power Property for Exponents
If a and b are real numbers, $b \neq 0$, and m is a counting number, then
Equation:

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

To raise a fraction to a power, raise the numerator and denominator to that power.

An example with numbers may help you understand this property:
Equation:

$$
\begin{aligned}
\left(\frac{2}{3}\right)^3 &= \frac{2^3}{3^3} \\
\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3} &= \frac{8}{27} \\
\frac{8}{27} &= \frac{8}{27}\checkmark
\end{aligned}
$$

Example:

Exercise:

Problem: Simplify: ⓐ $\left(\frac{3}{7}\right)^2$ ⓑ $\left(\frac{b}{3}\right)^4$ ⓒ $\left(\frac{k}{j}\right)^3$.

Solution:

Solution

ⓐ

	$\left(\frac{3}{7}\right)^2$
Use the Quotient Property, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.	$\frac{3^2}{7^2}$
Simplify.	$\frac{9}{49}$

ⓑ

	$\left(\frac{b}{3}\right)^4$
Use the Quotient Property, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.	$\frac{b^4}{3^4}$
Simplify.	$\frac{b^4}{81}$

ⓒ

	$\left(\frac{k}{j}\right)^3$
Raise the numerator and denominator to the third power.	$\frac{k^3}{j^3}$

Note:
Exercise:

Problem: Simplify: ⓐ $\left(\frac{5}{8}\right)^2$ ⓑ $\left(\frac{p}{10}\right)^4$ ⓒ $\left(\frac{m}{n}\right)^7$.

Solution:

ⓐ $\frac{25}{64}$ ⓑ $\frac{p^4}{10{,}000}$ ⓒ $\frac{m^7}{n^7}$

Note:
Exercise:

Problem: Simplify: ⓐ $\left(\frac{1}{3}\right)^3$ ⓑ $\left(\frac{-2}{q}\right)^3$ ⓒ $\left(\frac{w}{x}\right)^4$.

Solution:

ⓐ $\frac{1}{27}$ ⓑ $\frac{-8}{q^3}$ ⓒ $\frac{w^4}{x^4}$

Simplify Expressions by Applying Several Properties

We'll now summarize all the properties of exponents so they are all together to refer to as we simplify expressions using several properties. Notice that they are now defined for whole number exponents.

Note:
Summary of Exponent Properties
If a and b are real numbers, and m and n are whole numbers, then

Product Property	$a^m \cdot a^n = a^{m+n}$
Power Property	$(a^m)^n = a^{m \cdot n}$
Product to a Power	$(ab)^m = a^m b^m$
Quotient Property	

	$\frac{a^m}{b^m} = a^{m-n}, a \neq 0, m > n$ $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}, a \neq 0, n > m$
Zero Exponent Definition	$a^o = 1, a \neq 0$
Quotient to a Power Property	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$

Example:
Exercise:

Problem: Simplify: $\frac{\left(y^4\right)^2}{y^6}$.

Solution:
Solution

	$\frac{\left(y^4\right)^2}{y^6}$
Multiply the exponents in the numerator.	$\frac{y^8}{y^6}$
Subtract the exponents.	y^2

Note:
Exercise:

Problem: Simplify: $\frac{\left(m^5\right)^4}{m^7}$.

Solution:

m^{13}

Note:
Exercise:

Problem: Simplify: $\frac{\left(k^2\right)^6}{k^7}$.

Solution:

k^5

Example:
Exercise:

Problem: Simplify: $\frac{b^{12}}{\left(b^2\right)^6}$.

Solution:
Solution

	$\frac{b^{12}}{\left(b^2\right)^6}$
Multiply the exponents in the numerator.	$\frac{b^{12}}{b^{12}}$
Subtract the exponents.	b^0
Simplify.	1

Note:
Exercise:

Problem: Simplify: $\frac{n^{12}}{(n^3)^4}$.

Solution:

1

Note:
Exercise:

Problem: Simplify: $\frac{x^{15}}{(x^3)^5}$.

Solution:

1

Example:
Exercise:

Problem: Simplify: $\left(\frac{y^9}{y^4}\right)^2$.

Solution:
Solution

	$\left(\frac{y^9}{y^4}\right)^2$
Remember parentheses come before exponents.	$\left(y^5\right)^2$

Notice the bases are the same, so we can simplify inside the parentheses. Subtract the exponents.	
Multiply the exponents.	y^{10}

Note:
Exercise:

Problem: Simplify: $\left(\frac{r^5}{r^3}\right)^4$.

Solution:

r^8

Note:
Exercise:

Problem: Simplify: $\left(\frac{v^6}{v^4}\right)^3$.

Solution:

v^6

Example:
Exercise:

Problem: Simplify: $\left(\frac{j^2}{k^3}\right)^4$.

Solution:
Solution

Here we cannot simplify inside the parentheses first, since the bases are not the same.

	$\left(\frac{j^2}{k^3}\right)^4$
Raise the numberator and denominator to the third power using the Quotient to a Power Property, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.	
Use the Powert Property and simplify.	$\frac{j^8}{k^{12}}$

Note:
Exercise:

Problem: Simplify: $\left(\frac{a^3}{b^2}\right)^4$.

Solution:

$\frac{a^{12}}{b^8}$

Note:
Exercise:

Problem: Simplify: $\left(\frac{q^7}{r^5}\right)^3$.

Solution:

$\frac{q^{21}}{r^{15}}$

Example:
Exercise:

Problem: Simplify: $\left(\frac{2m^2}{5n}\right)^4$.

Solution:
Solution

	$\left(\frac{2m^2}{5n}\right)^4$
Raise the numberator and denominator to the fourth power, using the Quotient to a Power Property, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.	$\frac{(2m^2)^4}{(5n)^4}$
Raise each factor to the fourth power.	$\frac{(2m^2)^4}{(5n)^4}$
Use the Power Property and simplify.	$\frac{16m^8}{625n^4}$

Note:
Exercise:

Problem: Simplify: $\left(\frac{7x^3}{9y}\right)^2$.

Solution:

$\frac{49x^6}{81y^2}$

Note:
Exercise:

Problem: Simplify: $\left(\frac{3x^4}{7y}\right)^2$.

Solution:

$\frac{9x^8}{49y^2}$

Example:
Exercise:

Problem: Simplify: $\frac{\left(x^3\right)^4\left(x^2\right)^5}{\left(x^6\right)^5}$.

Solution:
Solution

	$\frac{\left(x^3\right)^4\left(x^2\right)^5}{\left(x^6\right)^5}$
Use the Power Property, $\left(a^m\right)^n = a^{m \cdot n}$.	$\frac{\left(x^{12}\right)\left(x^{10}\right)}{\left(x^{30}\right)}$
Add the exponents in the numerator.	$\frac{x^{22}}{x^{30}}$
Use the Quotient Property, $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$.	$\frac{1}{x^8}$

Note:

Exercise:

Problem: Simplify: $\frac{(a^2)^3(a^2)^4}{(a^4)^5}$.

Solution:

$\frac{1}{a^6}$

Note:
Exercise:

Problem: Simplify: $\frac{(p^3)^4(p^5)^3}{(p^7)^6}$.

Solution:

$\frac{1}{p^{15}}$

Example:
Exercise:

Problem: Simplify: $\frac{(10p^3)^2}{(5p)^3(2p^5)^4}$.

Solution:
Solution

	$\frac{(10p^3)^2}{(5p)^3(2p^5)^4}$
Use the Product to a Power Property, $(ab)^m = a^m b^m$.	

	$\frac{(10)^2(p^3)^2}{(5)^3(p)^3(2)^4(p^5)^4}$
Use the Power Property, $(a^m)^n = a^{m \cdot n}$.	$\frac{100p^6}{125p^3 \cdot 16p^{20}}$
Add the exponents in the denominator.	$\frac{100p^6}{125 \cdot 16p^{23}}$
Use the Quotient Property, $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$.	$\frac{100}{125 \cdot 16p^{17}}$
Simplify.	$\frac{1}{20p^{17}}$

Note:
Exercise:

Problem: Simplify: $\frac{(3r^3)^2(r^3)^7}{(r^3)^3}$.

Solution:

$9r^{18}$

Note:
Exercise:

Problem: Simplify: $\frac{(2x^4)^5}{(4x^3)^2(x^3)^5}$.

Solution:

$\frac{2}{x}$

Divide Monomials

You have now been introduced to all the properties of exponents and used them to simplify expressions. Next, you'll see how to use these properties to divide monomials. Later, you'll use them to divide polynomials.

Example:
Exercise:

Problem: Find the quotient: $56x^7 \div 8x^3$.

Solution:
Solution

	$56x^7 \div 8x^3$
Rewrite as a fraction.	$\frac{56x^7}{8x^3}$
Use fraction multiplication.	$\frac{56}{8} \cdot \frac{x^7}{x^3}$
Simplify and use the Quotient Property.	$7x^4$

Note:
Exercise:

Problem: Find the quotient: $42y^9 \div 6y^3$.

Solution:

$7y^6$

Note:
Exercise:

Problem: Find the quotient: $48z^8 \div 8z^2$.

Solution:

$6z^6$

Example:
Exercise:

Problem: Find the quotient: $\frac{45a^2b^3}{-5ab^5}$.

Solution:
Solution

	$\frac{45a^2b^3}{-5ab^5}$
Use fraction multiplication.	$\frac{45}{-5} \cdot \frac{a^2}{a} \cdot \frac{b^3}{b^5}$
Simplify and use the Quotient Property.	$-9 \cdot a \cdot \frac{1}{b^2}$
Multiply.	$-\frac{9a}{b^2}$

Note:
Exercise:

Problem: Find the quotient: $\frac{-72a^7b^3}{8a^{12}b^4}$.

Solution:

$-\frac{9}{a^5b}$

Note:
Exercise:

Problem: Find the quotient: $\frac{-63c^8d^3}{7c^{12}d^2}$.

Solution:

$\frac{-9d}{c^4}$

Example:
Exercise:

Problem: Find the quotient: $\frac{24a^5b^3}{48ab^4}$.

Solution:
Solution

	$\frac{24a^5b^3}{48ab^4}$
Use fraction multiplication.	$\frac{24}{48}\cdot\frac{a^5}{a}\cdot\frac{b^3}{b^4}$
Simplify and use the Quotient Property.	$\frac{1}{2}\cdot a^4\cdot\frac{1}{b}$
Multiply.	$\frac{a^4}{2b}$

Note:
Exercise:

Problem: Find the quotient: $\frac{16a^7b^6}{24ab^8}$.

Solution:

$\frac{2a^6}{3b^2}$

Note:
Exercise:

Problem: Find the quotient: $\frac{27p^4q^7}{-45p^{12}q}$.

Solution:

$-\frac{3q^6}{5p^8}$

Once you become familiar with the process and have practiced it step by step several times, you may be able to simplify a fraction in one step.

Example:
Exercise:

Problem: Find the quotient: $\frac{14x^7y^{12}}{21x^{11}y^6}$.

Solution:
Solution

Be very careful to simplify $\frac{14}{21}$ by dividing out a common factor, and to simplify the variables by subtracting their exponents.

	$\frac{14x^7y^{12}}{21x^{11}y^6}$
Simplify and use the Quotient Property.	$\frac{2y^6}{3x^4}$

Note:
Exercise:

Problem: Find the quotient: $\frac{28x^5y^{14}}{49x^9y^{12}}$.

Solution:

$\frac{4y^2}{7x^4}$

Note:
Exercise:

Problem: Find the quotient: $\frac{30m^5n^{11}}{48m^{10}n^{14}}$.

Solution:

$\frac{5}{8m^5n^3}$

In all examples so far, there was no work to do in the numerator or denominator before simplifying the fraction. In the next example, we'll first find the product of two monomials in the numerator before we simplify the fraction. This follows the order of operations. Remember, a fraction bar is a grouping symbol.

Example:
Exercise:

Problem: Find the quotient: $\frac{\left(6x^2y^3\right)\left(5x^3y^2\right)}{\left(3x^4y^5\right)}$.

Solution:
Solution

	$\frac{\left(6x^2y^3\right)\left(5x^3y^2\right)}{\left(3x^4y^5\right)}$
Simplify the numerator.	$\frac{30x^5y^5}{3x^4y^5}$
Simplify.	$10x$

Note:
Exercise:

Problem: Find the quotient: $\frac{\left(6a^4b^5\right)\left(4a^2b^5\right)}{12a^5b^8}$.

Solution:

$2ab^2$

Note:
Exercise:

Problem: Find the quotient: $\frac{\left(-12x^6y^9\right)\left(-4x^5y^8\right)}{-12x^{10}y^{12}}$.

Solution:

$-4xy^5$

Note:
Access these online resources for additional instruction and practice with dividing monomials:

- Rational Expressions
- Dividing Monomials
- Dividing Monomials 2

Key Concepts

- **Quotient Property for Exponents:**
  - If a is a real number, $a \neq 0$, and m, n are whole numbers, then:
    $\frac{a^m}{a^n} = a^{m-n}, m > n \text{ and } \frac{a^m}{a^n} = \frac{1}{a^{m-n}}, n > m$
- **Zero Exponent**
  - If a is a non-zero number, then $a^0 = 1$.
- **Quotient to a Power Property for Exponents**:
  - If a and b are real numbers, $b \neq 0$, and m is a counting number, then:
    $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$
  - To raise a fraction to a power, raise the numerator and denominator to that power.

- **Summary of Exponent Properties**
  - If a, b are real numbers and m, n are whole numbers, then

$$\begin{array}{lrcl} \textbf{Product Property} & a^m \cdot a^n & = & a^{m+n} \\ \textbf{Power Property} & (a^m)^n & = & a^{m \cdot n} \\ \textbf{Product to a Power} & (ab)^m & = & a^m b^m \\ \textbf{Quotient Property} & \frac{a^m}{b^m} & = & a^{m-n}, a \neq 0, m > n \\ & \frac{a^m}{a^n} & = & \frac{1}{a^{n-m}}, a \neq 0, n > m \\ \textbf{Zero Exponent Definition} & a^o & = & 1, a \neq 0 \\ \textbf{Quotient to a Power Property} & \left(\frac{a}{b}\right)^m & = & \frac{a^m}{b^m}, b \neq 0 \end{array}$$

Practice Makes Perfect

Simplify Expressions Using the Quotient Property for Exponents

In the following exercises, simplify.

Exercise:

Problem: ⓐ $\frac{x^{18}}{x^3}$ ⓑ $\frac{5^{12}}{5^3}$

Exercise:

Problem: ⓐ $\frac{y^{20}}{y^{10}}$ ⓑ $\frac{7^{16}}{7^2}$

Solution:

ⓐ y^{10} ⓑ 7^{14}

Exercise:

Problem: ⓐ $\frac{p^{21}}{p^7}$ ⓑ $\frac{4^{16}}{4^4}$

Exercise:

Problem: ⓐ $\frac{u^{24}}{u^3}$ ⓑ $\frac{9^{15}}{9^5}$

Solution:

ⓐ u^{21} ⓑ 9^{10}

Exercise:

Problem: ⓐ $\frac{q^{18}}{q^{36}}$ ⓑ $\frac{10^2}{10^3}$

Exercise:

Problem: ⓐ $\frac{t^{10}}{t^{40}}$ ⓑ $\frac{8^3}{8^5}$

Solution:

ⓐ $\frac{1}{t^{30}}$ ⓑ $\frac{1}{64}$

Exercise:

Problem: ⓐ $\frac{b}{b^9}$ ⓑ $\frac{4}{4^6}$

Exercise:

Problem: ⓐ $\frac{x}{x^7}$ ⓑ $\frac{10}{10^3}$

Solution:

ⓐ $\frac{1}{x^6}$ ⓑ $\frac{1}{100}$

Simplify Expressions with Zero Exponents

In the following exercises, simplify.

Exercise:

Problem: ⓐ 20^0
ⓑ b^0

Exercise:

Problem: ⓐ 13^0
ⓑ k^0

Solution:

ⓐ 1 ⓑ 1

Exercise:

Problem: ⓐ -27^0
ⓑ $-\left(27^0\right)$

Exercise:

Problem: ⓐ -15^0
ⓑ $-\left(15^0\right)$

Solution:

ⓐ -1 ⓑ -1

Exercise:

Problem: ⓐ $(25x)^0$
ⓑ $25x^0$

Exercise:

Problem: ⓐ $(6y)^0$
ⓑ $6y^0$

Solution:

ⓐ 1 ⓑ 6

Exercise:

Problem: ⓐ $(12x)^0$
ⓑ $\left(-56p^4q^3\right)^0$

Exercise:

Problem: ⓐ $7y^0(17y)^0$ ⓑ $\left(-93c^7d^{15}\right)^0$

Solution:

ⓐ 7 ⓑ 1

Exercise:

Problem: ⓐ $12n^0 - 18m^0$ ⓑ $(12n)^0 - (18m)^0$

Exercise:

Problem: ⓐ $15r^0 - 22s^0$ ⓑ $(15r)^0 - (22s)^0$

Solution:

ⓐ -7 ⓑ 0

Simplify Expressions Using the Quotient to a Power Property

In the following exercises, simplify.

Exercise:

Problem: ⓐ $\left(\frac{3}{4}\right)^3$ ⓑ $\left(\frac{p}{2}\right)^5$ ⓒ $\left(\frac{x}{y}\right)^6$

Exercise:

Problem: ⓐ $\left(\frac{2}{5}\right)^2$ ⓑ $\left(\frac{x}{3}\right)^4$ ⓒ $\left(\frac{a}{b}\right)^5$

Solution:

ⓐ $\frac{4}{25}$ ⓑ $\frac{x^4}{81}$ ⓒ $\frac{a^5}{b^5}$

Exercise:

Problem: ⓐ $\left(\frac{a}{3b}\right)^4$ ⓑ $\left(\frac{5}{4m}\right)^2$

Exercise:

Problem: ⓐ $\left(\frac{x}{2y}\right)^3$ ⓑ $\left(\frac{10}{3q}\right)^4$

Solution:

ⓐ $\frac{x^3}{8y^3}$ ⓑ $\frac{10{,}000}{81q^4}$

Simplify Expressions by Applying Several Properties

In the following exercises, simplify.

Exercise:

Problem: $\frac{\left(a^2\right)^3}{a^4}$

Exercise:

Problem: $\frac{\left(p^3\right)^4}{p^5}$

Solution:

p^7

Exercise:

Problem: $\frac{\left(y^3\right)^4}{y^{10}}$

Exercise:

Problem: $\frac{\left(x^4\right)^5}{x^{15}}$

Solution:

x^5

Exercise:

Problem: $\frac{u^6}{(u^3)^2}$

Exercise:

Problem: $\frac{v^{20}}{(v^4)^5}$

Solution:

1

Exercise:

Problem: $\frac{m^{12}}{(m^8)^3}$

Exercise:

Problem: $\frac{n^8}{(n^6)^4}$

Solution:

$\frac{1}{n^{16}}$

Exercise:

Problem: $\left(\frac{p^9}{p^3}\right)^5$

Exercise:

Problem: $\left(\frac{q^8}{q^2}\right)^3$

Solution:

q^{18}

Exercise:

Problem: $\left(\frac{r^2}{r^6}\right)^3$

Exercise:

Problem: $\left(\frac{m^4}{m^7}\right)^4$

Solution:

$\frac{1}{m^{12}}$

Exercise:

Problem: $\left(\frac{p}{r^{11}}\right)^2$

Exercise:

Problem: $\left(\frac{a}{b^6}\right)^3$

Solution:

$\frac{a^3}{b^{18}}$

Exercise:

Problem: $\left(\frac{w^5}{x^3}\right)^8$

Exercise:

Problem: $\left(\frac{y^4}{z^{10}}\right)^5$

Solution:

$\frac{y^{20}}{z^{50}}$

Exercise:

Problem: $\left(\frac{2j^3}{3k}\right)^4$

Exercise:

Problem: $\left(\frac{3m^5}{5n}\right)^3$

Solution:

$\frac{27m^{15}}{125n^3}$

Exercise:

Problem: $\left(\frac{3c^2}{4d^6}\right)^3$

Exercise:

Problem: $\left(\frac{5u^7}{2v^3}\right)^4$

Solution:

$\frac{625u^{28}}{16v^{12}}$

Exercise:

Problem: $\left(\frac{k^2k^8}{k^3}\right)^2$

Exercise:

Problem: $\left(\frac{j^2j^5}{j^4}\right)^3$

Solution:

j^9

Exercise:

Problem: $\frac{\left(t^2\right)^5\left(t^4\right)^2}{\left(t^3\right)^7}$

Exercise:

Problem: $\frac{(q^3)^6(q^2)^3}{(q^4)^8}$

Solution:

$\frac{1}{q^8}$

Exercise:

Problem: $\frac{(-2p^2)^4(3p^4)^2}{(-6p^3)^2}$

Exercise:

Problem: $\frac{(-2k^3)^2(6k^2)^4}{(9k^4)^2}$

Solution:

$64k^6$

Exercise:

Problem: $\frac{(-4m^3)^2(5m^4)^3}{(-10m^6)^3}$

Exercise:

Problem: $\frac{(-10n^2)^3(4n^5)^2}{(2n^8)^2}$

Solution:

$-4{,}000$

Divide Monomials

In the following exercises, divide the monomials.

Exercise:

Problem: $56b^8 \div 7b^2$

Exercise:

Problem: $63v^{10} \div 9v^2$

Solution:

$7v^8$

Exercise:

Problem: $-88y^{15} \div 8y^3$

Exercise:

Problem: $-72u^{12} \div 12u^4$

Solution:

$-6u^8$

Exercise:

Problem: $\frac{45a^6b^8}{-15a^{10}b^2}$

Exercise:

Problem: $\frac{54x^9y^3}{-18x^6y^{15}}$

Solution:

$-\frac{3x^3}{y^{12}}$

Exercise:

Problem: $\frac{15r^4s^9}{18r^9s^2}$

Exercise:

Problem: $\frac{20m^8n^4}{30m^5n^9}$

Solution:

$\frac{-2m^3}{3n^5}$

Exercise:

Problem: $\frac{18a^4b^8}{-27a^9b^5}$

Exercise:

Problem: $\frac{45x^5y^9}{-60x^8y^6}$

Solution:

$\frac{-3y^3}{4x^3}$

Exercise:

Problem: $\frac{64q^{11}r^9s^3}{48q^6r^8s^5}$

Exercise:

Problem: $\frac{65a^{10}b^8c^5}{42a^7b^6c^8}$

Solution:

$\frac{65a^3b^2}{42c^3}$

Exercise:

Problem: $\frac{\left(10m^5n^4\right)\left(5m^3n^6\right)}{25m^7n^5}$

Exercise:

Problem: $\frac{\left(-18p^4q^7\right)\left(-6p^3q^8\right)}{-36p^{12}q^{10}}$

Solution:

$\frac{-3q^5}{p^5}$

Exercise:

Problem: $\frac{(6a^4b^3)(4ab^5)}{(12a^2b)(a^3b)}$

Exercise:

Problem: $\frac{(4u^2v^5)(15u^3v)}{(12u^3v)(u^4v)}$

Solution:

$\frac{5v^4}{u^2}$

Mixed Practice

Exercise:

Problem:
ⓐ $24a^5 + 2a^5$
ⓑ $24a^5 - 2a^5$
ⓒ $24a^5 \cdot 2a^5$
ⓓ $24a^5 \div 2a^5$

Exercise:

Problem:
ⓐ $15n^{10} + 3n^{10}$
ⓑ $15n^{10} - 3n^{10}$
ⓒ $15n^{10} \cdot 3n^{10}$
ⓓ $15n^{10} \div 3n^{10}$

Solution:

ⓐ $18n^{10}$
ⓑ $12n^{10}$
ⓒ $45n^{20}$
ⓓ 5

Exercise:

Problem:
ⓐ $p^4 \cdot p^6$
ⓑ $(p^4)^6$

Exercise:

Problem: ⓐ $q^5 \cdot q^3$
ⓑ $\left(q^5\right)^3$

Solution:

ⓐ q^8
ⓑ q^{15}

Exercise:

Problem: ⓐ $\frac{y^3}{y}$
ⓑ $\frac{y}{y^3}$

Exercise:

Problem: ⓐ $\frac{z^6}{z^5}$
ⓑ $\frac{z^5}{z^6}$

Solution:

ⓐ z ⓑ $\frac{1}{z}$

Exercise:

Problem: $\left(8x^5\right)(9x) \div 6x^3$

Exercise:

Problem: $(4y)\left(12y^7\right) \div 8y^2$

Solution:

$6y^6$

Exercise:

Problem: $\frac{27a^7}{3a^3} + \frac{54a^9}{9a^5}$

Exercise:

Problem: $\frac{32c^{11}}{4c^5} + \frac{42c^9}{6c^3}$

Solution:

$15c^6$

Exercise:

Problem: $\frac{32y^5}{8y^2} - \frac{60y^{10}}{5y^7}$

Exercise:

Problem: $\frac{48x^6}{6x^4} - \frac{35x^9}{7x^7}$

Solution:

$3x^2$

Exercise:

Problem: $\frac{63r^6s^3}{9r^4s^2} - \frac{72r^2s^2}{6s}$

Exercise:

Problem: $\frac{56y^4z^5}{7y^3z^3} - \frac{45y^2z^2}{5y}$

Solution:

$-yz^2$

Everyday Math

Exercise:

Problem:

Memory One megabyte is approximately 10^6 bytes. One gigabyte is approximately 10^9 bytes. How many megabytes are in one gigabyte?

Exercise:

Problem:

Memory One gigabyte is approximately 10^9 bytes. One terabyte is approximately 10^{12} bytes. How many gigabytes are in one terabyte?

Solution:

10^3

Writing Exercises

Exercise:

Problem:

Jennifer thinks the quotient $\frac{a^{24}}{a^6}$ simplifies to a^4. What is wrong with her reasoning?

Exercise:

Problem:

Maurice simplifies the quotient $\frac{d^7}{d}$ by writing $\frac{\cancel{d}^7}{\cancel{d}} = 7$. What is wrong with his reasoning?

Solution:

Answers will vary.

Exercise:

Problem:

When Drake simplified -3^0 and $(-3)^0$ he got the same answer. Explain how using the Order of Operations correctly gives different answers.

Exercise:

Problem:

Robert thinks x^0 simplifies to 0. What would you say to convince Robert he is wrong?

Solution:

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
simplify expressions using the Quotient Property for Exponents.			
simplify expressions with zero exponents.			
simplify expressions using the Quotient to a Power Property.			
simplify expressions by applying several properties.			
divide monomials.			

ⓑ On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Divide Polynomials: ASE
By the end of this section, you will be able to:

- Divide a polynomial by a monomial
- Divide a polynomial by a binomial

Divide a Polynomial by a Monomial

In the last section, you learned how to divide a monomial by a monomial. As you continue to build up your knowledge of polynomials the next procedure is to divide a polynomial of two or more terms by a monomial.

The method we'll use to divide a polynomial by a monomial is based on the properties of fraction addition. So we'll start with an example to review fraction addition.

The sum,	$\frac{y}{5} + \frac{2}{5}$,
simplifies to	$\frac{y+2}{5}$.

Now we will do this in reverse to split a single fraction into separate fractions.

We'll state the fraction addition property here just as you learned it and in reverse.

Note:
Fraction Addition
If $a, b,$ and c are numbers where $c \neq 0$, then
Equation:

$$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c} \quad \text{and} \quad \frac{a+b}{c}=\frac{a}{c}+\frac{b}{c}$$

We use the form on the left to add fractions and we use the form on the right to divide a polynomial by a monomial.

For example,	$\frac{y+2}{5}$
can be written	$\frac{y}{5}+\frac{2}{5}$.

We use this form of fraction addition to divide polynomials by monomials.

Note:
Division of a Polynomial by a Monomial
To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.

Example:
Exercise:

Problem: Find the quotient: $\frac{7y^2+21}{7}$.

Solution:
Solution

	$\frac{7y^2+21}{7}$
Divide each term of the numerator by the denominator.	$\frac{7y^2}{7}+\frac{21}{7}$
Simplify each fraction.	y^2+3

Note:
Exercise:

Problem: Find the quotient: $\frac{8z^2+24}{4}$.

Solution:

$2z^2+6$

Note:
Exercise:

Problem: Find the quotient: $\frac{18z^2-27}{9}$.

Solution:

$2z^2-3$

Remember that division can be represented as a fraction. When you are asked to divide a polynomial by a monomial and it is not already in fraction

form, write a fraction with the polynomial in the numerator and the monomial in the denominator.

Example:
Exercise:

Problem: Find the quotient: $\left(18x^3 - 36x^2\right) \div 6x$.

Solution:
Solution

	$\left(18x^3 - 36x^2\right) \div 6x$
Rewrite as a fraction.	$\frac{18x^3 - 36x^2}{6x}$
Divide each term of the numerator by the denominator.	$\frac{18x^3}{6x} - \frac{36x^2}{6x}$
Simplify.	$3x^2 - 6x$

Note:
Exercise:

Problem: Find the quotient: $\left(27b^3 - 33b^2\right) \div 3b$.

Solution:

$9b^2 - 11b$

Note:
Exercise:

Problem: Find the quotient: $\left(25y^3 - 55y^2\right) \div 5y$.

Solution:

$5y^2 - 11y$

When we divide by a negative, we must be extra careful with the signs.

Example:
Exercise:

Problem: Find the quotient: $\frac{12d^2 - 16d}{-4}$.

Solution:
Solution

	$\frac{12d^2 - 16d}{-4}$

Divide each term of the numerator by the denominator.	$\frac{12d^2}{-4} - \frac{16d}{-4}$
Simplify. Remember, subtracting a negative is like adding a positive!	$-3d^2 + 4d$

Note:
Exercise:

Problem: Find the quotient: $\frac{25y^2-15y}{-5}$.

Solution:

$-5y^2 + 3y$

Note:
Exercise:

Problem: Find the quotient: $\frac{42b^2-18b}{-6}$.

Solution:

$-7b^2 + 3b$

Example:
Exercise:

Problem: Find the quotient: $\frac{105y^5+75y^3}{5y^2}$.

Solution:
Solution

	$\frac{105y^5+75y^3}{5y^2}$
Separate the terms.	$\frac{105y^5}{5y^2} + \frac{75y^3}{5y^2}$
Simplify.	$21y^3 + 15y$

Note:
Exercise:

Problem: Find the quotient: $\frac{60d^7+24d^5}{4d^3}$.

Solution:

$15d^4 + 6d^2$

Note:
Exercise:

Problem: Find the quotient: $\frac{216p^7-48p^5}{6p^3}$.

Solution:

$36p^4 - 8p^2$

Example:
Exercise:

Problem: Find the quotient: $\left(15x^3y - 35xy^2\right) \div (-5xy)$.

Solution:
Solution

	$\left(15x^3y - 35xy^2\right) \div (-5xy)$
Rewrite as a fraction.	$\frac{15x^3y-35xy^2}{-5xy}$
Separate the terms.	$\frac{15x^3y}{-5xy} - \frac{35xy^2}{-5xy}$
Simplify.	$-3x^2 + 7y$

Note:

Exercise:

Problem: Find the quotient: $\left(32a^2b - 16ab^2\right) \div (-8ab)$.

Solution:

$-4a + 2b$

Note:
Exercise:

Problem: Find the quotient: $\left(-48a^8b^4 - 36a^6b^5\right) \div \left(-6a^3b^3\right)$.

Solution:

$8a^5b + 6a^3b^2$

Example:
Exercise:

Problem: Find the quotient: $\frac{36x^3y^2+27x^2y^2-9x^2y^3}{9x^2y}$.

Solution:
Solution

	$\frac{36x^3y^2+27x^2y^2-9x^2y^3}{9x^2y}$

Separate the terms.	$\frac{36x^3y^2}{9x^2y} + \frac{27x^2y^2}{9x^2y} - \frac{9x^2y^3}{9x^2y}$
Simplify.	$4xy + 3y - y^2$

Note:
Exercise:

Problem: Find the quotient: $\frac{40x^3y^2+24x^2y^2-16x^2y^3}{8x^2y}$.

Solution:

$5xy + 3y - 2y^2$

Note:
Exercise:

Problem: Find the quotient: $\frac{35a^4b^2+14a^4b^3-42a^2b^4}{7a^2b^2}$.

Solution:

$5a^2 + 2a^2b - 6b^2$

Example:
Exercise:

Problem: Find the quotient: $\frac{10x^2+5x-20}{5x}$.

Solution:
Solution

	$\frac{10x^2+5x-20}{5x}$
Separate the terms.	$\frac{10x^2}{5x} + \frac{5x}{5x} - \frac{20}{5x}$
Simplify.	$2x + 1 + \frac{4}{x}$

Note:
Exercise:

Problem: Find the quotient: $\frac{18c^2+6c-9}{6c}$.

Solution:

$3c + 1 - \frac{3}{2c}$

Note:
Exercise:

Problem: Find the quotient: $\frac{10d^2-5d-2}{5d}$.

Solution:

$2d - 1 - \frac{2}{5d}$

Divide a Polynomial by a Binomial

To divide a polynomial by a binomial, we follow a procedure very similar to long division of numbers. So let’s look carefully the steps we take when we divide a 3-digit number, 875, by a 2-digit number, 25.

We write the long division	$25\overline{)875}$
We divide the first two digits, 87, by 25.	$\begin{array}{r} 3 \\ 25\overline{)875} \end{array}$
We multiply 3 times 25 and write the product under the 87.	$\begin{array}{r} 3 \\ 25\overline{)875} \\ \underline{75} \end{array}$
Now we subtract 75 from 87.	

	$\begin{array}{r} 3 \\ 25\overline{)875} \\ -75 \\ \hline 12 \end{array}$
Then we bring down the third digit of the dividend, 5.	$\begin{array}{r} 3 \\ 25\overline{)875} \\ -75 \\ \hline 125 \end{array}$
Repeat the process, dividing 25 into 125.	$\begin{array}{r} 35 \\ 25\overline{)875} \\ -75 \\ \hline 125 \\ -125 \\ \hline \end{array}$

We check division by multiplying the quotient by the divisor.

If we did the division correctly, the product should equal the dividend.
Equation:

$$35 \cdot 25$$
$$875 \checkmark$$

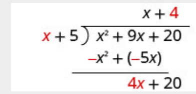
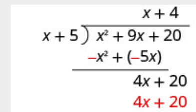
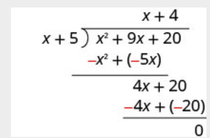
Now we will divide a trinomial by a binomial. As you read through the example, notice how similar the steps are to the numerical example above.

Example:
Exercise:

Problem: Find the quotient: $\left(x^2 + 9x + 20\right) \div (x + 5)$.

Solution:
Solution

	$(x^2 + 9x + 20) \div (x + 5)$
Write it as a long division problem.	
Be sure the dividend is in standard form.	$x + 5\overline{)\,x^2 + 9x + 20}$
Divide x^2 by x. It may help to ask yourself, "What do I need to multiply x by to get x^2?"	
Put the answer, x, in the quotient over the x term.	$\begin{array}{r} x \\ x + 5\overline{)\,x^2 + 9x + 20} \end{array}$
Multiply x times $x + 5$. Line up the like terms under the dividend.	$\begin{array}{r} x \\ x + 5\overline{)\,x^2 + 9x + 20} \\ \underline{x^2 + 5x} \end{array}$
Subtract $x^2 + 5x$ from $x^2 + 9x$.	
You may find it easier to change the signs and then add. Then bring down the last term, 20.	$\begin{array}{r} x \\ x + 5\overline{)\,x^2 + 9x + 20} \\ \underline{-x^2 + (-5x)} \\ 4x + 20 \end{array}$
Divide $4x$ by x. It may help to ask yourself, "What do I need to	

multiply x by to get $4x$?"	
Put the answer, 4, in the quotient over the constant term.	
Multiply 4 times $x + 5$.	
Subtract $4x + 20$ from $4x + 20$.	
Check:	
Multiply the quotient by the divisor.	
$(x + 4)(x + 5)$	
You should get the dividend.	
$x^2 + 9x + 20$✓	

Note:
Exercise:

Problem: Find the quotient: $\left(y^2 + 10y + 21\right) \div (y + 3)$.

Solution:

$y + 7$

Note:
Exercise:

Problem: Find the quotient: $\left(m^2 + 9m + 20\right) \div (m + 4)$.

Solution:

$m + 5$

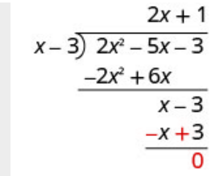
When the divisor has subtraction sign, we must be extra careful when we multiply the partial quotient and then subtract. It may be safer to show that we change the signs and then add.

Example:
Exercise:

Problem: Find the quotient: $\left(2x^2 - 5x - 3\right) \div (x - 3)$.

Solution:
Solution

	$(2x^2-5x-3)\div(x-3)$
Write it as a long division problem.	
Be sure the dividend is in standard form.	$x-3\overline{)2x^2-5x-3}$
Divide $2x^2$ by x. Put the answer, $2x$, in the quotient over the x term.	$\begin{array}{r} 2x \\ x-3\overline{)2x^2-5x-3} \end{array}$
Multiply $2x$ times $x-3$. Line up the like terms under the dividend.	$\begin{array}{r} 2x \\ x-3\overline{)2x^2-5x-3} \\ \underline{2x^2-6x} \end{array}$
Subtract $2x^2-6x$ from $2x^2-5x$. Change the signs and then add. Then bring down the last term.	$\begin{array}{r} 2x \\ x-3\overline{)\ 2x^2-5x-3} \\ \underline{-2x^2+6x} \\ x-3 \end{array}$
Divide x by x. Put the answer, 1, in the quotient over the constant term.	$\begin{array}{r} 2x+1 \\ x-3\overline{)\ 2x^2-5x-3} \\ \underline{-2x^2-(-6x)} \\ x-3 \end{array}$
Multiply 1 times $x-3$.	$\begin{array}{r} 2x+1 \\ x-3\overline{)\ 2x^2-5x-3} \\ \underline{-2x^2+6x} \\ x-3 \\ \underline{x-3} \end{array}$
Subtract $x-3$ from $x-3$ by changing the signs and adding.	

	
To check, multiply $(x - 3)(2x + 1)$.	
The result should be $2x^2 - 5x - 3$.	

Note:
Exercise:

Problem: Find the quotient: $\left(2x^2 - 3x - 20\right) \div (x - 4)$.

Solution:

$2x + 5$

Note:
Exercise:

Problem: Find the quotient: $\left(3x^2 - 16x - 12\right) \div (x - 6)$.

Solution:

$3x + 2$

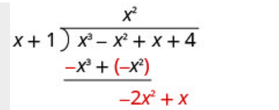
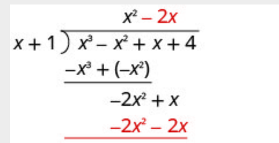
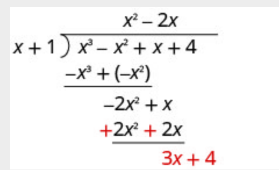
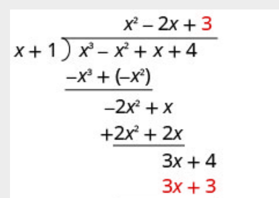
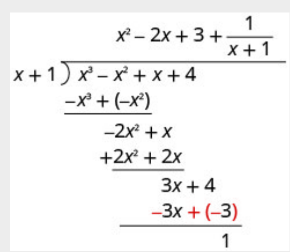
When we divided 875 by 25, we had no remainder. But sometimes division of numbers does leave a remainder. The same is true when we divide polynomials. In [link], we'll have a division that leaves a remainder. We write the remainder as a fraction with the divisor as the denominator.

Example:
Exercise:

Problem: Find the quotient: $\left(x^3 - x^2 + x + 4\right) \div (x+1)$.

Solution:
Solution

	$(x^3 - x^2 + x + 4) \div (x + 1)$
Write it as a long division problem.	
Be sure the dividend is in standard form.	$x + 1\overline{)\,x^3 - x^2 + x + 4}$
Divide x^3 by x. Put the answer, x^2, in the quotient over the x^2 term. Multiply x^2 times $x + 1$. Line up the like terms under the dividend.	$\begin{array}{l} \qquad\qquad x^2 \\ x + 1\overline{)\,x^3 - x^2 + x + 4} \\ \qquad\ \underline{x^3 + x^2} \end{array}$
Subtract $x^3 + x^2$ from $x^3 - x^2$ by changing the signs and adding.	

Then bring down the next term.	
Divide $-2x^2$ by x. Put the answer, $-2x$, in the quotient over the x term. Multiply $-2x$ times $x + 1$. Line up the like terms under the dividend.	
Subtract $-2x^2 - 2x$ from $-2x^2 + x$ by changing the signs and adding. Then bring down the last term.	
Divide $3x$ by x. Put the answer, 3, in the quotient over the constant term. Multiply 3 times $x + 1$. Line up the like terms under the dividend.	
Subtract $3x + 3$ from $3x + 4$ by changing the signs and adding. Write the remainder as a fraction with the divisor as the denominator.	
To check, multiply $(x+1)\left(x^2 - 2x + 3 + \frac{1}{x+1}\right)$. The result should be $x^3 - x^2 + x + 4$.	

Note:
Exercise:

Problem: Find the quotient: $\left(x^3+5x^2+8x+6\right)\div(x+2)$.

Solution:

$x^2+3x+2+\frac{2}{x+2}$

Note:
Exercise:

Problem: Find the quotient: $\left(2x^3+8x^2+x-8\right)\div(x+1)$.

Solution:

$2x^2+6x-5-\frac{3}{x+1}$

Look back at the dividends in [link], [link], and [link]. The terms were written in descending order of degrees, and there were no missing degrees. The dividend in [link] will be x^4-x^2+5x-2. It is missing an x^3 term. We will add in $0x^3$ as a placeholder.

Example:
Exercise:

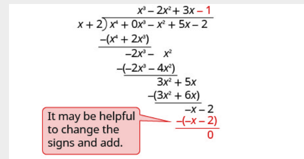
Problem: Find the quotient: $\left(x^4-x^2+5x-2\right)\div(x+2)$.

Solution:

Solution

Notice that there is no x^3 term in the dividend. We will add $0x^3$ as a placeholder.

	$(x^4 - x^2 + 5x - 2) \div (x + 2)$
Write it as a long division problem. Be sure the dividend is in standard form with placeholders for missing terms.	$x + 2\overline{)\,x^4 - 0x^3 - x^2 + 5x - 2}$
Divide x^4 by x. Put the answer, x^3, in the quotient over the x^3 term. Multiply x^3 times $x + 2$. Line up the like terms. Subtract and then bring down the next term.	x^3 $x + 2\overline{)\,x^4 + 0x^3 - x^2 + 5x - 2}$ $\underline{-(x^4 + 2x^3)}$ $-2x^3 - x^2$ It may be helpful to change the signs and add.
Divide $-2x^3$ by x. Put the answer, $-2x^2$, in the quotient over the x^2 term. Multiply $-2x^2$ times $x + 1$. Line up the like terms. Subtract and bring down the next term.	$x^3 - 2x^2$ $x + 2\overline{)\,x^4 + 0x^3 - x^2 + 5x - 2}$ $\underline{-(x^4 + 2x^3)}$ $-2x^3 - x^2$ $\underline{-(-2x^3 - 4x^2)}$ $3x^2 + 5x$ It may be helpful to change the signs and add.
Divide $3x^2$ by x. Put the answer, $3x$, in the quotient over the x term. Multiply $3x$ times $x + 1$. Line up the like	$x^3 - 2x^2 + 3x$ $x + 2\overline{)\,x^4 + 0x^3 - x^2 + 5x - 2}$ $\underline{-(x^4 + 2x^3)}$ $-2x^3 - x^2$ $\underline{-(-2x^3 - 4x^2)}$ $3x^2 + 5x$ $\underline{-(3x^2 + 6x)}$ $-x - 2$ It may be helpful to change the signs and add.

terms. Subtract and bring down the next term.	
Divide −*x* by *x*. Put the answer, −1, in the quotient over the constant term. Multiply −1 times *x* + 1. Line up the like terms. Change the signs, add.	
To check, multiply $(x+2)\left(x^3-2x^2+3x-1\right)$.	
The result should be x^4-x^2+5x-2.	

Note:
Exercise:

Problem: Find the quotient: $\left(x^3+3x+14\right) \div (x+2)$.

Solution:

x^2-3x+7 x^2-2x+7

Note:
Exercise:

Problem: Find the quotient: $\left(x^4-3x^3-1000\right) \div (x+5)$.

Solution:

$x^3 - 8x^2 + 40x - 200$

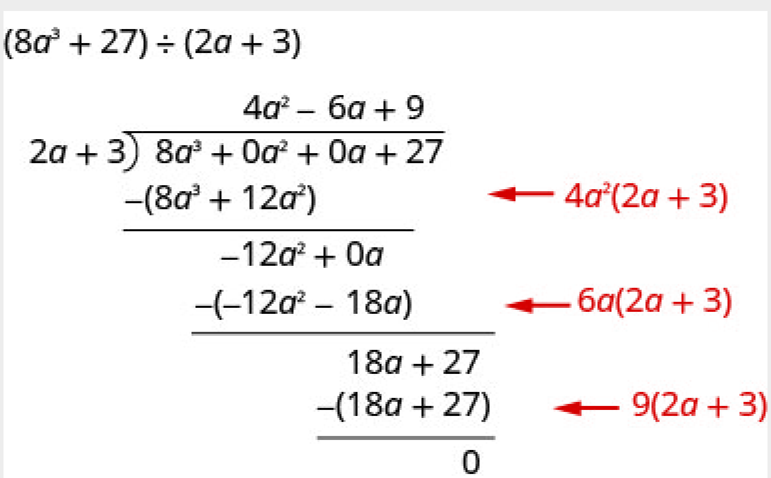
In [link], we will divide by $2a - 3$. As we divide we will have to consider the constants as well as the variables.

Example:
Exercise:

Problem: Find the quotient: $\left(8a^3 + 27\right) \div (2a + 3)$.

Solution:
Solution

This time we will show the division all in one step. We need to add two placeholders in order to divide.



To check, multiply $(2a + 3)\left(4a^2 - 6a + 9\right)$.

The result should be $8a^3 + 27$.

Note:
Exercise:

Problem: Find the quotient: $\left(x^3 - 64\right) \div (x - 4)$.

Solution:

$x^2 + 4x + 16$

Note:
Exercise:

Problem: Find the quotient: $\left(125x^3 - 8\right) \div (5x - 2)$.

Solution:

$25x^2 + 10x + 4$

Note:
Access these online resources for additional instruction and practice with dividing polynomials:

- Divide a Polynomial by a Monomial
- Divide a Polynomial by a Monomial 2
- Divide Polynomial by Binomial

Key Concepts

- **Fraction Addition**

- If $a, b,$ and c are numbers where $c \neq 0$, then $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$ and $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$

- **Division of a Polynomial by a Monomial**
  - To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.

Practice Makes Perfect

In the following exercises, divide each polynomial by the monomial.

Exercise:

Problem: $\frac{45y+36}{9}$

Exercise:

Problem: $\frac{30b+75}{5}$

Solution:

$6b + 15$

Exercise:

Problem: $\frac{8d^2-4d}{2}$

Exercise:

Problem: $\frac{42x^2-14x}{7}$

Solution:

$6x^2 - 2x$

Exercise:

Problem: $\left(16y^2-20y\right)\div 4y$

Exercise:

Problem: $\left(55w^2-10w\right)\div 5w$

Solution:

$11w-2$

Exercise:

Problem: $\left(9n^4+6n^3\right)\div 3n$

Exercise:

Problem: $\left(8x^3+6x^2\right)\div 2x$

Solution:

$4x^2+3x$

Exercise:

Problem: $\frac{18y^2-12y}{-6}$

Exercise:

Problem: $\frac{20b^2-12b}{-4}$

Solution:

$-5b^2+3b$

Exercise:

Problem: $\frac{35a^4+65a^2}{-5}$

Exercise:

Problem: $\frac{51m^4+72m^3}{-3}$

Solution:

$-17m^4 - 24m^3$

Exercise:

Problem: $\frac{310y^4-200y^3}{5y^2}$

Exercise:

Problem: $\frac{412z^8-48z^5}{4z^3}$

Solution:

$103z^5 - 12z^2$

Exercise:

Problem: $\frac{46x^3+38x^2}{2x^2}$

Exercise:

Problem: $\frac{51y^4+42y^2}{3y^2}$

Solution:

$17y^2 + 14$

Exercise:

Problem: $\left(24p^2 - 33p\right) \div (-3p)$

Exercise:

Problem: $\left(35x^4 - 21x\right) \div (-7x)$

Solution:

$-5x^3 + 3$

Exercise:

Problem: $\left(63m^4 - 42m^3\right) \div \left(-7m^2\right)$

Exercise:

Problem: $\left(48y^4 - 24y^3\right) \div \left(-8y^2\right)$

Solution:

$-6y^2 + 3y$

Exercise:

Problem: $\left(63a^2b^3 + 72ab^4\right) \div (9ab)$

Exercise:

Problem: $\left(45x^3y^4 + 60xy^2\right) \div (5xy)$

Solution:

$9x^2y^3 + 12y$

Exercise:

Problem: $\frac{52p^5q^4+36p^4q^3-64p^3q^2}{4p^2q}$

Exercise:

Problem: $\frac{49c^2d^2-70c^3d^3-35c^2d^4}{7cd^2}$

Solution:

$7c-10c^2d-5cd^2$

Exercise:

Problem: $\frac{66x^3y^2-110x^2y^3-44x^4y^3}{11x^2y^2}$

Exercise:

Problem: $\frac{72r^5s^2+132r^4s^3-96r^3s^5}{12r^2s^2}$

Solution:

$6r^3+11r^2s-8rs^3$

Exercise:

Problem: $\frac{4w^2+2w-5}{2w}$

Exercise:

Problem: $\frac{12q^2+3q-1}{3q}$

Solution:

$4q+1-\frac{1}{3q}$

Exercise:

Problem: $\frac{10x^2+5x-4}{-5x}$

Exercise:

Problem: $\frac{20y^2+12y-1}{-4y}$

Solution:

$-5y-3+\frac{1}{4y}$

Exercise:

Problem: $\frac{36p^3+18p^2-12p}{6p^2}$

Exercise:

Problem: $\frac{63a^3-108a^2+99a}{9a^2}$

Solution:

$7a-12+\frac{11}{a}$

Divide a Polynomial by a Binomial

In the following exercises, divide each polynomial by the binomial.

Exercise:

Problem: $\left(y^2+7y+12\right)\div(y+3)$

Exercise:

Problem: $\left(d^2+8d+12\right)\div(d+2)$

Solution:

$d+6$

Exercise:

Problem: $\left(x^2-3x-10\right) \div (x+2)$

Exercise:

Problem: $\left(a^2-2a-35\right) \div (a+5)$

Solution:

$a-7$

Exercise:

Problem: $\left(t^2-12t+36\right) \div (t-6)$

Exercise:

Problem: $\left(x^2-14x+49\right) \div (x-7)$

Solution:

$x-7$

Exercise:

Problem: $\left(6m^2-19m-20\right) \div (m-4)$

Exercise:

Problem: $\left(4x^2-17x-15\right) \div (x-5)$

Solution:

$4x+3$

Exercise:

Problem: $\left(q^2+2q+20\right) \div (q+6)$

Exercise:

Problem: $\left(p^2+11p+16\right) \div (p+8)$

Solution:

$p+3-\frac{8}{p+8}$

Exercise:

Problem: $\left(y^2-3y-15\right) \div (y-8)$

Exercise:

Problem: $\left(x^2+2x-30\right) \div (x-5)$

Solution:

$x+7+\frac{5}{x-5}$

Exercise:

Problem: $\left(3b^3+b^2+2\right) \div (b+1)$

Exercise:

Problem: $\left(2n^3-10n+24\right) \div (n+3)$

Solution:

$2n^2-6n+8$

Exercise:

Problem: $\left(2y^3-6y-36\right)\div(y-3)$

Exercise:

Problem: $\left(7q^3-5q-2\right)\div(q-1)$

Solution:

$7q^2+7q+2$

Exercise:

Problem: $\left(z^3+1\right)\div(z+1)$

Exercise:

Problem: $\left(m^3+1000\right)\div(m+10)$

Solution:

$m^2-10m+100$

Exercise:

Problem: $\left(a^3-125\right)\div(a-5)$

Exercise:

Problem: $\left(x^3-216\right)\div(x-6)$

Solution:

$x^2+6x+36$

Exercise:

Problem: $\left(64x^3 - 27\right) \div (4x - 3)$

Exercise:

Problem: $\left(125y^3 - 64\right) \div (5y - 4)$

Solution:

$25y^2 + 20x + 16$

Everyday Math

Exercise:

Problem:

Average cost Pictures Plus produces digital albums. The company’s average cost (in dollars) to make x albums is given by the expression $\frac{7x+500}{x}$.

ⓐ Find the quotient by dividing the numerator by the denominator.
ⓑ What will the average cost (in dollars) be to produce 20 albums?

Exercise:

Problem:

Handshakes At a company meeting, every employee shakes hands with every other employee. The number of handshakes is given by the expression $\frac{n^2-n}{2}$, where n represents the number of employees. How many handshakes will there be if there are 10 employees at the meeting?

Solution:

45

Writing Exercises

Exercise:

Problem:

James divides $48y+6$ by 6 this way: $\frac{48y+\cancel{6}}{\cancel{6}}=48y$. What is wrong with his reasoning?

Exercise:

Problem:

Divide $\frac{10x^2+x-12}{2x}$ and explain with words how you get each term of the quotient.

Solution:

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
divide a polynomial by a monomial.			
divide a polynomial by a binomial.			

ⓑ After reviewing this checklist, what will you do to become confident for all goals?

Integer Exponents and Scientific Notation: ASE
By the end of this section, you will be able to:

- Use the definition of a negative exponent
- Simplify expressions with integer exponents
- Convert from decimal notation to scientific notation
- Convert scientific notation to decimal form
- Multiply and divide using scientific notation

Use the Definition of a Negative Exponent

We saw that the Quotient Property for Exponents introduced earlier in this chapter, has two forms depending on whether the exponent is larger in the numerator or the denominator.

Note:
Quotient Property for Exponents
If a is a real number, $a \neq 0$, and m and n are whole numbers, then
Equation:

$$\frac{a^m}{a^n} = a^{m-n}, m > n \quad \text{and} \quad \frac{a^m}{a^n} = \frac{1}{a^{n-m}}, n > m$$

What if we just subtract exponents regardless of which is larger?

Let's consider $\frac{x^2}{x^5}$.

We subtract the exponent in the denominator from the exponent in the numerator.
Equation:

$$\frac{x^2}{x^5}$$
$$x^{2-5}$$
$$x^{-3}$$

We can also simplify $\frac{x^2}{x^5}$ by dividing out common factors:

$$\frac{\cancel{x} \cdot \cancel{x}}{\cancel{x} \cdot \cancel{x} \cdot x \cdot x \cdot x}$$
$$\frac{1}{x^3}$$

This implies that $x^{-3} = \frac{1}{x^3}$ and it leads us to the definition of a *negative exponent*.

Note:
Negative Exponent
If n is an integer and $a \neq 0$, then $a^{-n} = \frac{1}{a^n}$.

The negative exponent tells us we can re-write the expression by taking the reciprocal of the base and then changing the sign of the exponent.

Any expression that has negative exponents is not considered to be in simplest form. We will use the definition of a negative exponent and other properties of exponents to write the expression with only positive exponents.

For example, if after simplifying an expression we end up with the expression x^{-3}, we will take one more step and write $\frac{1}{x^3}$. The answer is considered to be in simplest form when it has only positive exponents.

Example:
Exercise:

Problem: Simplify: ⓐ 4^{-2} ⓑ 10^{-3}.

Solution:
Solution

ⓐ	4^{-2}
Use the definition of a negative exponent, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{4^2}$
Simplify.	$\frac{1}{16}$
ⓑ	10^{-3}
Use the definition of a negative exponent, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{10^3}$
Simplify.	$\frac{1}{1000}$

Note:
Exercise:

Problem: Simplify: ⓐ 2^{-3} ⓑ 10^{-7}.

Solution:

ⓐ $\frac{1}{8}$ ⓑ $\frac{1}{10^7}$

Note:
Exercise:

Problem: Simplify: ⓐ 3^{-2} ⓑ 10^{-4}.

Solution:

ⓐ $\frac{1}{9}$ ⓑ $\frac{1}{10,000}$

In [link] we raised an integer to a negative exponent. What happens when we raise a fraction to a negative exponent? We'll start by looking at what happens to a fraction whose numerator is one and whose denominator is an integer raised to a negative exponent.

	$\frac{1}{a^{-n}}$
Use the definition of a negative exponent, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{\frac{1}{a^n}}$
Simplify the complex fraction.	$1 \cdot \frac{a^n}{1}$
Multiply.	a^n

This leads to the Property of Negative Exponents.

Note:
Property of Negative Exponents
If n is an integer and $a \neq 0$, then $\frac{1}{a^{-n}} = a^n$.

Example:
Exercise:

Problem: Simplify: ⓐ $\frac{1}{y^{-4}}$ ⓑ $\frac{1}{3^{-2}}$.

Solution:
Solution

ⓐ	$\frac{1}{y^{-4}}$
Use the property of a negative exponent, $\frac{1}{a^{-n}} = a^n$.	y^4
ⓑ	$\frac{1}{3^{-2}}$
Use the property of a negative exponent, $\frac{1}{a^{-n}} = a^n$.	3^2
Simplify.	9

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{1}{p^{-8}}$ ⓑ $\frac{1}{4^{-3}}$.

Solution:

ⓐ p^8 ⓑ 64

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{1}{q^{-7}}$ ⓑ $\frac{1}{2^{-4}}$.

Solution:

ⓐ q^7 ⓑ 16

Suppose now we have a fraction raised to a negative exponent. Let's use our definition of negative exponents to lead us to a new property.

	$\left(\frac{3}{4}\right)^{-2}$
Use the definition of a negative exponent, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{\left(\frac{3}{4}\right)^2}$
Simplify the denominator.	$\frac{1}{\frac{9}{16}}$

Simplify the complex fraction.	$\frac{16}{9}$
But we know that $\frac{16}{9}$ is $\left(\frac{4}{3}\right)^2$.	
This tells us that:	$\left(\frac{3}{4}\right)^{-2} = \left(\frac{4}{3}\right)^2$

To get from the original fraction raised to a negative exponent to the final result, we took the reciprocal of the base—the fraction—and changed the sign of the exponent.

This leads us to the *Quotient to a Negative Power Property*.

Note:
Quotient to a Negative Exponent Property
If a and b are real numbers, $a \neq 0, b \neq 0$, and n is an integer, then $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$.

Example:
Exercise:

Problem: Simplify: ⓐ $\left(\frac{5}{7}\right)^{-2}$ ⓑ $\left(-\frac{2x}{y}\right)^{-3}$.

Solution:
Solution

ⓐ	$\left(\frac{5}{7}\right)^{-2}$
Use the Quotient to a Negative Exponent Property, $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$.	
Take the reciprocal of the fraction and change the sign of the exponent.	$\left(\frac{7}{5}\right)^2$
Simplify.	$\frac{49}{25}$
ⓑ	$\left(-\frac{2x}{y}\right)^{-3}$
Use the Quotient to a Negative Exponent Property, $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$.	
Take the reciprocal of the fraction and change the sign of the exponent.	$\left(-\frac{y}{2x}\right)^3$
Simplify.	$-\frac{y^3}{8x^3}$

Note:
Exercise:

Problem: Simplify: ⓐ $\left(\frac{2}{3}\right)^{-4}$ ⓑ $\left(-\frac{6m}{n}\right)^{-2}$.

Solution:

ⓐ $\frac{81}{16}$ ⓑ $\frac{n^2}{36m^2}$

Note:
Exercise:

Problem: Simplify: ⓐ $\left(\frac{3}{5}\right)^{-3}$ ⓑ $\left(-\frac{a}{2b}\right)^{-4}$.

Solution:

ⓐ $\frac{125}{27}$ ⓑ $\frac{16b^4}{a^4}$

When simplifying an expression with exponents, we must be careful to correctly identify the base.

Example:
Exercise:

Problem: Simplify: ⓐ $(-3)^{-2}$ ⓑ -3^{-2} ⓒ $\left(-\frac{1}{3}\right)^{-2}$ ⓓ $-\left(\frac{1}{3}\right)^{-2}$.

Solution:
Solution

ⓐ Here the exponent applies to the base -3.	$(-3)^{-2}$
Take the reciprocal of the base and change the sign of the exponent.	$\frac{1}{(-3)^{-2}}$
Simplify.	$\frac{1}{9}$
ⓑ The expression -3^{-2} means "find the opposite of 3^{-2}." Here the exponent	-3^{-2}

applies to the base $\left(-\frac{1}{3}\right)$.	
Rewrite as a product with -1.	$-1 \cdot 3^{-2}$
Take the reciprocal of the base and change the sign of the exponent.	$-1 \cdot \frac{1}{3^2}$
Simplify.	$-\frac{1}{9}$
ⓒ Here the exponent applies to the base $\left(-\frac{1}{3}\right)$.	$\left(-\frac{1}{3}\right)^{-2}$
Take the reciprocal of the base and change the sign of the exponent.	$\left(-\frac{3}{1}\right)^{2}$
Simplify.	9
ⓓ The expression $-\left(\frac{1}{3}\right)^{-2}$ means "find the opposite of $\left(\frac{1}{3}\right)^{-2}$." Here the exponent applies to the base $\left(\frac{1}{3}\right)$.	
Rewrite as a product with -1.	$-1 \cdot \left(\frac{1}{3}\right)^{-2}$
Take the reciprocal of the base and change the sign of the exponent.	$-1 \cdot \left(\frac{3}{1}\right)^{2}$
Simplify.	-9

Note:
Exercise:

Problem: Simplify: ⓐ $(-5)^{-2}$ ⓑ -5^{-2} ⓒ $\left(-\frac{1}{5}\right)^{-2}$ ⓓ $-\left(\frac{1}{5}\right)^{-2}$.

Solution:

ⓐ $\frac{1}{25}$ ⓑ $-\frac{1}{25}$ ⓒ 25 ⓓ -25

Note:
Exercise:

Problem: Simplify: ⓐ $(-7)^{-2}$ ⓑ -7^{-2}, ⓒ $\left(-\frac{1}{7}\right)^{-2}$ ⓓ $-\left(\frac{1}{7}\right)^{-2}$.

Solution:

ⓐ $\frac{1}{49}$ ⓑ $-\frac{1}{49}$ ⓒ 49 ⓓ -49

We must be careful to follow the Order of Operations. In the next example, parts (a) and (b) look similar, but the results are different.

Example:
Exercise:

Problem: Simplify: ⓐ $4 \cdot 2^{-1}$ ⓑ $(4 \cdot 2)^{-1}$.

Solution:
Solution

ⓐ Do exponents before multiplication.	$4 \cdot 2^{-1}$
Use $a^{-n} = \frac{1}{a^n}$.	$4 \cdot \frac{1}{2^1}$
Simplify.	2
ⓑ	$(4 \cdot 2)^{-1}$
Simplify inside the parentheses first.	$(8)^{-1}$
Use $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{8^1}$
Simplify.	$\frac{1}{8}$

Note:
Exercise:

Problem: Simplify: ⓐ $6 \cdot 3^{-1}$ ⓑ $(6 \cdot 3)^{-1}$.

Solution:

ⓐ 2 ⓑ $\frac{1}{18}$

Note:
Exercise:

Problem: Simplify: ⓐ $8 \cdot 2^{-2}$ ⓑ $(8 \cdot 2)^{-2}$.

Solution:

ⓐ 2 ⓑ $\frac{1}{16}$

When a variable is raised to a negative exponent, we apply the definition the same way we did with numbers. We will assume all variables are non-zero.

Example:
Exercise:

Problem: Simplify: ⓐ x^{-6} ⓑ $\left(u^4\right)^{-3}$.

Solution:
Solution

ⓐ

	x^{-6}
Use the definition of a negative exponent, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{x^6}$

ⓑ

	$\left(u^4\right)^{-3}$
Use the definition of a negative exponent, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{\left(u^4\right)^3}$
Simplify.	$\frac{1}{u^{12}}$

Note:
Exercise:

Problem: Simplify: ⓐ y^{-7} ⓑ $\left(z^3\right)^{-5}$.

Solution:

ⓐ $\frac{1}{y^7}$ ⓑ $\frac{1}{z^{15}}$

Note:
Exercise:

Problem: Simplify: ⓐ p^{-9} ⓑ $\left(q^4\right)^{-6}$.

Solution:

ⓐ $\frac{1}{p^9}$ ⓑ $\frac{1}{q^{24}}$

When there is a product and an exponent we have to be careful to apply the exponent to the correct quantity. According to the Order of Operations, we simplify expressions in parentheses before applying exponents. We'll see how this works in the next example.

Example:
Exercise:

Problem: Simplify: ⓐ $5y^{-1}$ ⓑ $(5y)^{-1}$ ⓒ $(-5y)^{-1}$.

Solution:
Solution

ⓐ

	$5y^{-1}$
Notice the exponent applies to just the base y. Take the reciprocal of y and change the sign of the exponent.	$5 \cdot \frac{1}{y^1}$
Simplify.	$\frac{5}{y}$

ⓑ

	$(5y)^{-1}$
Here the parentheses make the exponent apply to the base $5y$. Take the reciprocal of $5y$ and change the sign of the exponent.	$\frac{1}{(5y)^1}$
Simplify.	$\frac{1}{5y}$

ⓒ

	$(-5y)^{-1}$
The base here is $-5y$. Take the reciprocal of $-5y$ and change the sign of the exponent.	$\frac{1}{(-5y)^1}$
Simplify.	$\frac{1}{-5y}$
Use $\frac{a}{-b} = -\frac{a}{b}$.	$-\frac{1}{5y}$

Note:
Exercise:

Problem: Simplify: ⓐ $8p^{-1}$ ⓑ $(8p)^{-1}$ ⓒ $(-8p)^{-1}$.

Solution:

ⓐ $\frac{8}{p}$ ⓑ $\frac{1}{8p}$ ⓒ $-\frac{1}{8p}$

Note:
Exercise:

Problem: Simplify: ⓐ $11q^{-1}$ ⓑ $(11q)^{-1}-(11q)^{-1}$ ⓒ $(-11q)^{-1}$.

Solution:

ⓐ $\frac{1}{11q}$ ⓑ $\frac{1}{11q}-\frac{1}{11q}$ ⓒ $-\frac{1}{11q}$

With negative exponents, the Quotient Rule needs only one form $\frac{a^m}{a^n}=a^{m-n}$, for $a\neq 0$. When the exponent in the denominator is larger than the exponent in the numerator, the exponent of the quotient will be negative.

Simplify Expressions with Integer Exponents

All of the exponent properties we developed earlier in the chapter with whole number exponents apply to integer exponents, too. We restate them here for reference.

Note:
Summary of Exponent Properties
If a and b are real numbers, and m and n are integers, then
Equation:

$$\begin{array}{lrcl}
\textbf{Product Property} & a^m\cdot a^n & = & a^{m+n}\\
\textbf{Power Property} & (a^m)^n & = & a^{m\cdot n}\\
\textbf{Product to a Power} & (ab)^m & = & a^m b^m\\
\textbf{Quotient Property} & \frac{a^m}{a^n} & = & a^{m-n}, a\neq 0\\
\textbf{Zero Exponent Property} & a^0 & = & 1, a\neq 0\\
\textbf{Quotient to a Power Property} & \left(\frac{a}{b}\right)^m & = & \frac{a^m}{b^m},\ \ b\neq 0\\
\textbf{Properties of Negative Exponents} & a^{-n} & = & \frac{1}{a^n}\ \text{ and }\ \frac{1}{a^{-n}}=a^n\\
\textbf{Quotient to a Negative Exponent} & \left(\frac{a}{b}\right)^{-n} & = & \left(\frac{b}{a}\right)^n
\end{array}$$

Example:
Exercise:

Problem: Simplify: ⓐ $x^{-4} \cdot x^{6}$ ⓑ $y^{-6} \cdot y^{4}$ ⓒ $z^{-5} \cdot z^{-3}$.

Solution:
Solution

ⓐ

	$x^{-4} \cdot x^{6}$
Use the Product Property, $a^m \cdot a^n = a^{m+n}$.	x^{-4+6}
Simplify.	x^{2}

ⓑ

	$y^{-6} \cdot y^{4}$
Notice the same bases, so add the exponents.	y^{-6+4}
Simplify.	y^{-2}
Use the definition of a negative exponent, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{y^2}$

ⓒ

	$z^{-5} \cdot z^{-3}$
Add the exponents, since the bases are the same.	z^{-5-3}
Simplify.	z^{-8}
Take the reciprocal and change the sign of the exponent, using the definition of a negative exponent.	$\frac{1}{z^8}$

Note:
Exercise:

Problem: Simplify: ⓐ $x^{-3} \cdot x^{7}$ ⓑ $y^{-7} \cdot y^{2}$ ⓒ $z^{-4} \cdot z^{-5}$.

Solution:

ⓐ x^4 ⓑ $\frac{1}{y^5}$ ⓒ $\frac{1}{z^9}$

Note:
Exercise:

Problem: Simplify: ⓐ $a^{-1} \cdot a^{6}$ ⓑ $b^{-8} \cdot b^{4}$ ⓒ $c^{-8} \cdot c^{-7}$.

Solution:

ⓐ a^5 ⓑ $\frac{1}{b^4}$ ⓒ $\frac{1}{c^{15}}$

In the next two examples, we'll start by using the Commutative Property to group the same variables together. This makes it easier to identify the like bases before using the Product Property.

Example:
Exercise:

Problem: Simplify: $\left(m^4n^{-3}\right)\left(m^{-5}n^{-2}\right)$.

Solution:
Solution

	$\left(m^4n^{-3}\right)\left(m^{-5}n^{-2}\right)$
Use the Commutative Property to get like bases together.	$m^4m^{-5}\cdot n^{-2}n^{-3}$
Add the exponents for each base.	$m^{-1}\cdot n^{-5}$
Take reciprocals and change the signs of the exponents.	$\frac{1}{m^1}\cdot\frac{1}{n^5}$
Simplify.	$\frac{1}{mn^5}$

Note:
Exercise:

Problem: Simplify: $\left(p^6q^{-2}\right)\left(p^{-9}q^{-1}\right)$.

Solution:

$\frac{1}{p^3q^3}$

Note:
Exercise:

Problem: Simplify: $\left(r^5s^{-3}\right)\left(r^{-7}s^{-5}\right)$.

Solution:

$\frac{1}{r^2s^8}$

If the monomials have numerical coefficients, we multiply the coefficients, just like we did earlier.

Example:
Exercise:

Problem: Simplify: $\left(2x^{-6}y^{8}\right)\left(-5x^{5}y^{-3}\right)$.

Solution:
Solution

	$\left(2x^{-6}y^{8}\right)\left(-5x^{5}y^{-3}\right)$
Rewrite with the like bases together.	$2\left(-5\right)\cdot\left(x^{-6}x^{5}\right)\cdot\left(y^{8}y^{-3}\right)$
Multiply the coefficients and add the exponents of each variable.	$-10\cdot x^{-1}\cdot y^{5}$
Use the definition of a negative exponent, $a^{-n}=\frac{1}{a^{n}}$.	$-10\cdot\frac{1}{x^{1}}\cdot y^{5}$
Simplify.	$\frac{-10y^{5}}{x}$

Note:
Exercise:

Problem: Simplify: $\left(3u^{-5}v^{7}\right)\left(-4u^{4}v^{-2}\right)$.

Solution:

$-\frac{12v^{5}}{u}$

Note:
Exercise:

Problem: Simplify: $\left(-6c^{-6}d^{4}\right)\left(-5c^{-2}d^{-1}\right)$.

Solution:

$\frac{30d^{3}}{c^{8}}$

In the next two examples, we’ll use the Power Property and the Product to a Power Property.

Example:
Exercise:

Problem: Simplify: $\left(6k^{3}\right)^{-2}$.

Solution:

Solution

	$\left(6k^3\right)^{-2}$
Use the Product to a Power Property, $(ab)^m = a^m b^m$.	$(6)^{-2}\left(k^3\right)^{-2}$
Use the Power Property, $(a^m)^n = a^{m \cdot n}$.	$6^{-2}k^{-6}$
Use the Definition of a Negative Exponent, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{6^2} \cdot \frac{1}{k^6}$
Simplify.	$\frac{1}{36k^6}$

Note:
Exercise:

Problem: Simplify: $\left(-4x^4\right)^{-2}$.

Solution:

$\frac{1}{16x^8}$

Note:
Exercise:

Problem: Simplify: $\left(2b^3\right)^{-4}$.

Solution:

$\frac{1}{16b^{12}}$

Example:
Exercise:

Problem: Simplify: $\left(5x^{-3}\right)^2$.

Solution:
Solution

	$\left(5x^{-3}\right)^2$
Use the Product to a Power Property, $(ab)^m = a^m b^m$.	$5^2\left(x^{-3}\right)^2$
Simplify 5^2 and multiply the exponents of x using the Power Property, $(a^m)^n = a^{m \cdot n}$.	$25 \cdot x^{-6}$
Rewrite x^{-6} by using the Definition of a Negative Exponent, $a^{-n} = \frac{1}{a^n}$.	$25 \cdot \frac{1}{x^6}$
Simplify.	$\frac{25}{x^6}$

Note:
Exercise:

Problem: Simplify: $\left(8a^{-4}\right)^2$.

Solution:

$\frac{64}{a^8}$

Note:
Exercise:

Problem: Simplify: $\left(2c^{-4}\right)^3$.

Solution:

$\frac{8}{c^{12}}$

To simplify a fraction, we use the Quotient Property and subtract the exponents.

Example:
Exercise:

Problem: Simplify: $\frac{r^5}{r^{-4}}$.

Solution:
Solution

	$\frac{r^5}{r^{-4}}$
Use the Quotient Property, $\frac{a^m}{a^n} = a^{m-n}$.	$r^{5-(-4)}$
Simplify.	r^9

Note:
Exercise:

Problem: Simplify: $\frac{x^8}{x^{-3}}$.

Solution:

x^{11}

Note:
Exercise:

Problem: Simplify: $\frac{y^8}{y^{-6}}$.

Solution:

y^{13}

Convert from Decimal Notation to Scientific Notation

Remember working with place value for whole numbers and decimals? Our number system is based on powers of 10. We use tens, hundreds, thousands, and so on. Our decimal numbers are also based on powers of tens—tenths, hundredths, thousandths, and so on. Consider the numbers 4,000 and 0.004. We know that 4,000 means $4 \times 1{,}000$ and 0.004 means $4 \times \frac{1}{1{,}000}$.

If we write the 1000 as a power of ten in exponential form, we can rewrite these numbers in this way:
Equation:

$4{,}000$	0.004
$4 \times 1{,}000$	$4 \times \frac{1}{1{,}000}$
4×10^3	$4 \times \frac{1}{10^3}$
	4×10^{-3}

When a number is written as a product of two numbers, where the first factor is a number greater than or equal to one but less than 10, and the second factor is a power of 10 written in exponential form, it is said to be in *scientific notation.*

Note:
Scientific Notation
A number is expressed in **scientific notation** when it is of the form
Equation:

$$a \times 10^n \text{ where } 1 \leq a < 10 \text{ and } n \text{ is an integer}$$

It is customary in scientific notation to use as the $\times$ multiplication sign, even though we avoid using this sign elsewhere in algebra.

If we look at what happened to the decimal point, we can see a method to easily convert from decimal notation to scientific notation.

$4000. = 4 \times 10^3$	$0.004 = 4 \times 10^{-3}$
$4000. = 4 \times 10^3$	$0.004 = 4 \times 10^{-3}$
Moved the decimal point 3 places to the left.	Moved the decimal point 3 places to the right.

In both cases, the decimal was moved 3 places to get the first factor between 1 and 10.

The power of 10 is positive when the number is larger than 1:	$4{,}000 = 4 \times 10^3$
The power of 10 is negative when the number is between 0 and 1:	$0.004 = 4 \times 10^{-3}$

Example:
How to Convert from Decimal Notation to Scientific Notation
Exercise:

Problem: Write in scientific notation: 37,000.

Solution:
Solution

Step 1. Move the decimal point so that the first factor is greater than or equal to 1 but less than 10.	Remember, there is a decimal at the end of 37,000. Move the decimal after the 3. 3.700 is between 1 and 10.	37,000.
Step 2. Count the number of decimal places, *n*, that the decimal point was moved.	The decimal point was moved 4 places to the left.	37000.

Step 3. Write the number as a product with a power of 10. If the original number is: Greater than 1, the power of 10 will be 10^n. Between 0 and 1, the power of 10 will be 10^{-n}.	37,000 is greater than 1 so the power of 10 will have exponent 4.	3.7×10^4
Step 4. Check.	Check to see if your answer makes sense.	10^4 is 10,000 and 10,000 times 3.7 will be 37,000. $37{,}000 = 3.7 \times 10^4$

Note:
Exercise:

Problem: Write in scientific notation: 96,000.

Solution:

9.6×10^4

Note:
Exercise:

Problem: Write in scientific notation: 48,300.

Solution:

4.83×10^4

Note:
Convert from decimal notation to scientific notation

Move the decimal point so that the first factor is greater than or equal to 1 but less than 10.
Count the number of decimal places, n, that the decimal point was moved.
Write the number as a product with a power of 10. If the original number is:

- greater than 1, the power of 10 will be 10^n.
- between 0 and 1, the power of 10 will be 10^{-n}.

Check.

Example:
Exercise:

Problem: Write in scientific notation: 0.0052.

Solution:
Solution

The original number, 0.0052, is between 0 and 1 so we will have a negative power of 10.

	0.0052
Move the decimal point to get 5.2, a number between 1 and 10.	0.0052
Count the number of decimal places the point was moved.	3 places
Write as a product with a power of 10.	5.2×10^{-3}
Check.	
5.2×10^{-3} $5.2 \times \frac{1}{10^3}$ $5.2 \times \frac{1}{1000}$ 5.2×0.001	
0.0052	$0.0052 = 5.2 \times 10^{-3}$

Note:
Exercise:

Problem: Write in scientific notation: 0.0078.

Solution:

7.8×10^{-3}

Note:
Exercise:

Problem: Write in scientific notation: 0.0129.

Solution:

1.29×10^{-2}

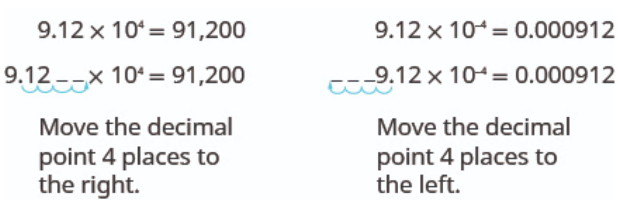
Convert Scientific Notation to Decimal Form

How can we convert from scientific notation to decimal form? Let's look at two numbers written in scientific notation and see.
Equation:

$$\begin{array}{cc} 9.12 \times 10^4 & 9.12 \times 10^{-4} \\ 9.12 \times 10{,}000 & 9.12 \times 0.0001 \\ 91{,}200 & 0.000912 \end{array}$$

If we look at the location of the decimal point, we can see an easy method to convert a number from scientific notation to decimal form.
Equation:

$$9.12 \times 10^4 = 91{,}200 \qquad 9.12 \times 10^{-4} = 0.000912$$



In both cases the decimal point moved 4 places. When the exponent was positive, the decimal moved to the right. When the exponent was negative, the decimal point moved to the left.

Example:
How to Convert Scientific Notation to Decimal Form
Exercise:

Problem: Convert to decimal form: 6.2×10^3.

Solution:
Solution

Step 1. Determine the exponent, *n*, on the factor 10.	The exponent is 3.	6.2×10^3
Step 2. Move the decimal *n* places, adding zeros if needed. If the exponent is positive, move the decimal point *n* places to the right. If the exponent is negative, move the decimal point $\vert n \vert$ places to the left.	The exponent is positive, so move the decimal point 3 places to the right. We need to add 2 zeros as placeholders.	6.200. 6,200
Step 3. Check to see if your answer makes sense.		10^3 is 1000 and 1000 times 6.2 will be 6,200. $6.2 \times 10^3 = 6{,}200$

Note:
Exercise:

Problem: Convert to decimal form: 1.3×10^3.

Solution:

1,300

Note:
Exercise:

Problem: Convert to decimal form: 9.25×10^4.

Solution:

92,500

The steps are summarized below.

Note:
Convert scientific notation to decimal form.
To convert scientific notation to decimal form:

Determine the exponent, n, on the factor 10.
Move the decimal n places, adding zeros if needed.

- If the exponent is positive, move the decimal point n places to the right.
- If the exponent is negative, move the decimal point $|n|$ places to the left.

Check.

Example:
Exercise:

Problem: Convert to decimal form: 8.9×10^{-2}.

Solution:
Solution

	8.9×10^{-2}
Determine the exponent, n, on the factor 10.	The exponent is −2.
Since the exponent is negative, move the decimal point 2 places to the left.	8 . 9
Add zeros as needed for placeholders.	$8.9 \times 10^{-2} = 0.089$

Note:
Exercise:

Problem: Convert to decimal form: 1.2×10^{-4}.

Solution:

0.00012

Note:
Exercise:

Problem: Convert to decimal form: 7.5×10^{-2}.

Solution:

0.075

Multiply and Divide Using Scientific Notation

Astronomers use very large numbers to describe distances in the universe and ages of stars and planets. Chemists use very small numbers to describe the size of an atom or the charge on an electron. When scientists perform calculations with very large or very small numbers, they use scientific notation. Scientific notation provides a way for the calculations to be done without writing a lot of zeros. We will see how the Properties of Exponents are used to multiply and divide numbers in scientific notation.

Example:
Exercise:

Problem: Multiply. Write answers in decimal form: $\left(4 \times 10^5\right)\left(2 \times 10^{-7}\right)$.

Solution:
Solution

	$\left(4 \times 10^5\right)\left(2 \times 10^{-7}\right)$
Use the Commutative Property to rearrange the factors.	$4 \cdot 2 \cdot 10^5 \cdot 10^{-7}$
Multiply.	8×10^{-2}
Change to decimal form by moving the decimal two places left.	0.08

Note:
Exercise:

Problem: Multiply $\left(3 \times 10^6\right)\left(2 \times 10^{-8}\right)$. Write answers in decimal form.

Solution:

0.06

Note:
Exercise:

Problem: Multiply $\left(3 \times 10^{-2}\right)\left(3 \times 10^{-1}\right)$. Write answers in decimal form.

Solution:

0.009

Example:
Exercise:

Problem: Divide. Write answers in decimal form: $\frac{9 \times 10^{3}}{3 \times 10^{-2}}$.

Solution:
Solution

	$\frac{9 \times 10^{3}}{3 \times 10^{-2}}$
Separate the factors, rewriting as the product of two fractions.	$\frac{9}{3} \times \frac{10^{3}}{10^{-2}}$
Divide.	3×10^{5}
Change to decimal form by moving the decimal five places right.	300,000

Note:
Exercise:

Problem: Divide $\frac{8 \times 10^{4}}{2 \times 10^{-1}}$. Write answers in decimal form.

Solution:

400,000

Note:
Exercise:

Problem: Divide $\frac{8 \times 10^{2}}{4 \times 10^{-2}}$. Write answers in decimal form.

Solution:

20,000

Note:
Access these online resources for additional instruction and practice with integer exponents and scientific notation:

- Negative Exponents
- Scientific Notation
- Scientific Notation 2

Key Concepts

- **Property of Negative Exponents**
  - If n is a positive integer and $a \neq 0$, then $\frac{1}{a^{-n}} = a^n$
- **Quotient to a Negative Exponent**
  - If a, b are real numbers, $b \neq 0$ and n is an integer , then $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$
- **To convert a decimal to scientific notation:**

  Move the decimal point so that the first factor is greater than or equal to 1 but less than 10.
  Count the number of decimal places, n, that the decimal point was moved.
  Write the number as a product with a power of 10. If the original number is:
  - greater than 1, the power of 10 will be 10^n
  - between 0 and 1, the power of 10 will be 10^{-n}

  Check.
- **To convert scientific notation to decimal form:**

  Determine the exponent, n, on the factor 10.
  Move the decimal n places, adding zeros if needed.
  - If the exponent is positive, move the decimal point n places to the right.
  - If the exponent is negative, move the decimal point $|n|$ places to the left.

  Check.

Section Exercises

Practice Makes Perfect

Use the Definition of a Negative Exponent

In the following exercises, simplify.
Exercise:

Problem: ⓐ 4^{-2} ⓑ 10^{-3}

Exercise:

Problem: ⓐ 3^{-4} ⓑ 10^{-2}

Solution:

ⓐ $\frac{1}{81}$ ⓑ $\frac{1}{100}$

Exercise:

Problem: ⓐ 5^{-3} ⓑ 10^{-5}

Exercise:

Problem: ⓐ 2^{-8} ⓑ 10^{-2}

Solution:

ⓐ $\frac{1}{256}$ ⓑ $\frac{1}{100}$

Exercise:

Problem: ⓐ $\frac{1}{c^{-5}}$ ⓑ $\frac{1}{3^{-2}}$

Exercise:

Problem: ⓐ $\frac{1}{c^{-5}}$ ⓑ $\frac{1}{5^{-2}}$

Solution:

ⓐ c^5 ⓑ 25

Exercise:

Problem: ⓐ $\frac{1}{q^{-10}}$ ⓑ $\frac{1}{10^{-3}}$

Exercise:

Problem: ⓐ $\frac{1}{t^{-9}}$ ⓑ $\frac{1}{10^{-4}}$

Solution:

ⓐ t^9 ⓑ 10000

Exercise:

Problem: ⓐ $\left(\frac{5}{8}\right)^{-2}$ ⓑ $\left(-\frac{3m}{n}\right)^{-2}$

Exercise:

Problem: ⓐ $\left(\frac{3}{10}\right)^{-2}$ ⓑ $\left(-\frac{2}{cd}\right)^{-3}$

Solution:

ⓐ $\frac{100}{9}$ ⓑ $-\frac{c^3d^3}{8}$

Exercise:

Problem: ⓐ $\left(\frac{4}{9}\right)^{-3}$ ⓑ $\left(-\frac{u^2}{2v}\right)^{-5}$

Exercise:

Problem: ⓐ $\left(\frac{7}{2}\right)^{-3}$ ⓑ $\left(-\frac{3}{xy^2}\right)^{-3}$

Solution:

ⓐ $\frac{8}{343}$ ⓑ $-\frac{x^3y^6}{27}$

Exercise:

ⓐ $(-5)^{-2}$
ⓑ -5^{-2}
ⓒ $\left(-\frac{1}{5}\right)^{-2}$
Problem: ⓓ $-\left(\frac{1}{5}\right)^{-2}$

Exercise:

ⓐ $(-7)^{-2}$
ⓑ-7^{-2}
ⓒ $\left(-\frac{1}{7}\right)^{-2}$
Problem: ⓓ $-\left(\frac{1}{7}\right)^{-2}$

Solution:

ⓐ $\frac{1}{49}$ ⓑ $-\frac{1}{49}$ ⓒ 49 ⓓ -49

Exercise:

ⓐ -3^{-3}
ⓑ $\left(-\frac{1}{3}\right)^{-3}$
ⓒ $-\left(\frac{1}{3}\right)^{-3}$
Problem: ⓓ $(-3)^{-3}$

Exercise:

ⓐ -5^{-3}
ⓑ $\left(-\frac{1}{5}\right)^{-3}$
ⓒ $-\left(\frac{1}{5}\right)^{-3}$
Problem: ⓓ $(-5)^{-3}$

Solution:

ⓐ $-\frac{1}{125}$ ⓑ -125 ⓒ -125 ⓓ $-\frac{1}{125}$

Exercise:

ⓐ $3 \cdot 5^{-1}$
Problem: ⓑ $(3 \cdot 5)^{-1}$

Exercise:

ⓐ $2 \cdot 5^{-1}$
Problem: ⓑ $(2 \cdot 5)^{-1}$

Solution:

ⓐ $\frac{2}{5}$ ⓑ $\frac{1}{10}$

Exercise:

Problem: ⓐ $4 \cdot 5^{-2}$ ⓑ $(4 \cdot 5)^{-2}$

Exercise:

Problem: ⓐ $3 \cdot 4^{-2}$ ⓑ $(3 \cdot 4)^{-2}$

Solution:

ⓐ $\frac{3}{16}$ ⓑ $\frac{1}{144}$

Exercise:

Problem: ⓐ m^{-4} ⓑ $\left(x^3\right)^{-4}$

Exercise:

Problem: ⓐ b^{-5} ⓑ $\left(k^2\right)^{-5}$

Solution:

ⓐ $\frac{1}{b^5}$ ⓑ $\frac{1}{k^{10}}$

Exercise:

Problem: ⓐ p^{-10} ⓑ $\left(q^6\right)^{-8}$

Exercise:

Problem: ⓐ s^{-8} ⓑ $\left(a^9\right)^{-10}$

Solution:

ⓐ $\frac{1}{s^8}$ ⓑ $\frac{1}{a^{90}}$

Exercise:

Problem: ⓐ $7n^{-1}$
ⓑ $(7n)^{-1}$
ⓒ $(-7n)^{-1}$

Exercise:

Problem: ⓐ $6r^{-1}$
ⓑ $(6r)^{-1}$
ⓒ $(-6r)^{-1}$

Solution:

ⓐ $\frac{6}{r}$ ⓑ $\frac{1}{6r}$ ⓒ $-\frac{1}{6r}$

Exercise:

Problem: ⓐ $(3p)^{-2}$
ⓑ $3p^{-2}$
ⓒ $-3p^{-2}$

Exercise:

Problem: ⓐ $(2q)^{-4}$
ⓑ $2q^{-4}$
ⓒ $-2q^{-4}$

Solution:

ⓐ $\frac{1}{16q^4}$ ⓑ $\frac{2}{q^4}$ ⓒ $-\frac{2}{q^4}$

Simplify Expressions with Integer Exponents

In the following exercises, simplify.

Exercise:

Problem: ⓐ b^4b^{-8}
ⓑ $r^{-2}r^5$
ⓒ $x^{-7}x^{-3}$

Exercise:

Problem: ⓐ $s^3 \cdot s^{-7}$
ⓑ $q^{-8} \cdot q^3$
ⓒ $y^{-2} \cdot y^{-5}$

Solution:

ⓐ $\frac{1}{s^4}$ ⓑ $\frac{1}{q^5}$ ⓒ $\frac{1}{y^7}$

Exercise:

Problem: ⓐ $a^3 \cdot a^{-3}$
ⓑ $a \cdot a^3$
ⓒ $a \cdot a^{-3}$

Exercise:

Problem: ⓐ $y^5 \cdot y^{-5}$
ⓑ $y \cdot y^5$
ⓒ $y \cdot y^{-5}$

Solution:

ⓐ 1 ⓑ y^6 ⓒ $\frac{1}{y^4}$

Exercise:

Problem: $p^5 \cdot p^{-2} \cdot p^{-4}$

Exercise:

Problem: $x^4 \cdot x^{-2} \cdot x^{-3}$

Solution:

$\frac{1}{x}$

Exercise:

Problem: $\left(w^4 x^{-5}\right) \left(w^{-2} x^{-4}\right)$

Exercise:

Problem: $\left(m^3 n^{-3}\right) \left(m^{-5} n^{-1}\right)$

Solution:

$\frac{1}{m^2 n^4}$

Exercise:

Problem: $\left(uv^{-2}\right) \left(u^{-5} v^{-3}\right)$

Exercise:

Problem: $\left(pq^{-4}\right) \left(p^{-6} q^{-3}\right)$

Solution:

$\frac{1}{p^5q^7}$

Exercise:

Problem: $\left(-6c^{-3}d^9\right)\left(2c^4d^{-5}\right)$

Exercise:

Problem: $\left(-2j^{-5}k^8\right)\left(7j^2k^{-3}\right)$

Solution:

$-\frac{14k^5}{j^3}$

Exercise:

Problem: $\left(-4r^{-2}s^{-8}\right)\left(9r^4s^3\right)$

Exercise:

Problem: $\left(-5m^4n^6\right)\left(8m^{-5}n^{-3}\right)$

Solution:

$-\frac{40n^3}{m}$

Exercise:

Problem: $\left(5x^2\right)^{-2}$

Exercise:

Problem: $\left(4y^3\right)^{-3}$

Solution:

$\frac{1}{64y^9}$

Exercise:

Problem: $\left(3z^{-3}\right)^2$

Exercise:

Problem: $\left(2p^{-5}\right)^2$

Solution:

$\frac{4}{p^{10}}$

Exercise:

Problem: $\frac{t^9}{t^{-3}}$

Exercise:

Problem: $\frac{n^5}{n^{-2}}$

Solution:

n^7

Exercise:

Problem: $\frac{x^{-7}}{x^{-3}}$

Exercise:

Problem: $\frac{y^{-5}}{y^{-10}}$

Solution:

y^5

Convert from Decimal Notation to Scientific Notation

In the following exercises, write each number in scientific notation.

Exercise:

Problem: 57,000

Exercise:

Problem: 340,000

Solution:

3.4×10^5

Exercise:

Problem: 8,750,000

Exercise:

Problem: 1,290,000

Solution:

1.29×10^6

Exercise:

Problem: 0.026

Exercise:

Problem: 0.041

Solution:

4.1×10^{-2}

Exercise:

Problem: 0.00000871

Exercise:

Problem: 0.00000103

Solution:

1.03×10^{-6}

Convert Scientific Notation to Decimal Form

In the following exercises, convert each number to decimal form.

Exercise:

Problem: 5.2×10^{2}

Exercise:

Problem: 8.3×10^{2}

Solution:

830

Exercise:

Problem: 7.5×10^{6}

Exercise:

Problem: 1.6×10^{10}

Solution:

16,000,000,000

Exercise:

Problem: 2.5×10^{-2}

Exercise:

Problem: 3.8×10^{-2}

Solution:

0.038

Exercise:

Problem: 4.13×10^{-5}

Exercise:

Problem: 1.93×10^{-5}

Solution:

0.0000193

Multiply and Divide Using Scientific Notation

In the following exercises, multiply. Write your answer in decimal form.

Exercise:

Problem: $\left(3 \times 10^{-5}\right)\left(3 \times 10^{9}\right)$

Exercise:

Problem: $\left(2 \times 10^{2}\right)\left(1 \times 10^{-4}\right)$

Solution:

0.02

Exercise:

Problem: $\left(7.1 \times 10^{-2}\right)\left(2.4 \times 10^{-4}\right)$

Exercise:

Problem: $\left(3.5 \times 10^{-4}\right)\left(1.6 \times 10^{-2}\right)$

Solution:

5.6×10^{-6}

In the following exercises, divide. Write your answer in decimal form.

Exercise:

Problem: $\frac{7 \times 10^{-3}}{1 \times 10^{-7}}$

Exercise:

Problem: $\frac{5 \times 10^{-2}}{1 \times 10^{-10}}$

Solution:

500,000,000

Exercise:

Problem: $\frac{6 \times 10^{4}}{3 \times 10^{-2}}$

Exercise:

Problem: $\frac{8 \times 10^{6}}{4 \times 10^{-1}}$

Solution:

20,000,000

Everyday Math

Exercise:

Problem:

The population of the United States on July 4, 2010 was almost 310,000,000. Write the number in scientific notation.

Exercise:

Problem:

The population of the world on July 4, 2010 was more than 6,850,000,000. Write the number in scientific notation

Solution:

6.85×10^{9}.

Exercise:

Problem:

The average width of a human hair is 0.0018 centimeters. Write the number in scientific notation.

Exercise:

Problem:

The probability of winning the 2010 Megamillions lottery was about 0.0000000057. Write the number in scientific notation.

Solution:

5.7×10^{-10}

Exercise:

Problem:

In 2010, the number of Facebook users each day who changed their status to ‘engaged’ was 2×10^4. Convert this number to decimal form.

Exercise:

Problem:

At the start of 2012, the US federal budget had a deficit of more than $\$1.5 \times 10^{13}$. Convert this number to decimal form.

Solution:

15,000,000,000,000

Exercise:

Problem:

The concentration of carbon dioxide in the atmosphere is 3.9×10^{-4}. Convert this number to decimal form.

Exercise:

Problem:

The width of a proton is 1×10^{-5} of the width of an atom. Convert this number to decimal form.

Solution:

0.00001

Exercise:

Problem:

Health care costs The Centers for Medicare and Medicaid projects that consumers will spend more than $4 trillion on health care by 2017.

ⓐ Write 4 trillion in decimal notation.
ⓑ Write 4 trillion in scientific notation.

Exercise:

Problem:

Coin production In 1942, the U.S. Mint produced 154,500,000 nickels. Write 154,500,000 in scientific notation.

Solution:

1.545×10^8

Exercise:

Problem:

Distance The distance between Earth and one of the brightest stars in the night star is 33.7 light years. One light year is about 6,000,000,000,000 (6 trillion), miles.

ⓐ Write the number of miles in one light year in scientific notation.
ⓑUse scientific notation to find the distance between Earth and the star in miles. Write the answer in scientific notation.

Exercise:

Problem:

Debt At the end of fiscal year 2015 the gross United States federal government debt was estimated to be approximately $18,600,000,000,000 ($18.6 trillion), according to the Federal Budget. The population of the United States was approximately 300,000,000 people at the end of fiscal year 2015.

ⓐ Write the debt in scientific notation.
ⓑ Write the population in scientific notation.
ⓒ Find the amount of debt per person by using scientific notation to divide the debt by the population. Write the answer in scientific notation.

Solution:

ⓐ 1.86×10^{13} ⓑ 3×10^{8} ⓒ 6.2×10^{4}

Writing Exercises

Exercise:

Problem:

ⓐ Explain the meaning of the exponent in the expression 2^3.
ⓑ Explain the meaning of the exponent in the expression 2^{-3}.

Exercise:

Problem:

When you convert a number from decimal notation to scientific notation, how do you know if the exponent will be positive or negative?

Solution:

answers will vary

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
use the definition of a negative exponent.			
simplify expressions with integer exponents.			
convert from decimal notation to scientific notation.			
convert scientific notation to decimal form.			
multiply and divide using scientific notation.			

ⓑ Overall, after looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

Chapte Review Exercises

Add and Subtract Polynomials

Identify Polynomials, Monomials, Binomials and Trinomials

In the following exercises, determine if each of the following polynomials is a monomial, binomial, trinomial, or other polynomial.

Exercise:

ⓐ $11c^4 - 23c^2 + 1$
ⓑ $9p^3 + 6p^2 - p - 5$
ⓒ $\frac{3}{7}x + \frac{5}{14}$
ⓓ 10
Problem: ⓔ $2y - 12$

Exercise:

ⓐ $a^2 - b^2$
ⓑ $24d^3$
ⓒ $x^2 + 8x - 10$
ⓓ $m^2n^2 - 2mn + 6$
Problem: ⓔ $7y^3 + y^2 - 2y - 4$

Solution:

ⓐ binomial ⓑ monomial ⓒ trinomial ⓓ trinomial ⓔ other polynomial

Determine the Degree of Polynomials

In the following exercises, determine the degree of each polynomial.

Exercise:

Problem:

ⓐ $3x^2 + 9x + 10$
ⓑ $14a^2bc$
ⓒ $6y + 1$

ⓓ $n^3 - 4n^2 + 2n - 8$
ⓔ -19

Exercise:

Problem:

ⓐ $5p^3 - 8p^2 + 10p - 4$
ⓑ $-20q^4$
ⓒ $x^2 + 6x + 12$
ⓓ $23r^2s^2 - 4rs + 5$
ⓔ 100

Solution:

ⓐ 3 ⓑ 4 ⓒ 2 ⓓ 4 ⓔ 0

Add and Subtract Monomials

In the following exercises, add or subtract the monomials.

Exercise:

Problem: $5\text{y}^3 + 8y^3$

Exercise:

Problem: $-14k + 19k$

Solution:

$5k$

Exercise:

Problem: $12q - (-6q)$

Exercise:

Problem: $-9c - 18c$

Solution:

$-27c$

Exercise:

Problem: $12\text{x} - 4y - 9x$

Exercise:

Problem: $3m^2 + 7n^2 - 3m^2$

Solution:

$7n^2$

Exercise:

Problem: $6x^2y - 4x + 8xy^2$

Exercise:

Problem: $13\text{a} + b$

Solution:

$13\text{a} + b$

Add and Subtract Polynomials

In the following exercises, add or subtract the polynomials.

Exercise:

Problem: $(5x^2 + 12x + 1) + (6x^2 - 8x + 3)$

Exercise:

Problem: $(9p^2 - 5p + 3) + (4p^2 - 4)$

Solution:

$13p^2 - 5p - 1$

Exercise:

Problem: $(10m^2 - 8m - 1) - (5m^2 + m - 2)$

Exercise:

Problem: $(7y^2 - 8y) - (y - 4)$

Solution:

$7y^2 - 9y + 4$

Exercise:

Problem: Subtract $\left(3s^2 + 10\right)$ from $\left(15s^2 - 2s + 8\right)$

Exercise:

Problem: Find the sum of $\left(a^2 + 6a + 9\right)$ and $\left(5a^3 - 7\right)$

Solution:

$5a^3 + a^2 + 6a + 2$

Evaluate a Polynomial for a Given Value of the Variable

In the following exercises, evaluate each polynomial for the given value.
Exercise:

Problem: Evaluate $3y^2 - y + 1$ when:

ⓐ $y = 5$
ⓑ $y = -1$
ⓒ $y = 0$

Exercise:

Problem: Evaluate $10 - 12x$ when:

ⓐ $x = 3$
ⓑ $x = 0$
ⓒ $x = -1$

Solution:

ⓐ -26 ⓑ 10 ⓒ 22

Exercise:

Problem:

Randee drops a stone off the 200 foot high cliff into the ocean. The polynomial $-16t^2 + 200$ gives the height of a stone t seconds after it is dropped from the cliff. Find the height after $t = 3$ seconds.

Exercise:

Problem:

A manufacturer of stereo sound speakers has found that the revenue received from selling the speakers at a cost of p dollars each is given by the polynomial $-4p^2 + 460p$. Find the revenue received when $p = 75$ dollars.

Solution:

12,000

Use Multiplication Properties of Exponents

Simplify Expressions with Exponents

In the following exercises, simplify.
Exercise:

Problem: 10^4

Exercise:

Problem: 17^1

Solution:

17

Exercise:

Problem: $\left(\frac{2}{9}\right)^2$

Exercise:

Problem: $(0.5)^3$

Solution:

0.125

Exercise:

Problem: $(-2)^6$

Exercise:

Problem: -2^6

Solution:

-64

Simplify Expressions Using the Product Property for Exponents

In the following exercises, simplify each expression.

Exercise:

Problem: $x^4 \cdot x^3$

Exercise:

Problem: $p^{15} \cdot p^{16}$

Solution:

p^{31}

Exercise:

Problem: $4^{10} \cdot 4^6$

Exercise:

Problem: $8 \cdot 8^5$

Solution:

8^6

Exercise:

Problem: $n \cdot n^2 \cdot n^4$

Exercise:

Problem: $y^c \cdot y^3$

Solution:

y^{c+3}

Simplify Expressions Using the Power Property for Exponents

In the following exercises, simplify each expression.

Exercise:

Problem: $(m^3)^5$

Exercise:

Problem: $(5^3)^2$

Solution:

5^6

Exercise:

Problem: $\left(y^4\right)^x$

Exercise:

Problem: $(3^r)^s$

Solution:

3^{rs}

Simplify Expressions Using the Product to a Power Property

In the following exercises, simplify each expression.

Exercise:

Problem: $(4a)^2$

Exercise:

Problem: $(-5y)^3$

Solution:

$-125y^3$

Exercise:

Problem: $(2mn)^5$

Exercise:

Problem: $(10xyz)^3$

Solution:

$1000x^3y^3z^3$

Simplify Expressions by Applying Several Properties

In the following exercises, simplify each expression.

Exercise:

Problem: $\left(p^2\right)^5 \cdot \left(p^3\right)^6$

Exercise:

Problem: $(4a^3b^2)^3$

Solution:

$64a^9b^6$

Exercise:

Problem: $(5x)^2(7x)$

Exercise:

Problem: $(2q^3)^4(3q)^2$

Solution:

$144q^{14}$

Exercise:

Problem: $\left(\frac{1}{3}x^2\right)^2\left(\frac{1}{2}x\right)^3$

Exercise:

Problem: $\left(\frac{2}{5}m^2n\right)^3$

Solution:

$\frac{8}{125}m^6n^3$

Multiply Monomials

In the following exercises, multiply the monomials.
Exercise:

Problem: $\left(-15x^2\right)\left(6x^4\right)$

Exercise:

Problem: $(-9n^7)(-16n)$

Solution:

$144n^8$

Exercise:

Problem: $(7p^5q^3)(8pq^9)$

Exercise:

Problem: $\left(\frac{5}{9}ab^2\right)\left(27ab^3\right)$

Solution:

$15a^2b^5$

Multiply Polynomials

Multiply a Polynomial by a Monomial

In the following exercises, multiply.
Exercise:

Problem: $7(a+9)$

Exercise:

Problem: $-4(y+13)$

Solution:

$-4y-52$

Exercise:

Problem: $-5(r-2)$

Exercise:

Problem: $p(p+3)$

Solution:

p^2+3p

Exercise:

Problem: $-m(m+15)$

Exercise:

Problem: $-6u(2u+7)$

Solution:

$-12u^2-42u$

Exercise:

Problem: $9(b^2+6b+8)$

Exercise:

Problem: $3q^2(q^2-7q+6)$

Solution:

$3q^4-21q^3+18q^2$

Exercise:

Problem: $(5z-1)z$

Exercise:

Problem: $(b-4)\cdot 11$

Solution:

$11b-44$

Multiply a Binomial by a Binomial

In the following exercises, multiply the binomials using: ⓐ the Distributive Property, ⓑ the FOIL method, ⓒ the Vertical Method.

Exercise:

Problem: $(x-4)\,(x+10)$

Exercise:

Problem: $(6y-7)\,(2y-5)$

Solution:

ⓐ $12y^2-44y+35$ ⓑ $12y^2-44y+35$ ⓒ $12y^2-44y+35$

In the following exercises, multiply the binomials. Use any method.

Exercise:

Problem: $(x+3)(x+9)$

Exercise:

Problem: $(y-4)(y-8)$

Solution:

$y^2-12y+32$

Exercise:

Problem: $(p-7)(p+4)$

Exercise:

Problem: $(q+16)(q-3)$

Solution:

$q^2+13q-48$

Exercise:

Problem: $(5m-8)(12m+1)$

Exercise:

Problem: $(u^2+6)(u^2-5)$

Solution:

u^4+u^2-30

Exercise:

Problem: $(9x-y)\,(6x-5)$

Exercise:

Problem: $(8mn+3)(2mn-1)$

Solution:

$16m^2n^2-2mn-3$

Multiply a Trinomial by a Binomial

In the following exercises, multiply using ⓐ the Distributive Property, ⓑ the Vertical Method.
Exercise:

Problem: $(n+1)\left(n^2+5n-2\right)$

Exercise:

Problem: $(3x-4)\left(6x^2+x-10\right)$

Solution:

ⓐ $18x^3-21x^2-34x+40$ ⓑ $18x^3-21x^2-34x+40$

In the following exercises, multiply. Use either method.
Exercise:

Problem: $(y-2)(y^2-8y+9)$

Exercise:

Problem: $(7m+1)(m^2-10m-3)$

Solution:

$7m^3-69m^2-31m-3$

Special Products

Square a Binomial Using the Binomial Squares Pattern

In the following exercises, square each binomial using the Binomial Squares Pattern.
Exercise:

Problem: $(c+11)^2$

Exercise:

Problem: $(q-15)^2$

Solution:

$q^2 - 30q + 225$

Exercise:

Problem: $\left(x + \frac{1}{3}\right)^2$

Exercise:

Problem: $(8u + 1)^2$

Solution:

$64u^2 + 16u + 1$

Exercise:

Problem: $\left(3n^3 - 2\right)^2$

Exercise:

Problem: $(4a - 3b)^2$

Solution:

$16a^2 - 24ab + 9b^2$

Multiply Conjugates Using the Product of Conjugates Pattern

In the following exercises, multiply each pair of conjugates using the Product of Conjugates Pattern.

Exercise:

Problem: $(s - 7)(s + 7)$

Exercise:

Problem: $\left(y + \frac{2}{5}\right)\left(y - \frac{2}{5}\right)$

Solution:

$y^2 - \frac{4}{25}$

Exercise:

Problem: $(12c + 13)(12c - 13)$

Exercise:

Problem: $(6 - r)(6 + r)$

Solution:

$36 - r^2$

Exercise:

Problem: $\left(u+\frac{3}{4}v\right)\left(u-\frac{3}{4}v\right)$

Exercise:

Problem: $(5p^4-4q^3)(5p^4+4q^3)$

Solution:

$25p^8-16q^6$

Recognize and Use the Appropriate Special Product Pattern

In the following exercises, find each product.

Exercise:

Problem: $(3m+10)^2$

Exercise:

Problem: $(6a+11)(6a-11)$

Solution:

$36a^2-121$

Exercise:

Problem: $(5x+y)(x-5y)$

Exercise:

Problem: $(c^4+9d)^2$

Solution:

$c^8+18c^4d+81d^2$

Exercise:

Problem: $(p^5+q^5)(p^5-q^5)$

Exercise:

Problem: $(a^2+4b)(4b-a^2)$

Solution:

16msupb^2-a^4

[Divide Monomials]

Simplify Expressions Using the Quotient Property for Exponents

In the following exercises, simplify.
Exercise:

Problem: $\frac{u^{24}}{u^6}$

Exercise:

Problem: $\frac{10^{25}}{10^5}$

Solution:

10^{20}

Exercise:

Problem: $\frac{3^4}{3^6}$

Exercise:

Problem: $\frac{v^{12}}{v^{48}}$

Solution:

$\frac{1}{v^{36}}$

Exercise:

Problem: $\frac{x}{x^5}$

Exercise:

Problem: $\frac{5}{5^8}$

Solution:

$\frac{1}{5^7}$

Simplify Expressions with Zero Exponents

In the following exercises, simplify.
Exercise:

Problem: 75^0

Exercise:

Problem: x^0

Solution:

1

Exercise:

Problem: -12^0

Exercise:

Problem: $\left(-12^0\right)(-12)^0$

Solution:

1

Exercise:

Problem: $25x^0$

Exercise:

Problem: $(25x)^0$

Solution:

1

Exercise:

Problem: $19n^0 - 25m^0$

Exercise:

Problem: $(19n)^0 - (25m)^0$

Solution:

0

Simplify Expressions Using the Quotient to a Power Property

In the following exercises, simplify.

Exercise:

Problem: $\left(\frac{2}{5}\right)^3$

Exercise:

Problem: $\left(\frac{m}{3}\right)^4$

Solution:

$\frac{m^4}{81}$

Exercise:

Problem: $\left(\frac{r}{s}\right)^8$

Exercise:

Problem: $\left(\frac{x}{2y}\right)^6$

Solution:

$\frac{x^6}{64y^6}$

Simplify Expressions by Applying Several Properties

In the following exercises, simplify.

Exercise:

Problem: $\frac{\left(x^3\right)^5}{x^9}$

Exercise:

Problem: $\frac{n^{10}}{\left(n^5\right)^2}$

Solution:

1

Exercise:

Problem: $\left(\frac{q^6}{q^8}\right)^3$

Exercise:

Problem: $\left(\frac{r^8}{r^3}\right)^4$

Solution:

r^{20}

Exercise:

Problem: $\left(\frac{c^2}{d^5}\right)^9$

Exercise:

Problem: $\left(\frac{3x^4}{2y^2}\right)^5$

Solution:

$\frac{243x^{20}}{32y^{10}}$

Exercise:

Problem: $\left(\frac{v^3v^9}{v^6}\right)^4$

Exercise:

Problem: $\frac{\left(3n^2\right)^4\left(-5n^4\right)^3}{\left(-2n^5\right)^2}$

Solution:

$-\frac{10{,}125n^{10}}{4}$

Divide Monomials

In the following exercises, divide the monomials.
Exercise:

Problem: $-65y^{14} \div 5y^2$

Exercise:

Problem: $\frac{64a^5b^9}{-16a^{10}b^3}$

Solution:

$-\frac{4b^6}{a^5}$

Exercise:

Problem: $\frac{144x^{15}y^8z^3}{18x^{10}y^2z^{12}}$

Exercise:

Problem: $\frac{\left(8p^6q^2\right)\left(9p^3q^5\right)}{16p^8q^7}$

Solution:

$\frac{9p}{2}$

Divide Polynomials

Divide a Polynomial by a Monomial

In the following exercises, divide each polynomial by the monomial.
Exercise:

Problem: $\frac{42z^2-18z}{6}$

Exercise:

Problem: $\left(35x^2-75x\right) \div 5x$

Solution:

$7x-15$

Exercise:

Problem: $\frac{81n^4+105n^2}{-3}$

Exercise:

Problem: $\frac{550p^6-300p^4}{10p^3}$

Solution:

$55p^3-30p$

Exercise:

Problem: $\left(63xy^3+56x^2y^4\right) \div (7xy)$

Exercise:

Problem: $\frac{96a^5b^2-48a^4b^3-56a^2b^4}{8ab^2}$

Solution:

$12a^4-6a^3b-7ab^2$

Exercise:

Problem: $\frac{57m^2-12m+1}{-3m}$

Exercise:

Problem: $\frac{105y^5+50y^3-5y}{5y^3}$

Solution:

$21y^2+10-\frac{1}{y^2}$

Divide a Polynomial by a Binomial

In the following exercises, divide each polynomial by the binomial.

Exercise:

Problem: $\left(k^2-2k-99\right)\div(k+9)$

Exercise:

Problem: $\left(v^2-16v+64\right)\div(v-8)$

Solution:

$v-8$

Exercise:

Problem: $\left(3x^2-8x-35\right)\div(x-5)$

Exercise:

Problem: $\left(n^2-3n-14\right)\div(n+3)$

Solution:

$n-6+\frac{4}{n+3}$

Exercise:

Problem: $\left(4m^3+m-5\right)\div(m-1)$

Exercise:

Problem: $\left(u^3-8\right)\div(u-2)$

Solution:

u^2+2u+4

Integer Exponents and Scientific Notation

Use the Definition of a Negative Exponent

In the following exercises, simplify.

Exercise:

Problem: 9^{-2}

Exercise:

Problem: $(-5)^{-3}$

Solution:

$-\frac{1}{125}$

Exercise:

Problem: $3 \cdot 4^{-3}$

Exercise:

Problem: $(6u)^{-3}$

Solution:

$\frac{1}{216u^3}$

Exercise:

Problem: $\left(\frac{2}{5}\right)^{-1}$

Exercise:

Problem: $\left(\frac{3}{4}\right)^{-2}$

Solution:

$\frac{16}{9}$

Simplify Expressions with Integer Exponents

In the following exercises, simplify.

Exercise:

Problem: $p^{-2} \cdot p^8$

Exercise:

Problem: $q^{-6} \cdot q^{-5}$

Solution:

$\frac{1}{q^{11}}$

Exercise:

Problem: $\left(c^{-2}d\right)\left(c^{-3}d^{-2}\right)$

Exercise:

Problem: $\left(y^8\right)^{-1}$

Solution:

$\frac{1}{y^8}$

Exercise:

Problem: $\left(q^{-4}\right)^{-3}$

Exercise:

Problem: $\frac{a^8}{a^{12}}$

Solution:

$\frac{1}{a^4}$

Exercise:

Problem: $\frac{n^5}{n^{-4}}$

Exercise:

Problem: $\frac{r^{-2}}{r^{-3}}$

Solution:

r

Convert from Decimal Notation to Scientific Notation

In the following exercises, write each number in scientific notation.

Exercise:

Problem: 8,500,000

Exercise:

Problem: 0.00429

Solution:

4.29×10^{-3}

Exercise:

Problem: The thickness of a dime is about 0.053 inches.

Exercise:

Problem: In 2015, the population of the world was about 7,200,000,000 people.

Solution:

7.2×10^{9}

Convert Scientific Notation to Decimal Form

In the following exercises, convert each number to decimal form.

Exercise:

Problem: 3.8×10^{5}

Exercise:

Problem: 1.5×10^{10}

Solution:

$15{,}000{,}000{,}000$

Exercise:

Problem: 9.1×10^{-7}

Exercise:

Problem: 5.5×10^{-1}

Solution:

0.55

Multiply and Divide Using Scientific Notation

In the following exercises, multiply and write your answer in decimal form.

Exercise:

Problem: $\left(2 \times 10^{5}\right)\left(4 \times 10^{-3}\right)$

Exercise:

Problem: $\left(3.5 \times 10^{-2}\right)\left(6.2 \times 10^{-1}\right)$

Solution:

0.0217

In the following exercises, divide and write your answer in decimal form.

Exercise:

Problem: $\frac{8 \times 10^{5}}{4 \times 10^{-1}}$

Exercise:

Problem: $\frac{9 \times 10^{-5}}{3 \times 10^{2}}$

Solution:

0.0000003

Chapter Practice Test

Exercise:

Problem: For the polynomial $10x^4 + 9y^2 - 1$
ⓐ Is it a monomial, binomial, or trinomial?
ⓑ What is its degree?

In the following exercises, simplify each expression.

Exercise:

Problem: $\left(12a^2 - 7a + 4\right) + \left(3a^2 + 8a - 10\right)$

Solution:

$15a^2 + a - 6$

Exercise:

Problem: $\left(9p^2 - 5p + 1\right) - \left(2p^2 - 6\right)$

Exercise:

Problem: $\left(-\frac{2}{5}\right)^3$

Solution:

$-\frac{8}{125}$

Exercise:

Problem: $u \cdot u^4$

Exercise:

Problem: $\left(4a^3b^5\right)^2$

Solution:

$16a^6b^{10}$

Exercise:

Problem: $\left(-9r^4s^5\right)\left(4rs^7\right)$

Exercise:

Problem: $3k\left(k^2 - 7k + 13\right)$

Solution:

$3k^3 - 21k^2 + 39k$

Exercise:

Problem: $(m+6)(m+12)$

Exercise:

Problem: $(v-9)(9v-5)$

Solution:

$9v^2-86v+45$

Exercise:

Problem: $(4c-11)(3c-8)$

Exercise:

Problem: $(n-6)\left(n^2-5n+4\right)$

Solution:

$n^3-11n^2+34n-24$

Exercise:

Problem: $(2x-15y)(5x+7y)$

Exercise:

Problem: $(7p-5)(7p+5)$

Solution:

$49p^2-25$

Exercise:

Problem: $(9v-2)^2$

Exercise:

Problem: $\frac{3^8}{3^{10}}$

Solution:

$\frac{1}{9}$

Exercise:

Problem: $\left(\frac{m^4\cdot m}{m^3}\right)^6$

Exercise:

Problem: $\left(87x^{15}y^{3}z^{22}\right)^{0}$

Solution:

1

Exercise:

Problem: $\frac{80c^{8}d^{2}}{16cd^{10}}$

Exercise:

Problem: $\frac{12x^{2}+42x-6}{2x}$

Solution:

$6x+21-\frac{3}{x}$

Exercise:

Problem: $\left(70xy^{4}+95x^{3}y\right)\div 5xy$

Exercise:

Problem: $\frac{64x^{3}-1}{4x-1}$

Solution:

$16x^{2}+4x+1$

Exercise:

Problem: $\left(y^{2}-5y-18\right)\div(y+3)$

Exercise:

Problem: 5^{-2}

Solution:

$\frac{1}{25}$

Exercise:

Problem: $(4m)^{-3}$

Exercise:

Problem: $q^{-4}\cdot q^{-5}$

Solution:

$\frac{1}{q^9}$

Exercise:

Problem: $\frac{n^{-2}}{n^{-10}}$

Exercise:

Problem: Convert 83,000,000 to scientific notation.

Solution:

8.3×10^7

Exercise:

Problem: Convert 6.91×10^{-5} to decimal form.

In the following exercises, simplify, and write your answer in decimal form.

Exercise:

Problem: $\left(3.4 \times 10^9\right)\left(2.2 \times 10^{-5}\right)$

Solution:

74,800

Exercise:

Problem: $\frac{8.4 \times 10^{-3}}{4 \times 10^3}$

Exercise:

Problem:

A helicopter flying at an altitude of 1000 feet drops a rescue package. The polynomial $-16t^2 + 1000$ gives the height of the package t seconds a after it was dropped. Find the height when $t = 6$ seconds.

Solution:

424 feet

Glossary

negative exponent
: If n is a positive integer and $a \neq 0$, then $a^{-n} = \frac{1}{a^n}$.

scientific notation
: A number is expressed in scientific notation when it is of the form $a \times 10^n$ where $a \geq 1$ and $a < 10$ and n is an integer.

Introduction
class="introduction"

The Sydney Harbor Bridge is one of Australia’s most photographed landmarks. It is the world’s largest steel arch bridge with the top of the bridge standing 134 meters above the harbor. Can you see why it is known by the locals as the “Coathanger” ?

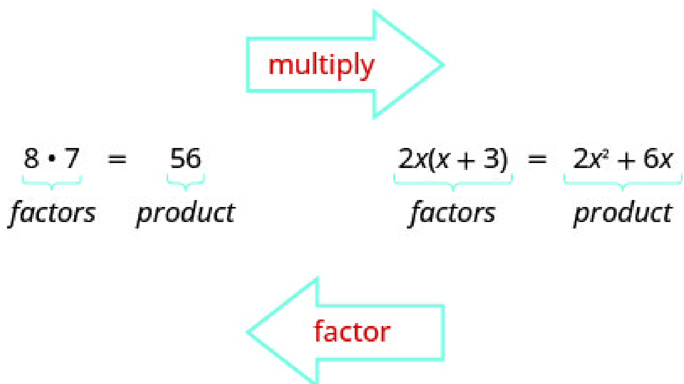
Quadratic expressions may be used to model physical properties of a large bridge, the trajectory of a baseball or rocket, and revenue and profit of a business. By factoring these expressions, specific characteristics of the model can be identified. In this chapter, you will explore the process of factoring expressions and see how factoring is used to solve certain types of equations.

Greatest Common Factor and Factor by Grouping: ASE
By the end of this section, you will be able to:

- Find the greatest common factor of two or more expressions
- Factor the greatest common factor from a polynomial
- Factor by grouping

Find the Greatest Common Factor of Two or More Expressions

Earlier we multiplied factors together to get a product. Now, we will be reversing this process; we will start with a product and then break it down into its factors. Splitting a product into factors is called **factoring**.



We have learned how to factor numbers to find the least common multiple (LCM) of two or more numbers. Now we will factor expressions and find the **greatest common factor** of two or more expressions. The method we use is similar to what we used to find the LCM.

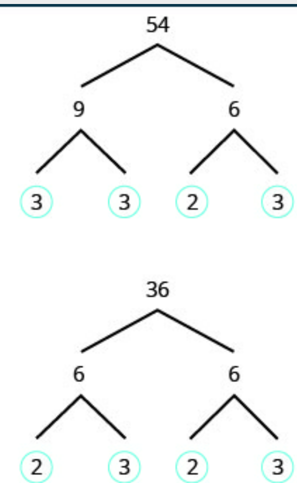
> **Note:**
> Greatest Common Factor
> The greatest common factor (GCF) of two or more expressions is the largest expression that is a factor of all the expressions.

First we'll find the GCF of two numbers.

> **Example:**
> **How to Find the Greatest Common Factor of Two or More Expressions**
> **Exercise:**

Problem: Find the GCF of 54 and 36.

Solution:
Solution

Step 1. Factor each coefficient into primes. Write all variables with exponents in expanded form.	Factor 54 and 36.	
Step 2. In each column, circle the common factors.	Circle the 2, 3, and 3 that are shared by both numbers.	$36 = 2 \cdot 2 \cdot 3 \cdot 3$ $18 = 2 \cdot \quad 3 \cdot 3 \cdot 3$
Step 3. Bring down the common factors that all expressions share.	Bring down the 2, 3, and 3, and then multiply.	$\text{GCF} = 2 \cdot \quad 3 \cdot 3$
Step 4. Multiply the factors.		$\text{GCF} = 18$ The GCF of 54 and 36 is 18.

Notice that, because the GCF is a factor of both numbers, 54 and 36 can be written as multiples of 18.

Equation:

$$54 = 18 \cdot 3$$
$$36 = 18 \cdot 2$$

Note:
Exercise:

Problem: Find the GCF of 48 and 80.

Solution:

16

Note:
Exercise:

Problem: Find the GCF of 18 and 40.

Solution:

2

We summarize the steps we use to find the GCF below.

Note:
Find the Greatest Common Factor (GCF) of two expressions.

Factor each coefficient into primes. Write all variables with exponents in expanded form.
List all factors—matching common factors in a column. In each column, circle the common factors.
Bring down the common factors that all expressions share.
Multiply the factors.

In the first example, the GCF was a constant. In the next two examples, we will get variables in the greatest common factor.

Example:
Exercise:

Problem: Find the greatest common factor of $27x^3$ and $18x^4$.

Solution:
Solution

Factor each coefficient into primes and write the variables with exponents in expanded form. Circle the common factors in each column.	$27x^3 = 3 \cdot 3 \cdot 3 \cdot x \cdot x \cdot x$ $18x^4 = 2 \cdot 3 \cdot 3 \cdot x \cdot x \cdot x \cdot x$
Bring down the common factors.	$\text{GCF} = 3 \cdot 3 \cdot x \cdot x \cdot x$
Multiply the factors.	$\text{GCF} = 9x^3$
	The GCF of $27x^3$ and $18x^4$ is $9x^3$.

Note:
Exercise:

Problem: Find the GCF: $12x^2$, $18x^3$.

Solution:

$3x^2$

Note:
Exercise:

Problem: Find the GCF: $16y^2, 24y^3$.

Solution:

$8y^2$

Example:
Exercise:

Problem: Find the GCF of $4x^2y, 6xy^3$.

Solution:
Solution

Factor each coefficient into primes and write the variables with exponents in expanded form. Circle the common factors in each column.	$4x^2y = 2 \cdot 2 \quad x \cdot x \cdot y$ $6xy^3 = 2 \cdot 3 \cdot x \cdot y \cdot y \cdot y$
Bring down the common factors.	$\text{GCF} = 2 \cdot x \cdot y$
Multiply the factors.	$\text{GCF} = 2xy$
	The GCF of $4x^2y$ and $6xy^3$ is 2 xy.

Note:
Exercise:

Problem: Find the GCF: $6ab^4, 8a^2b$.

Solution:

$2ab$

Note:
Exercise:

Problem: Find the GCF: $9m^5n^2, 12m^3n$.

Solution:

$3m^3n$

Example:
Exercise:

Problem: Find the GCF of: $21x^3, 9x^2, 15x$.

Solution:
Solution

Factor each coefficient into primes and write the variables with exponents in expanded form. Circle the common factors in each column.	$21x^3 = 3 \cdot 7 \cdot x \cdot x \cdot x$ $9x^2 = 3 \cdot 3 \cdot x \cdot x$ $15x = 3 \cdot 5 \cdot x$
Bring down the common factors.	GCF = $3 \cdot x$
Multiply the factors.	GCF = $3x$

	The GCF of $21x^3$, $9x^2$ and $15x$ is $3x$.

Note:
Exercise:

Problem: Find the greatest common factor: $25m^4, 35m^3, 20m^2$.

Solution:

$5m^2$

Note:
Exercise:

Problem: Find the greatest common factor: $14x^3, 70x^2, 105x$.

Solution:

$7x$

Factor the Greatest Common Factor from a Polynomial

Just like in arithmetic, where it is sometimes useful to represent a number in factored form (for example, 12 as $2 \cdot 6$ or $3 \cdot 4$), in algebra, it can be useful to represent a polynomial in factored form. One way to do this is by finding the GCF of all the terms. Remember, we multiply a polynomial by a monomial as follows:

Equation:

$$\begin{array}{cc} 2\left(x+7\right) & \text{factors} \\ 2\cdot x+2\cdot 7 & \\ 2x+14 & \text{product} \end{array}$$

Now we will start with a product, like $2x + 14$, and end with its factors, $2\left(x + 7\right)$. To do this we apply the Distributive Property "in reverse."

We state the Distributive Property here just as you saw it in earlier chapters and "in reverse."

Note:
Distributive Property
If a, b, c are real numbers, then
Equation:

$$a\left(b + c\right) = ab + ac \qquad \text{and} \qquad ab + ac = a\left(b + c\right)$$

The form on the left is used to multiply. The form on the right is used to factor.

So how do you use the Distributive Property to factor a polynomial? You just find the GCF of all the terms and write the polynomial as a product!

Example:
How to Factor the Greatest Common Factor from a Polynomial
Exercise:

Problem: Factor: $4x + 12$.

Solution:
Solution

Step 1. Find the GCF of all the terms of the polynomial.	Find the GCF of $4x$ and 12.	$4x = 2 \cdot 2 \cdot \quad \cdot x$ $12 = 2 \cdot 2 \cdot 3$ GCF $= 2 \cdot 2$ GCF $= 4$
Step 2. Rewrite each term as a product using the GCF.	Rewrite $4x$ and 12 as products of their GCF, 4. $4x = 4 \cdot x$ $12 = 4 \cdot 3$	$4x + 12$ $4 \cdot x + 4 \cdot 3$

Step 3. Use the “reverse” Distributive Property to factor the expression.		$4(x+3)$
Step 4. Check by multiplying the factors.		$4(x+3)$ $4 \cdot x + 4 \cdot 3$ $4x + 12$ ✓

Note:
Exercise:

Problem: Factor: $6a+24$.

Solution:

$6\,(a+4)$

Note:
Exercise:

Problem: Factor: $2b+14$.

Solution:

$2\,(b+7)$

Note:
Factor the greatest common factor from a polynomial.

Find the GCF of all the terms of the polynomial.
Rewrite each term as a product using the GCF.
Use the “reverse” Distributive Property to factor the expression.
Check by multiplying the factors.

Note:
Factor as a Noun and a Verb
We use “factor” as both a noun and a verb.

Noun	7 is a factor of 14
Verb	factor 3 from $3a + 3$

Example:
Exercise:

Problem: Factor: $5a + 5$.

Solution:
Solution

Find the GCF of $5a$ and 5.	$5a = 5 \cdot a$ $5 = 5$ $\text{GCF} = 5$
	$5a + 5$
Rewrite each term as a product using the GCF.	$5 \cdot a + 5 \cdot 1$
Use the Distributive Property "in reverse" to factor the GCF.	$5(a + 1)$
Check by mulitplying the factors to get the orginal polynomial.	

$5(a+1)$	
$5 \cdot a + 5 \cdot 1$	
$5a + 5 \checkmark$	

Note:
Exercise:

Problem: Factor: $14x + 14$.

Solution:

$14\left(x+1\right)$

Note:
Exercise:

Problem: Factor: $12p + 12$.

Solution:

$12\left(p+1\right)$

The expressions in the next example have several factors in common. Remember to write the GCF as the product of all the common factors.

Example:
Exercise:

Problem: Factor: $12x - 60$.

Solution:
Solution

Find the GCF of 12*x* and 60.	$12x = 2 \cdot 2 \cdot 3 \cdot x$ $60 = 2 \cdot 2 \cdot 3 \cdot 5$ $\text{GCF} = 2 \cdot 2 \cdot 3$ $\text{GCF} = 12$
	$12x - 60$
Rewrite each term as a product using the GCF.	$12 \cdot x - 12 \cdot 5$
Factor the GCF.	$12(x - 5)$
Check by mulitplying the factors.	
$12(x-5)$	
$12 \cdot x - 12 \cdot 5$	
$12x - 60$✓	

Note:
Exercise:

Problem: Factor: $18u - 36$.

Solution:

$8\left(u - 2\right)$

Note:

Exercise:

Problem: Factor: $30y - 60$.

Solution:

$30\left(y - 2\right)$

Now we'll factor the greatest common factor from a trinomial. We start by finding the GCF of all three terms.

Example:
Exercise:

Problem: Factor: $4y^2 + 24y + 28$.

Solution:
Solution

We start by finding the GCF of all three terms.

Find the GCF of $4y^2$, $24y$ and 28.	$4y^2 = 2 \cdot 2 \cdot y \cdot y$ $24y = 2 \cdot 2 \cdot 2 \cdot 3 \cdot y$ $28 = 2 \cdot 2 \cdot 7$ $\text{GCF} = 2 \cdot 2$ $\text{GCF} = 4$
	$4y^2 + 24y + 28$

Rewrite each term as a product using the GCF.	$4 \cdot y^2 + 4 \cdot 6y + 4 \cdot 7$
Factor the GCF.	$4(y^2 + 6y + 7)$
Check by mulitplying.	
$4\left(y^2 + 6y + 7\right)$	
$4 \cdot y^2 + 4 \cdot 6y + 4 \cdot 7$	
$4y^2 + 24y + 28$✓	

Note:
Exercise:

Problem: Factor: $5x^2 - 25x + 15$.

Solution:

$5\left(x^2 - 5x + 3\right)$

Note:
Exercise:

Problem: Factor: $3y^2 - 12y + 27$.

Solution:

$3\left(y^2 - 4y + 9\right)$

Example:
Exercise:

Problem: Factor: $5x^3 - 25x^2$.

Solution:
Solution

Find the GCF of $5x^3$ and $25x^2$.	$5x^3 = 5 \cdot x \cdot x \cdot x$ $25x^2 = 5 \cdot 5 \cdot x \cdot x$ $\text{GCF} = 5 \cdot x \cdot x$ $\text{GCF} = 5x^2$
	$5x^3 - 25x^2$
Rewrite each term.	$5x^2 \cdot x - 5x^2 \cdot 5$
Factor the GCF.	$5x^2(x - 5)$
Check.	
$5x^2(x - 5)$	
$5x^2 \cdot x - 5x^2 \cdot 5$	
$5x^3 - 25x^2$✓	

Note:
Exercise:

Problem: Factor: $2x^3 + 12x^2$.

Solution:

$2x^2 (x + 6)$

Note:
Exercise:

Problem: Factor: $6y^3 - 15y^2$.

Solution:

$3y^2 (2y - 5)$

Example:
Exercise:

Problem: Factor: $21x^3 - 9x^2 + 15x$.

Solution:
Solution

In a previous example we found the GCF of $21x^3, 9x^2, 15x$ to be $3x$.

	$21x^3 - 9x^2 + 15x$
Rewrite each term using the GCF, 3*x*.	$3x \cdot 7x^2 - 3x \cdot 3x + 3x \cdot 5$
Factor the GCF.	$3x(7x^2 - 3x + 5)$

Check.	
$3x\left(7x^2 - 3x + 5\right)$	
$3x \cdot 7x^2 - 3x \cdot 3x + 3x \cdot 5$	
$21x^3 - 9x^2 + 15x$✓	

Note:
Exercise:

Problem: Factor: $20x^3 - 10x^2 + 14x$.

Solution:

$2x\left(10x^2 - 5x + 7\right)$

Note:
Exercise:

Problem: Factor: $24y^3 - 12y^2 - 20y$.

Solution:

$4y\left(6y^2 - 3y - 5\right)$

Example:
Exercise:

Problem: Factor: $8m^3 - 12m^2n + 20mn^2$.

Solution:
Solution

Find the GCF of $8m^3$, $12m^2n$, $20mn^2$.	$8m^3 = 2 \cdot 2 \cdot 2 \quad m \cdot m \cdot m$ $12m^2n = 2 \cdot 2 \cdot 3 \cdot \quad m \cdot m \cdot n$ $20mn^2 = 2 \cdot 2 \cdot \quad 5 \cdot m \cdot \quad n \cdot n$ $\text{GCF} = 2 \cdot 2 \cdot \quad m$ $\text{GCF} = 4m$
	$8m^3 - 12m^2n + 20mn^2$
Rewrite each term.	$4m \cdot 2m^2 - 4m \cdot 3m\,n + 4m \cdot 5n^2$
Factor the GCF.	$4m(2m^2 - 3m\;n + 5n^2)$
Check.	
$4m\left(2m^2 - 3mn + 5n^2\right)$	
$4m \cdot 2m^2 - 4m \cdot 3mn + 4m \cdot 5n^2$	
$8m^3 - 12m^2n + 20mn^2$✓	

Note:
Exercise:

Problem: Factor: $9xy^2 + 6x^2y^2 + 21y^3$.

Solution:

$3y^2\left(3x + 2x^2 + 7y\right)$

Note:
Exercise:

Problem: Factor: $3p^3 - 6p^2q + 9pq^3$.

Solution:

$3p\left(p^2 - 2pq + 3q^2\right)$

When the leading coefficient is negative, we factor the negative out as part of the GCF.

Example:
Exercise:

Problem: Factor: $-8y - 24$.

Solution:
Solution

When the leading coefficient is negative, the GCF will be negative.

Ignoring the signs of the terms, we first find the GCF of $8y$ and 24 is 8. Since the expression $-8y - 24$ has a negative leading coefficient, we use -8 as the GCF.	$8y = 2 \cdot 2 \cdot 2 \cdot y$ $24 = 2 \cdot 2 \cdot 2 \cdot 3$ GCF $= 2 \cdot 2 \cdot 2$ GCF $= 8$
Rewrite each term using the GCF.	$-8y - 24$ $-8 \cdot y + (-8) \cdot 3$
Factor the GCF.	$-8(y + 3)$

Check.	
$-8(y+3)$	
$-8 \cdot y + (-8) \cdot 3$	
$-8y - 24 ✓$	

Note:
Exercise:

Problem: Factor: $-16z - 64$.

Solution:

$-8\,(8z+8)$

Note:
Exercise:

Problem: Factor: $-9y - 27$.

Solution:

$-9\,(y+3)$

Example:
Exercise:

Problem: Factor: $-6a^2 + 36a$.

Solution:
Solution

The leading coefficient is negative, so the GCF will be negative.?

Since the leading coefficient is negative, the GCF is negative, $-6a$.	$6a^2 = 2 \cdot 3 \cdot a \cdot a$ $36a = 2 \cdot 2 \cdot 3 \cdot 3 \cdot a$ $\text{GCF} = 2 \cdot 3 \cdot a$ $\text{GCF} = 6a$ $-6a^2 + 36a$
Rewrite each term using the GCF.	$-6a \cdot a - (-6a) \cdot 6$
Factor the GCF.	$-6a(a - 6)$
Check.	
$-6a(a-6)$	
$-6a \cdot a + (-6a)(-6)$	
$-6a^2 + 36a$✓	

Note:
Exercise:

Problem: Factor: $-4b^2 + 16b$.

Solution:

$-4b\,(b-4)$

Note:
Exercise:

Problem: Factor: $-7a^2 + 21a$.

Solution:

$-7a\,(a - 3)$

Example:
Exercise:

Problem: Factor: $5q\,(q + 7) - 6\,(q + 7)$.

Solution:
Solution

The GCF is the binomial $q + 7$.

	$5q(q + 7) - 6(q + 7)$
Factor the GCF, $(q + 7)$.	$(q + 7)(5q - 6)$
Check on your own by multiplying.	

Note:
Exercise:

Problem: Factor: $4m\,(m + 3) - 7\,(m + 3)$.

Solution:

$(m+3)(4m-7)$

Note:
Exercise:

Problem: Factor: $8n(n-4)+5(n-4)$.

Solution:

$(n-4)(8n+5)$

Factor by Grouping

When there is no common factor of all the terms of a polynomial, look for a common factor in just some of the terms. When there are four terms, a good way to start is by separating the polynomial into two parts with two terms in each part. Then look for the GCF in each part. If the polynomial can be factored, you will find a common factor emerges from both parts.

(Not all polynomials can be factored. Just like some numbers are prime, some polynomials are prime.)

Numbers can often be factored in more than one way. For example, 36 = 4 x 9 and 36 = 6 x 6. Yet, when completely factored into primes both will eventually give the same answer.

36 = 4 x 9 = 2 x 2 x 3 x 3 and 36 = 6 x 6 = 2 x 3 x 2 x 3 and so the prime factors are the same. The same is true in factoring expressions in algebra.

Example:
How to Factor by Grouping
Exercise:

Problem: Factor: $xy+3y+2x+6$.

Solution:
Solution

Step 1. Group terms with common factors.	Is there a greatest common factor of all four terms? No, so let's separate the first two terms from the second two.	$xy + 3y + 2x + 6$ $\underbrace{xy + 3y} + \underbrace{2x + 6}$
Step 2. Factor out the common factor in each group.	Factor the GCF from the first two terms. Factor the GCF from the second two terms.	$y(x + 3) + \underbrace{2x + 6}$ $y(x + 3) + 2(x + 3)$
Step 3. Factor the common factor from the expression.	Notice that each term has a common factor of $(x + 3)$. Factor out the common factor.	$y(x + 3) + 2(x + 3)$ $(x + 3)(y + 2)$
Step 4. Check.	Multiply $(x + 3)(y + 2)$. Is the product the original expression?	$(x + 3)(y + 2)$ $xy + 2x + 3y + 6$ $xy + 3y + 2x + 6$ ✓

Now try this equivalent problem where only two middle terms have switched location.
Factor: $xy + 2x + 3y + 6$.
Group the terms: (xy + 2x) + (3y + 6)
Factor the GCF from each group: x(y + 2) and 3(y + 2)
Factor the common factor (y + 2): (y + 2)(x + 3)
Notice that the answer is equivalent to the answer from the equivalent problem above.
The bottom line is that we don't have to worry about getting a different factorization depending on how we group the terms.

Note:
Exercise:

Problem: Factor: $xy + 8y + 3x + 24$.

Solution:

$(x + 8)(y + 3)$

Note:
Exercise:

Problem: Factor: $ab+7b+8a+56$.

Solution:

$(a+7)(b+8)$

Note:
Factor by grouping.

Group terms with common factors.
Factor out the common factor in each group.
Factor the common factor from the expression.
Check by multiplying the factors.

Example:
Exercise:

Problem: Factor: $x^2+3x-2x-6$.

Solution:
Solution

There is no GCF in all four terms.	$x^2+3x\ \ -2x-6$
Separate into two parts.	$x^2+3x\ \ -2x-6$
Factor the GCF from both parts. Be careful with the signs when factoring the GCF from the last two terms.	$x(x+3)-2(x+3)$ $(x+3)(x-2)$

Check on your own by multiplying.

Note:

Exercise:

Problem: Factor: $x^2 + 2x - 5x - 10$.

Solution:

$(x - 5)\,(x + 2)$

Note:
Exercise:

Problem: Factor: $y^2 + 4y - 7y - 28$.

Solution:

$(y + 4)\,(y - 7)$

Note:
Access these online resources for additional instruction and practice with greatest common factors (GFCs) and factoring by grouping.

- Greatest Common Factor (GCF)
- Factoring Out the GCF of a Binomial
- Greatest Common Factor (GCF) of Polynomials

Key Concepts

- **Finding the Greatest Common Factor (GCF):** To find the GCF of two expressions:

  Factor each coefficient into primes. Write all variables with exponents in expanded form.
  List all factors—matching common factors in a column. In each column, circle the common factors.
  Bring down the common factors that all expressions share.
  Multiply the factors as in[link].

- **Factor the Greatest Common Factor from a Polynomial:** To factor a greatest common factor from a polynomial:

  Find the GCF of all the terms of the polynomial.
  Rewrite each term as a product using the GCF.
  Use the ‘reverse’ Distributive Property to factor the expression.
  Check by multiplying the factors as in[link].

- **Factor by Grouping:** To factor a polynomial with 4 four or more terms

  Group terms with common factors.
  Factor out the common factor in each group.
  Factor the common factor from the expression.
  Check by multiplying the factors as in[link].

Practice Makes Perfect

Find the Greatest Common Factor of Two or More Expressions

In the following exercises, find the greatest common factor.
Exercise:

Problem: 8, 18

Solution:

2

Exercise:

Problem: 24, 40

Exercise:

Problem: 72, 162

Solution:

18

Exercise:

Problem: 150, 275

Exercise:

Problem: $10a$, 50

Solution:

10

Exercise:

Problem: $5b$, 30

Exercise:

Problem: $3x, 10x^2$

Solution:

x

Exercise:

Problem: $21b^2, 14b$

Exercise:

Problem: $8w^2, 24w^3$

Solution:

$8w^2$

Exercise:

Problem: $30x^2, 18x^3$

Exercise:

Problem: $10p^3q, 12pq^2$

Solution:

$2pq$

Exercise:

Problem: $8a^2b^3, 10ab^2$

Exercise:

Problem: $12m^2n^3, 30m^5n^3$

Solution:

$6m^2n^3$

Exercise:

Problem: $28x^2y^4, 42x^4y^4$

Exercise:

Problem: $10a^3, 12a^2, 14a$

Solution:

$2a$

Exercise:

Problem: $20y^3, 28y^2, 40y$

Exercise:

Problem: $35x^3, 10x^4, 5x^5$

Solution:

$5x^3$

Exercise:

Problem: $27p^2, 45p^3, 9p^4$

Factor the Greatest Common Factor from a Polynomial

In the following exercises, factor the greatest common factor from each polynomial.

Exercise:

Problem: $4x + 20$

Solution:

$4\left(x+5\right)$

Exercise:

Problem: $8y+16$

Exercise:

Problem: $6m+9$

Solution:

$3\left(2m+3\right)$

Exercise:

Problem: $14p+35$

Exercise:

Problem: $9q+9$

Solution:

$9\left(q+1\right)$

Exercise:

Problem: $7r+7$

Exercise:

Problem: $8m-8$

Solution:

$8\left(m-1\right)$

Exercise:

Problem: $4n-4$

Exercise:

Problem: $9n-63$

Solution:

$9\,(n-7)$

Exercise:

Problem: $45b-18$

Exercise:

Problem: $3x^2+6x-9$

Solution:

$3(x^2+2x-3)$

Exercise:

Problem: $4y^2+8y-4$

Exercise:

Problem: $8p^2+4p+2$

Solution:

$2(4p^2+2p+1)$

Exercise:

Problem: $10q^2+14q+20$

Exercise:

Problem: $8y^3+16y^2$

Solution:

$8y^2(y+2)$

Exercise:

Problem: $12x^3-10x$

Exercise:

Problem: $5x^3-15x^2+20x$

Solution:

$5x(x^2 - 3x + 4)$

Exercise:

Problem: $8m^2 - 40m + 16$

Exercise:

Problem: $12xy^2 + 18x^2y^2 - 30y^3$

Solution:

$6y^2(2x + 3x^2 - 5y)$

Exercise:

Problem: $21pq^2 + 35p^2q^2 - 28q^3$

Exercise:

Problem: $-2x - 4$

Solution:

$-2\left(x + 4\right)$

Exercise:

Problem: $-3b + 12$

Exercise:

Problem: $5x\left(x + 1\right) + 3\left(x + 1\right)$

Solution:

$\left(x + 1\right)\left(5x + 3\right)$

Exercise:

Problem: $2x\left(x - 1\right) + 9\left(x - 1\right)$

Exercise:

Problem: $3b\left(b-2\right)-13\left(b-2\right)$

Solution:

$\left(b-2\right)\left(3b-13\right)$

Exercise:

Problem: $6m\left(m-5\right)-7\left(m-5\right)$

Factor by Grouping

In the following exercises, factor by grouping.

Exercise:

Problem: $xy+2y+3x+6$

Solution:

$(y+3)(x+2)$

Exercise:

Problem: $mn+4n+6m+24$

Exercise:

Problem: $uv-9u+2v-18$

Solution:

$(u+2)(v-9)$

Exercise:

Problem: $pq-10p+8q-80$

Exercise:

Problem: $b^2+5b-4b-20$

Solution:

$(b-4)(b+5)$

Exercise:

Problem: $m^2 + 6m - 12m - 72$

Exercise:

Problem: $p^2 + 4p - 9p - 36$

Solution:

$(p - 9)(p + 4)$

Exercise:

Problem: $x^2 + 5x - 3x - 15$

Mixed Practice

In the following exercises, factor.

Exercise:

Problem: $-20x - 10$

Solution:

$-10\,(2x + 1)$

Exercise:

Problem: $5x^3 - x^2 + x$

Exercise:

Problem: $3x^3 - 7x^2 + 6x - 14$

Solution:

$(x^2 + 2)(3x - 7)$

Exercise:

Problem: $x^3 + x^2 - x - 1$

Exercise:

Problem: $x^2 + xy + 5x + 5y$

Solution:

$(x + y)(x + 5)$

Exercise:

Problem: $5x^3 - 3x^2 - 5x - 3$

Everyday Math

Exercise:

Problem:

Area of a rectangle The area of a rectangle with length 6 less than the width is given by the expression $w^2 - 6w$, where $w =$ width. Factor the greatest common factor from the polynomial.

Solution:

$w(w - 6)$

Exercise:

Problem:

Height of a baseball The height of a baseball t seconds after it is hit is given by the expression $-16t^2 + 80t + 4$. Factor the greatest common factor from the polynomial.

Writing Exercises

Exercise:

Problem: The greatest common factor of 36 and 60 is 12. Explain what this means.

Solution:

Answers will vary.

Exercise:

Problem:

What is the GCF of y^4, y^5, and y^{10}? Write a general rule that tells you how to find the GCF of y^a, y^b, and y^c.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
find the greatest common factor of two or more expressions.			
factor the greatest common factor from a polynomial.			
factor by grouping.			

ⓑ If most of your checks were:

…confidently. Congratulations! You have achieved your goals in this section! Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific!

…with some help. This must be addressed quickly as topics you do not master become potholes in your road to success. Math is sequential—every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is critical and you must not ignore it. You need to get help immediately or you will quickly be overwhelmed. See your instructor as soon as possible to discuss your situation. Together you can come up with a plan to get you the help you need.

Glossary

factoring
: Factoring is splitting a product into factors; in other words, it is the reverse process of multiplying.

greatest common factor

The greatest common factor is the largest expression that is a factor of two or more expressions is the greatest common factor (GCF).

Factor Quadratic Trinomials with Leading Coefficient 1: ASE
By the end of this section, you will be able to:

- Factor trinomials of the form $x^2 + bx + c$
- Factor trinomials of the form $x^2 + bxy + cy^2$

Factor Trinomials of the Form $x^2 + bx + c$

You have already learned how to multiply binomials using FOIL. Now you’ll need to “undo” this multiplication—to start with the product and end up with the factors. Let’s look at an example of multiplying binomials to refresh your memory.

$(x + 2)(x + 3)$ factors
F O I L
$x^2 + 3x + 2x + 6$
$x^2 + 5x + 6$ product

To factor the trinomial means to start with the product, $x^2 + 5x + 6$, and end with the factors, $(x + 2)(x + 3)$. You need to think about where each of the terms in the trinomial came from.

The *first term* came from multiplying the first term in each binomial. So to get x^2 in the product, each binomial must start with an *x*.
Equation:

$$\begin{array}{c} x^2 + 5x + 6 \\ (x \quad)(x \quad) \end{array}$$

The *last term* in the trinomial came from multiplying the last term in each binomial. So the last terms must multiply to 6.

What two numbers multiply to 6?

The factors of 6 could be 1 and 6, or 2 and 3. How do you know which pair to use?

Consider the *middle term*. It came from adding the outer and inner terms.

So the numbers that must have a product of 6 will need a sum of 5. We’ll test both possibilities and summarize the results in [link]—the table will be very helpful when you work with numbers that can be factored in many different ways.

Factors of 6	Sum of factors
$1, 6$	$1 + 6 = 7$
$2, 3$	$2 + 3 = 5$

We see that 2 and 3 are the numbers that multiply to 6 and add to 5. So we have the factors of $x^2 + 5x + 6$. They are $(x + 2)(x + 3)$.

Equation:

$$\begin{array}{ll} x^2 + 5x + 6 & \text{product} \\ (x + 2)(x + 3) & \text{factors} \end{array}$$

You should check this by multiplying.

Looking back, we started with $x^2 + 5x + 6$, which is of the form $x^2 + bx + c$, where $b = 5$ and $c = 6$. We factored it into two binomials of the form $(x + m)$ and $(x + n)$.

Equation:

$$\begin{array}{ll} x^2 + 5x + 6 & x^2 + bx + c \\ (x + 2)(x + 3) & (x + m)(x + n) \end{array}$$

To get the correct factors, we found two numbers *m* and *n* whose product is *c* and sum is *b*.

Example:
How to Factor Trinomials of the Form $x^2 + bx + c$
Exercise:

Problem: Factor: $x^2 + 7x + 12$.

Solution:
Solution

Step 1. Write the factors as two binomials with first terms *x*.	Write two sets of parentheses and put *x* as the first term.	$x^2 + 7x + 12$ $(x \quad)(x \quad)$
Step 2. Find two numbers *m and n* that multiply to *c*, $m \cdot n = c$ add to *b*, $m + n = b$	Find two numbers that multiply to 12 and add to 7. **Factors of 12** / **Sum of factors** 1, 12 / $1 + 12 = 13$ 2, 6 / $2 + 6 = 8$ 3, 4 / $3 + 4 = 7$*	
Step 3. Use *m and n* as the last terms of the factors.	Use 3 and 4 as the last terms of the binomials.	$(x + 3)(x + 4)$
Step 4. Check by multiplying the factors.		$(x + 3)(x + 4)$ $x^2 + 4x + 3x + 12$ $x^2 + 7x + 12$ ✓

Note:
Exercise:

Problem: Factor: $x^2 + 6x + 8$.

Solution:

$(x + 2)(x + 4)$

Note:
Exercise:

Problem: Factor: $y^2 + 8y + 15$.

Solution:

$(y + 3)(y + 5)$

Let's summarize the steps we used to find the factors.

Note:
Factor trinomials of the form $x^2 + bx + c$.

Write the factors as two binomials with first terms *x*: $(x \quad)(x \quad)$.
Find two numbers *m* and *n* that Multiply to *c*, $m \cdot n = c$ Add to *b*, $m + n = b$
Use *m* and *n* as the last terms of the factors: $(x + m)(x + n)$.
Check by multiplying the factors.

Example:
Exercise:

Problem: Factor: $u^2 + 11u + 24$.

Solution:
Solution

Notice that the variable is *u*, so the factors will have first terms *u*.

$$u^2 + 11u + 24$$

Write the factors as two binomials with first terms u. $(u \quad)(u \quad)$

Find two numbers that: multiply to 24 and add to 11.

Factors of 24	**Sum of factors**
$1, 24$	$1 + 24 = 25$

Factors of 24	Sum of factors
$2, 12$	$2 + 12 = 14$
$3, 8$	$3 + 8 = 11$*
$4, 6$	$4 + 6 = 10$

Use 3 and 8 as the last terms of the binomials. $(u + 3)\,(u + 8)$

Check.

$$(u + 3)\,(u + 8)$$
$$u^2 + 3u + 8u + 24$$
$$u^2 + 11u + 24\ \checkmark$$

Note:
Exercise:

Problem: Factor: $q^2 + 10q + 24$.

Solution:

$(q + 4)\,(q + 6)$

Note:
Exercise:

Problem: Factor: $t^2 + 14t + 24$.

Solution:

$(t + 2)\,(t + 12)$

Example:
Exercise:

Problem: Factor: $y^2 + 17y + 60$.

Solution:
Solution

$$y^2 + 17y + 60$$

Write the factors as two binomials with first terms y. $\quad (y \quad)(y \quad)$

Find two numbers that multiply to 60 and add to 17.

Factors of 60	**Sum of factors**
$1, 60$	$1 + 60 = 61$
$2, 30$	$2 + 30 = 32$
$3, 20$	$3 + 20 = 23$
$4, 15$	$4 + 15 = 19$
$5, 12$	$5 + 12 = 17$*
$6, 10$	$6 + 10 = 16$

Use 5 and 12 as the last terms. $\quad (y + 5)(y + 12)$

Check.

$$(y + 5)(y + 12)$$
$$(y^2 + 12y + 5y + 60)$$
$$(y^2 + 17y + 60)✓$$

Note:
Exercise:

Problem: Factor: $x^2 + 19x + 60$.

Solution:

$(x + 4)(x + 15)$

Note:
Exercise:

Problem: Factor: $v^2 + 23v + 60$.

Solution:

$(v + 3)(v + 20)$

Factor Trinomials of the Form $x^2 + bx + c$ with b Negative, c Positive

In the examples so far, all terms in the trinomial were positive. What happens when there are negative terms? Well, it depends which term is negative. Let's look first at trinomials with only the middle term negative.

Remember: To get a negative sum and a positive product, the numbers must both be negative.

Again, think about FOIL and where each term in the trinomial came from. Just as before,

- the first term, x^2, comes from the product of the two first terms in each binomial factor, *x* and *y*;
- the positive last term is the product of the two last terms
- the negative middle term is the sum of the outer and inner terms.

How do you get a *positive product* and a *negative sum*? With two negative numbers.

Example:
Exercise:

Problem: Factor: $t^2 - 11t + 28$.

Solution:
Solution

Again, with the positive last term, 28, and the negative middle term, $-11t$, we need two negative factors. Find two numbers that multiply 28 and add to -11.

$$t^2 - 11t + 28$$

Write the factors as two binomials with first terms t. $\quad (t \quad)(t \quad)$

Find two numbers that: multiply to 28 and add to -11.

Factors of 28	**Sum of factors**
$-1, -28$	$-1 + (-28) = -29$
$-2, -14$	$-2 + (-14) = -16$
$-4, -7$	$-4 + (-7) = -11$*

Use $-4, -7$ as the last terms of the binomials. $\quad (t-4)(t-7)$

Check.

$$(t-4)(t-7)$$
$$t^2 - 7t - 4t + 28$$
$$t^2 - 11t + 28 \checkmark$$

Note:

Exercise:

Problem: Factor: $u^2 - 9u + 18$.

Solution:

$(u-3)(u-6)$

Note:
Exercise:

Problem: Factor: $y^2 - 16y + 63$.

Solution:

$(y-7)(y-9)$

Factor Trinomials of the Form $x^2 + bx + c$ with c Negative

Now, what if the last term in the trinomial is negative? Think about FOIL. The last term is the product of the last terms in the two binomials. A negative product results from multiplying two numbers with opposite signs. You have to be very careful to choose factors to make sure you get the correct sign for the middle term, too.

Remember: To get a negative product, the numbers must have different signs.

Example:
Exercise:

Problem: Factor: $z^2 + 4z - 5$.

Solution:
Solution

To get a negative last term, multiply one positive and one negative. We need factors of -5 that add to positive 4.

Factors of -5	**Sum of factors**
$1, -5$	$1 + (-5) = -4$
$-1, 5$	$-1 + 5 = 4$*

Notice: We listed both $1, -5$ and $-1, 5$ to make sure we got the sign of the middle term correct.

	$z^2 + 4z - 5$
Factors will be two binomials with first terms z.	$(z \quad)(z \quad)$
Use $-1, 5$ as the last terms of the binomials.	$(z - 1)(z + 5)$
Check.	

$$(z - 1)(z + 5)$$
$$z^2 + 5z - 1z - 5$$
$$z^2 + 4z - 5 \checkmark$$

Note:
Exercise:

Problem: Factor: $h^2 + 4h - 12$.

Solution:

$(h - 2)(h + 6)$

Note:
Exercise:

Problem: Factor: $k^2 + k - 20$.

Solution:

$(k - 4)\,(k + 5)$

Let's make a minor change to the last trinomial and see what effect it has on the factors.

Example:
Exercise:

Problem: Factor: $z^2 - 4z - 5$.

Solution:
Solution

This time, we need factors of -5 that add to -4.

Factors of -5	**Sum of factors**
$1, -5$	$1 + (-5) = -4$*
$-1, 5$	$-1 + 5 = 4$

	$z^2 - 4z - 5$
Factors will be two binomials with first terms z.	$(z \quad)(z \quad)$
Use $1, -5$ as the last terms of the binomials.	$(z+1)(z-5)$

Check.

$$(z+1)(z-5)$$
$$z^2 - 5z + 1z - 5$$
$$z^2 - 4z - 5 \checkmark$$

Notice that the factors of $z^2 - 4z - 5$ are very similar to the factors of $z^2 + 4z - 5$. It is very important to make sure you choose the factor pair that results in the correct sign of the middle term.

Note:
Exercise:

Problem: Factor: $x^2 - 4x - 12$.

Solution:

$(x+2)(x-6)$

Note:
Exercise:

Problem: Factor: $y^2 - y - 20$.

Solution:

$(y+4)(y-5)$

Example:
Exercise:

Problem: Factor: $q^2 - 2q - 15$.

Solution:
Solution

	$q^2 - 2q - 15$
Factors will be two binomials with first terms q.	$(q \quad)(q \quad)$
You can use $3, -5$ as the last terms of the binomials.	$(q+3)(q-5)$

Factors of -15	Sum of factors
$1, -15$	$1 + (-15) = -14$
$-1, 15$	$-1 + 15 = 14$
$3, -5$	$3 + (-5) = -2$*
$-3, 5$	$-3 + 5 = 2$

Check.

$$(q+3)(q-5)$$

$$q^2 - 5q + 3q - 15$$

$$q^2 - 2q - 15 \checkmark$$

Note:
Exercise:

Problem: Factor: $r^2 - 3r - 40$.

Solution:

$(r+5)(r-8)$

Note:
Exercise:

Problem: Factor: $s^2-3s-10.$

Solution:

$(s+2)(s-5)$

Some trinomials are prime. The only way to be certain a trinomial is prime is to list all the possibilities and show that none of them work.

Example:
Exercise:

Problem: Factor: $y^2-6y+15.$

Solution:
Solution

	$y^2-6y+15$
Factors will be two binomials with first terms y.	$(y\quad)(y\quad)$

Factors of 15	Sum of factors
$-1, -15$	$-1 + (-15) = -16$
$-3, -5$	$-3 + (-5) = -8$

As shown in the table, none of the factors add to -6; therefore, the expression is prime.

Note:
Exercise:

Problem: Factor: $m^2 + 4m + 18$.

Solution:

prime

Note:
Exercise:

Problem: Factor: $n^2 - 10n + 12$.

Solution:

prime

Example:
Exercise:

Problem: Factor: $2x + x^2 - 48$.

Solution:

Solution

	$2x + x^2 - 48$
First we put the terms in decreasing degree order.	$x^2 + 2x - 48$
Factors will be two binomials with first terms x.	$(x \quad)(x \quad)$

As shown in the table, you can use $-6, 8$ as the last terms of the binomials.
Equation:

$$(x - 6)(x + 8)$$

Factors of -48	**Sum of factors**
$-1, 48$	$-1 + 48 = 47$
$-2, 24$ $-3, 16$ $-4, 12$ $-6, 8$	$-2 + 24 = 22$ $-3 + 16 = 13$ $-4 + 12 = 8$ $-6 + 8 = 2$

Check.

$$(x - 6)(x + 8)$$

$$x^2 - 6q + 8q - 48$$

$$x^2 + 2x - 48 \checkmark$$

Note:
Exercise:

Problem: Factor: $9m + m^2 + 18$.

Solution:

$(m+3)(m+6)$

Note:
Exercise:

Problem: Factor: $-7n+12+n^2$.

Solution:

$(n-3)(n-4)$

Let's summarize the method we just developed to factor trinomials of the form x^2+bx+c.

Note:
Factor trinomials.
When we factor a trinomial, we look at the signs of its terms first to determine the signs of the binomial factors.
Equation:

$$\begin{array}{c} x^2+bx+c \\ (x+m)(x+n) \end{array}$$

When *c* is positive, *m* and *n* have the same sign.
Equation:

$$\begin{array}{cc} b\,\text{positive} & b\,\text{negative} \\ m,n\,\text{positive} & m,n\,\text{negative} \\ x^2+5x+6 & x^2-6x+8 \\ (x+2)(x+3) & (x-4)(x-2) \\ \text{same signs} & \text{same signs} \end{array}$$

When *c* is negative, *m* and *n* have opposite signs.
Equation:

$$\begin{array}{cc} x^2 + x - 12 & x^2 - 2x - 15 \\ (x+4)(x-3) & (x-5)(x+3) \\ \text{opposite signs} & \text{opposite signs} \end{array}$$

Notice that, in the case when *m* and *n* have opposite signs, the sign of the one with the larger absolute value matches the sign of *b*.

Factor Trinomials of the Form $x^2 + bxy + cy^2$

Sometimes you'll need to factor trinomials of the form $x^2 + bxy + cy^2$ with two variables, such as $x^2 + 12xy + 36y^2$. The first term, x^2, is the product of the first terms of the binomial factors, $x \cdot x$. The y^2 in the last term means that the second terms of the binomial factors must each contain *y*. To get the coefficients *b* and *c*, you use the same process summarized in the previous objective.

Example:
Exercise:

Problem: Factor: $x^2 + 12xy + 36y^2$.

Solution:
Solution

	$x^2 + 12xy + 36y^2$
Note that the first terms are x, last terms contain y.	$(x \_ y)(x \_ y)$
Find the numbers that multiply to 36 and add to 12.	

Factors of 36	Sum of factors
1, 36	$1 + 36 = 37$
2, 18	$2 + 18 = 20$
3, 12	$3 + 12 = 15$
4, 9	$4 + 9 = 13$
6, 6	$6 + 6 = 12$*

Use 6 and 6 as the coefficients of the last terms. $(x + 6y)(x + 6y)$

Check your answer.

$$\begin{array}{c} (x + 6y)(x + 6y) \\ x^2 + 6xy + 6xy + 36y^2 \\ x^2 + 12xy + 36y^2 \checkmark \end{array}$$

Note:
Exercise:

Problem: Factor: $u^2 + 11uv + 28v^2$.

Solution:

$(u + 4v)(u + 7v)$

Note:
Exercise:

Problem: Factor: $x^2 + 13xy + 42y^2$.

Solution:

$(x+6y)(x+7y)$

Example:
Exercise:

Problem: Factor: $r^2-8rx-9s^2$.

Solution:
Solution

We need r in the first term of each binomial and s in the second term. The last term of the trinomial is negative, so the factors must have opposite signs.

$$\begin{array}{lr} & r^2-8rx-9s^2 \\ \text{Note that the first terms are } r, \text{ last terms contain } s. & (r\ \_\ s)(r\ \_\ s) \end{array}$$

Find the numbers that multiply to -9 and add to -8.

Factors of -9	**Sum of factors**
$1,-9$	$1+(-9)=-8^*$
$-1,9$	$-1+9=8$
$3,-3$	$3+(-3)=0$

Use $1, -9$ as coefficients of the last terms. $(r+s)(r-9s)$

Check your answer.

$$(r-9s)(r+s)$$
$$r^2+rs-9rs-9s^2$$
$$r^2-8rs-9s^2\ \checkmark$$

Note:
Exercise:

Problem: Factor: $a^2-11ab+10b^2$.

Solution:

$(a-b)(a-10b)$

Note:
Exercise:

Problem: Factor: $m^2-13mn+12n^2$.

Solution:

$(m-n)(m-12n)$

Example:
Exercise:

Problem: Factor: $u^2 - 9uv - 12v^2$.

Solution:
Solution

We need u in the first term of each binomial and v in the second term. The last term of the trinomial is negative, so the factors must have opposite signs.

	$u^2 - 9uv - 12v^2$
Note that the first terms are u, last terms contain v.	$(u \_ v)(u \_ v)$

Find the numbers that multiply to -12 and add to -9.

Factors of -12	**Sum of factors**
$1, -12$	$1 + (-12) = -11$
$-1, 12$	$-1 + 12 = 11$
$2, -6$	$2 + (-6) = -4$
$-2, 6$	$-2 + 6 = 4$
$3, -4$	$3 + (-4) = -1$
$-3, 4$	$-3 + 4 = 1$

Note there are no factor pairs that give us -9 as a sum. The trinomial is prime.

Note:
Exercise:

Problem: Factor: $x^2 - 7xy - 10y^2$.

Solution:

prime

Note:
Exercise:

Problem: Factor: $p^2 + 15pq + 20q^2$.

Solution:

prime

Key Concepts

- **Factor trinomials of the form** $x^2 + bx + c$

  Write the factors as two binomials with first terms*x*:$(x\quad)\,(x\quad)$.
  Find two numbers*m*and*n*thatMultiply to*c*,$m \cdot n = c$Add to*b*,$m + n = b$
  Use*m*and*n*as the last terms of the factors:$(x + m)(x + n)$.
  Check by multiplying the factors.

Practice Makes Perfect

Factor Trinomials of the Form $x^2 + bx + c$

In the following exercises, factor each trinomial of the form $x^2 + bx + c$.
Exercise:

Problem: $x^2 + 4x + 3$

Solution:

$(x + 1)(x + 3)$

Exercise:

Problem: $y^2 + 8y + 7$

Exercise:

Problem: $m^2 + 12m + 11$

Solution:

$(m + 1)(m + 11)$

Exercise:

Problem: $b^2 + 14b + 13$

Exercise:

Problem: $a^2 + 9a + 20$

Solution:

$(a + 4)(a + 5)$

Exercise:

Problem: $m^2 + 7m + 12$

Exercise:

Problem: $p^2 + 11p + 30$

Solution:

$(p + 5)(p + 6)$

Exercise:

Problem: $w^2 + 10x + 21$

Exercise:

Problem: $n^2 + 19n + 48$

Solution:

$(n + 3)(n + 16)$

Exercise:

Problem: $b^2 + 14b + 48$

Exercise:

Problem: $a^2 + 25a + 100$

Solution:

$(a + 5)(a + 20)$

Exercise:

Problem: $u^2 + 101u + 100$

Exercise:

Problem: $x^2 - 8x + 12$

Solution:

$(x - 2)(x - 6)$

Exercise:

Problem: $q^2 - 13q + 36$

Exercise:

Problem: $y^2 - 18x + 45$

Solution:

$(y - 3)(y - 15)$

Exercise:

Problem: $m^2 - 13m + 30$

Exercise:

Problem: $x^2 - 8x + 7$

Solution:

$(x-1)(x-7)$

Exercise:

Problem: $y^2 - 5y + 6$

Exercise:

Problem: $p^2 + 5p - 6$

Solution:

$(p-1)(p+6)$

Exercise:

Problem: $n^2 + 6n - 7$

Exercise:

Problem: $y^2 - 6y - 7$

Solution:

$\left(y+1\right)\left(y-7\right)$

Exercise:

Problem: $v^2 - 2v - 3$

Exercise:

Problem: $x^2 - x - 12$

Solution:

$(x - 4)(x + 1)$ $(x - 4)(x + 3)$

Exercise:

Problem: $r^2 - 2r - 8$

Exercise:

Problem: $a^2 - 3a - 28$

Solution:

$(a - 7)(a + 4)$

Exercise:

Problem: $b^2 - 13b - 30$

Exercise:

Problem: $w^2 - 5w - 36$

Solution:

$(w - 9)(w + 4)$

Exercise:

Problem: $t^2 - 3t - 54$

Exercise:

Problem: $x^2 + x + 5$

Solution:

prime

Exercise:

Problem: $x^2 - 3x - 9$

Exercise:

Problem: $8 - 6x + x^2$

Solution:

$(x - 4)(x - 2)$

Exercise:

Problem: $7x + x^2 + 6$

Exercise:

Problem: $x^2 - 12 - 11x$

Solution:

$(x - 12)(x + 1)$

Exercise:

Problem: $-11 - 10x + x^2$

Factor Trinomials of the Form $x^2 + bxy + cy^2$

In the following exercises, factor each trinomial of the form $x^2 + bxy + cy^2$.

Exercise:

Problem: $p^2 + 3pq + 2q^2$

Solution:

$(p + q)(p + 2q)$

Exercise:

Problem: $m^2 + 6mn + 5n^2$

Exercise:

Problem: $r^2 + 15rs + 36s^2$

Solution:

$(r + 3s)(r + 12s)$

Exercise:

Problem: $u^2 + 10uv + 24v^2$

Exercise:

Problem: $m^2 - 12mn + 20n^2$

Solution:

$(m - 2n)(m - 10n)$

Exercise:

Problem: $p^2 - 16pq + 63q^2$

Exercise:

Problem: $x^2 - 2xy - 80y^2$

Solution:

$(x + 8y)(x - 10y)$

Exercise:

Problem: $p^2 - 8pq - 65q^2$

Exercise:

Problem: $m^2 - 64mn - 65n^2$

Solution:

$(m + n)(m - 65n)$

Exercise:

Problem: $p^2 - 2pq - 35q^2$

Exercise:

Problem: $a^2 + 5ab - 24b^2$

Solution:

$(a + 8b)(a - 3b)$

Exercise:

Problem: $r^2 + 3rs - 28s^2$

Exercise:

Problem: $x^2 - 3xy - 14y^2$

Solution:

prime

Exercise:

Problem: $u^2 - 8uv - 24v^2$

Exercise:

Problem: $m^2 - 5mn + 30n^2$

Solution:

prime

Exercise:

Problem: $c^2 - 7cd + 18d^2$

Mixed Practice

In the following exercises, factor each expression.

Exercise:

Problem: $u^2 - 12u + 36$

Solution:

$(u-6)(u-6)$

Exercise:

Problem: $w^2+4w-32$

Exercise:

Problem: $x^2-14x-32$

Solution:

$(x+2)(x-16)$

Exercise:

Problem: $y^2+41y+40$

Exercise:

Problem: $r^2-20rs+64s^2$

Solution:

$(r-4s)(r-16s)$

Exercise:

Problem: $x^2-16xy+64y^2$

Exercise:

Problem: $k^2+34k+120$

Solution:

$(k+4)(k+30)$

Exercise:

Problem: $m^2+29m+120$

Exercise:

Problem: $y^2 + 10y + 15$

Solution:

prime

Exercise:

Problem: $z^2 - 3z + 28$

Exercise:

Problem: $m^2 + mn - 56n^2$

Solution:

$(m + 8n)(m - 7n)$

Exercise:

Problem: $q^2 - 29qr - 96r^2$

Exercise:

Problem: $u^2 - 17uv + 30v^2$

Solution:

$(u - 15v)(u - 2v)$

Exercise:

Problem: $m^2 - 31mn + 30n^2$

Exercise:

Problem: $c^2 - 8cd + 26d^2$

Solution:

prime

Exercise:

Problem: $r^2 + 11rs + 36s^2$

Everyday Math

Exercise:

Problem:

Consecutive integers Deirdre is thinking of two consecutive integers whose product is 56. The trinomial $x^2 + x - 56$ describes how these numbers are related. Factor the trinomial.

Solution:

$(x + 8)\,(x - 7)$

Exercise:

Problem:

Consecutive integers Deshawn is thinking of two consecutive integers whose product is 182. The trinomial $x^2 + x - 182$ describes how these numbers are related. Factor the trinomial.

Writing Exercises

Exercise:

Problem:

Many trinomials of the form $x^2 + bx + c$ factor into the product of two binomials $(x + m)(x + n)$. Explain how you find the values of *m* and *n*.

Solution:

Answers may vary

Exercise:

Problem:

How do you determine whether to use plus or minus signs in the binomial factors of a trinomial of the form x^2+bx+c where b and c may be positive or negative numbers?

Exercise:

Problem:

Will factored x^2-x-20 as $(x+5)(x-4)$. Bill factored it as $(x+4)(x-5)$. Phil factored it as $(x-5)(x-4)$. Who is correct? Explain why the other two are wrong.

Solution:

Answers may vary

Exercise:

Problem:

Look at [link], where we factored $y^2+17y+60$. We made a table listing all pairs of factors of 60 and their sums. Do you find this kind of table helpful? Why or why not?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
factor trinomials of the form x^2+bx+c.			
factor trinomials of the form $x^2+bxy+cy^2$.			

ⓑ After reviewing this checklist, what will you do to become confident for all goals?

Factor Quadratic Trinomials with Leading Coefficient Other than 1: ASE
By the end of this section, you will be able to:

- Recognize a preliminary strategy to factor polynomials completely
- Factor trinomials of the form $ax^2 + bx + c$ with a GCF
- Factor trinomials using trial and error
- Factor trinomials using the ‘ac’ method

Recognize a Preliminary Strategy for Factoring

Let’s summarize where we are so far with factoring polynomials. In the first two sections of this chapter, we used three methods of factoring: factoring the GCF, factoring by grouping, and factoring a trinomial by “undoing” FOIL. More methods will follow as you continue in this chapter, as well as later in your studies of algebra.

How will you know when to use each factoring method? As you learn more methods of factoring, how will you know when to apply each method and not get them confused? It will help to organize the factoring methods into a strategy that can guide you to use the correct method.

As you start to factor a polynomial, always ask first, “Is there a greatest common factor?” If there is, factor it first.

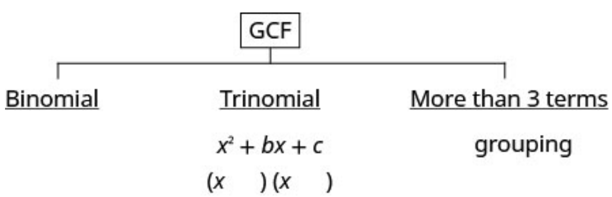
The next thing to consider is the type of polynomial. How many terms does it have? Is it a binomial? A trinomial? Or does it have more than three terms?

If it is a trinomial where the leading coefficient is one, $x^2 + bx + c$, use the “undo FOIL” method.

If it has more than three terms, try the grouping method. This is the only method to use for polynomials of more than three terms.

Some polynomials cannot be factored. They are called “prime.”

Below we summarize the methods we have so far. These are detailed in Choose a strategy to factor polynomials completely.



Note:
Choose a strategy to factor polynomials completely.

Is there a greatest common factor?

- Factor it out.

Is the polynomial a binomial, trinomial, or are there more than three terms?

- If it is a binomial, right now we have no method to factor it.
- If it is a trinomial of the form $x^2 + bx + c$: Undo FOIL $(x \quad)(x \quad)$

- If it has more than three terms: Use the grouping method.

Check by multiplying the factors.

Use the preliminary strategy to completely factor a polynomial. A polynomial is factored completely if, other than monomials, all of its factors are prime.

Example:
Exercise:

Problem: Identify the best method to use to factor each polynomial.

ⓐ $6y^2 - 72$
ⓑ $r^2 - 10r - 24$
ⓒ $p^2 + 5p + pq + 5q$

Solution:
Solution

ⓐ

	$6y^2 - 72$
Is there a greatest common factor?	Yes, 6.
Factor out the 6.	$6\left(y^2 - 12\right)$
Is it a binomial, trinomial, or are there more than 3 terms?	Binomial, we have no method to factor binomials yet.

ⓑ

	$r^2 - 10r - 24$
Is there a greatest common factor?	No, there is no common factor.
Is it a binomial, trinomial, or are there more than three terms?	Trinomial, with leading coefficient 1, so “undo” FOIL.

ⓒ

	$p^2 + 5p + pq + 5q$
Is there a greatest common factor?	No, there is no common factor.
Is it a binomial, trinomial, or are there more than three terms?	More than three terms, so factor using grouping.

Note:
Exercise:

Problem: Identify the best method to use to factor each polynomial:

ⓐ $4y^2+32$
ⓑ $y^2+10y+21$
ⓒ $yz+2y+3z+6$

Solution:

ⓐ no method ⓑ undo using FOIL ⓒ factor with grouping

Note:
Exercise:

Problem: Identify the best method to use to factor each polynomial:

ⓐ $ab+a+4b+4$
ⓑ $3k^2+15$
ⓒ p^2+9p+8

Solution:

ⓐ factor using grouping ⓑ no method ⓒ undo using FOIL

Factor Trinomials of the form *ax*² + *bx* + *c* with a GCF

Now that we have organized what we've covered so far, we are ready to factor trinomials whose leading coefficient is not 1, trinomials of the form ax^2+bx+c.

Remember to always check for a GCF first! Sometimes, after you factor the GCF, the leading coefficient of the trinomial becomes 1 and you can factor it by the methods in the last section. Let's do a few examples to see how this works.

Watch out for the signs in the next two examples.

Example:
Exercise:

Problem: Factor completely: $2n^2-8n-42$.

Solution:
Solution

Use the preliminary strategy.

Is there a greatest common factor?	$2n^2 - 8n - 42$
Yes, GCF $= 2$. Factor it out.	$2\left(n^2 - 4n - 21\right)$
Inside the parentheses, is it a binomial, trinomial, or are there more than three terms?	
It is a trinomial whose coefficient is 1, so undo FOIL.	$2\left(n \quad\right)\left(n \quad\right)$
Use 3 and -7 as the last terms of the binomials.	$2\left(n+3\right)\left(n-7\right)$

Factors of -21	Sum of factors
$1, -21$	$1 + (-21) = -20$
$3, -7$	$3 + (-7) = -4$*

Check.

$2\left(n+3\right)\left(n-7\right)$

$2(n^2 - 7n + 3n - 21)$

$2(n^2 - 4n - 21)$

$2n^2 - 8n - 42$ ✓

Note:
Exercise:

Problem: Factor completely: $4m^2 - 4m - 8$.

Solution:

$4\left(m+1\right)\left(m-2\right)$

Note:
Exercise:

Problem: Factor completely: $5k^2 - 15k - 50$.

Solution:

$5\left(k+2\right)\left(k-5\right)$

Example:
Exercise:

Problem: Factor completely: $4y^2 - 36y + 56$.

Solution:
Solution

Use the preliminary strategy.

Is there a greatest common factor?	$4y^2 - 36y + 56$
Yes, GCF = 4. Factor it.	$4\left(y^2 - 9y + 14\right)$
Inside the parentheses, is it a binomial, trinomial, or are there more than three terms?	
It is a trinomial whose coefficient is 1. So undo FOIL.	$4\left(y\quad\right)\left(y\quad\right)$
Use a table like the one below to find two numbers that multiply to 14 and add to -9.	
Both factors of 14 must be negative.	$4\left(y - 2\right)\left(y - 7\right)$

Factors of 14	**Sum of factors**
$-1, -14$	$-1 + (-14) = -15$
$-2, -7$	$-2 + (-7) = -9$*

Check.

$4\left(y - 2\right)\left(y - 7\right)$

$4(y^2 - 7y - 2y + 14)$

$4(y^2 - 9y + 14)$

$4y^2 - 36y + 42$ ✓

Note:
Exercise:

Problem: Factor completely: $3r^2 - 9r + 6$.

Solution:

$3\left(r - 1\right)\left(r - 2\right)$

Note:
Exercise:

Problem: Factor completely: $2t^2 - 10t + 12$.

Solution:

$2\left(t-2\right)\left(t-3\right)$

In the next example the GCF will include a variable.

Example:
Exercise:

Problem: Factor completely: $4u^3 + 16u^2 - 20u$.

Solution:
Solution

Use the preliminary strategy.

Is there a greatest common factor?	$4u^3 + 16u^2 - 20u$
Yes, GCF $= 4u$. Factor it.	$4u\left(u^2 + 4u - 5\right)$
Binomial, trinomial, or more than three terms?	
It is a trinomial. So "undo FOIL."	$4u\left(u \quad\right)\left(u \quad\right)$
Use a table like the table below to find two numbers that multiply to -5 and add to 4.	$4u\left(u-1\right)\left(u+5\right)$

Factors of -5	Sum of factors
$-1, 5$	$-1 + 5 = 4$*
$1, -5$	$1 + (-5) = -4$

Check.

$$4u\left(u-1\right)\left(u+5\right)$$

$$4u(u^2 + 5u - u - 5)$$

$$4u(u^2 + 4u - 5)$$

$$4u^3 + 16u^2 - 20u \checkmark$$

Note:
Exercise:

Problem: Factor completely: $5x^3 + 15x^2 - 20x$.

Solution:

$5x\left(x - 1\right)\left(x + 4\right)$

Note:
Exercise:

Problem: Factor completely: $6y^3 + 18y^2 - 60y$.

Solution:

$6y\left(y - 2\right)\left(y + 5\right)$

Factor Trinomials using Trial and Error

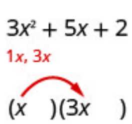
What happens when the leading coefficient is not 1 and there is no GCF? There are several methods that can be used to factor these trinomials. First we will use the Trial and Error method.

Let's factor the trinomial $3x^2 + 5x + 2$.

From our earlier work we expect this will factor into two binomials.
Equation:

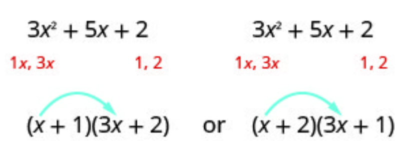
$$\begin{array}{c} 3x^2 + 5x + 2 \\ (\quad)(\quad) \end{array}$$

We know the first terms of the binomial factors will multiply to give us $3x^2$. The only factors of $3x^2$ are $1x, 3x$. We can place them in the binomials.

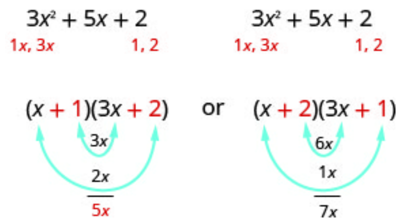


Check. Does $1x \cdot 3x = 3x^2$?

We know the last terms of the binomials will multiply to 2. Since this trinomial has all positive terms, we only need to consider positive factors. The only factors of 2 are 1 and 2. But we now have two cases to consider as it will make a difference if we write 1, 2, or 2, 1.



Which factors are correct? To decide that, we multiply the inner and outer terms.



Since the middle term of the trinomial is 5x, the factors in the first case will work. Let’s FOIL to check.

Equation:

$$\begin{array}{l}(x+1)\,(3x+2)\\ 3x^2+2x+3x+2\\ 3x^2+5x+2\,\checkmark\end{array}$$

Our result of the factoring is:

Equation:

$$\begin{array}{l}3x^2+5x+2\\ (x+1)\,(3x+2)\end{array}$$

Example:

How to Factor Trinomials of the Form ax^2+bx+c Using Trial and Error

Exercise:

Problem: Factor completely: $3y^2+22y+7$.

Solution:

Solution

Step 1. Write the trinomial in descending order.	The trinomial is already in descending order.	$3y^2+22y+7$
Step 2. Find all the factor pairs of the first term.	The only factors of $3y^2$ are $1y$, $3y$ Since there is only one pair, we can put them in the parentheses.	$3y^2+22y+7$ $1y,\ 3y$ $3y^2+22y+7$ $1y,\ 3y$ $(y\quad)(3y\quad)$
Step 3. Find all the factor pairs of the third term.	The only factors of 7 are 1, 7.	$3y^2+22y+7$ $1y,\ 3y \qquad 1,\ 7$ $(y\quad)(3y\quad)$

Step 4. Test all the possible combinations of the factors until the correct product is found.	$3y^2 + 22y + 7$ $1y, 3y \quad 1, 7$ $(y + 1)(3y + 7)$ $3y$ $7y$ $10y$ No. We need $22y$ $3y^2 + 22y + 7$ $1y, 3y \quad 1, 7$ $(y + 7)(3y + 1)$ $21y$ y $22y$	(see table below) $(y + 7)(3y + 1)$
Step 5. Check by multiplying.		$(y + 7)(3y + 1)$ $3y^2 + 22y + 7$ ✓

$3y^2 + 22y + 7$	
Possible factors	**Product**
$(y + 1)(3y + 7)$	$3y^2 + 10y + 7$
$(y + 7)(3y + 1)$	$3y^2 + 22y + 7$

Note:
Exercise:

Problem: Factor completely: $2a^2 + 5a + 3$.

Solution:

$(a + 1)(2a + 3)$

Note:
Exercise:

Problem: Factor completely: $4b^2 + 5b + 1$.

Solution:

$(b + 1)(4b + 1)$

Note:
Factor trinomials of the form $ax^2 + bx + c$ using trial and error.

Write the trinomial in descending order of degrees.
Find all the factor pairs of the first term.
Find all the factor pairs of the third term.
Test all the possible combinations of the factors until the correct product is found.
Check by multiplying.

When the middle term is negative and the last term is positive, the signs in the binomials must both be negative.

Example:
Exercise:

Problem: Factor completely: $6b^2 - 13b + 5$.

Solution:
Solution

The trinomial is already in descending order.	
Find the factors of the first term.	
Find the factors of the last term. Consider the signs. Since the last term, 5 is positive its factors must both be positive or both be negative. The coefficient of the middle term is negative, so we use the negative factors.	

Consider all the combinations of factors.

$6b^2 - 13b + 5$	
Possible factors	**Product**
$(b - 1)(6b - 5)$	$6b^2 - 11b + 5$
$(b - 5)(6b - 1)$	$6b^2 - 31b + 5$
$(2b - 1)(3b - 5)$	$6b^2 - 13b + 5$ *
$(2b - 5)(3b - 1)$	$6b^2 - 17b + 5$

The correct factors are those whose product is the original trinomial. $(2b-1)(3b-5)$

Check by multiplying.

$(2b-1)(3b-5)$
$6b^2-10b-3b+5$
$6b^2-13b+5$ ✓

Note:
Exercise:

Problem: Factor completely: $8x^2-13x+3$.

Solution:

$(2x-3)(4x-1)$

Note:
Exercise:

Problem: Factor completely: $10y^2-37y+7$.

Solution:

$(2y-7)(5y-1)$

When we factor an expression, we always look for a greatest common factor first. If the expression does not have a greatest common factor, there cannot be one in its factors either. This may help us eliminate some of the possible factor combinations.

Example:
Exercise:

Problem: Factor completely: $14x^2-47x-7$.

Solution:
Solution

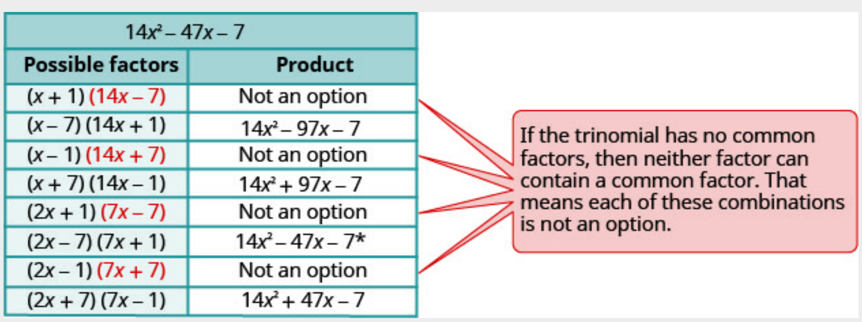
The trinomial is already in descending order.	$14x^2-47x-7$

Find the factors of the first term.	$14x^2 - 47x - 7$ $1x \cdot 14x$ $2x \cdot 7x$
Find the factors of the last term. Consider the signs. Since it is negative, one factor must be positive and one negative.	$14x^2 - 47x - 7$ $1x \cdot 14x$ $1, -7$ $2x \cdot 7x$ $-1, 7$

Consider all the combinations of factors. We use each pair of the factors of $14x^2$ with each pair of factors of -7.

Factors of $14x^2$	**Pair with**	**Factors of -7**
$x, 14x$		$1, -7$ $-7, 1$ (reverse order)
$x, 14x$		$-1, 7$ $7, -1$ (reverse order)
$2x, 7x$		$1, -7$ $-7, 1$ (reverse order)
$2x, 7x$		$-1, 7$ $7, -1$ (reverse order)

These pairings lead to the following eight combinations.



$14x^2 - 47x - 7$	
Possible factors	**Product**
$(x+1)(14x-7)$	Not an option
$(x-7)(14x+1)$	$14x^2 - 97x - 7$
$(x-1)(14x+7)$	Not an option
$(x+7)(14x-1)$	$14x^2 + 97x - 7$
$(2x+1)(7x-7)$	Not an option
$(2x-7)(7x+1)$	$14x^2 - 47x - 7$*
$(2x-1)(7x+7)$	Not an option
$(2x+7)(7x-1)$	$14x^2 + 47x - 7$

The correct factors are those whose product is the original trinomial. Check by multiplying. $(2x-7)(7x+1)$ $14x^2+2x-49x-7$ $14x^2-47x-7$ ✓	$(2x-7)(7x+1)$

Note:
Exercise:

Problem: Factor completely: $8a^2-3a-5$.

Solution:

$(a-1)(8a+5)$

Note:
Exercise:

Problem: Factor completely: $6b^2-b-15$.

Solution:

$(2b+3)(3b-5)$

Example:
Exercise:

Problem: Factor completely: $18n^2-37n+15$.

Solution:
Solution

The trinomial is already in descending order.	$18n^2-37n+15$
Find the factors of the first term.	$18n^2-37n+15$ $1n \cdot 18n$ $2n \cdot 9n$ $3n \cdot 6n$

Find the factors of the last term. Consider the signs. Since 15 is positive and the coefficient of the middle term is negative, we use the negative facotrs.	$18n^2 - 37n + 15$ $1n \cdot 18n$ $-1(-15)$ $2n \cdot 9n$ $-3(-5)$ $3n \cdot 6n$

Consider all the combinations of factors.

$18n^2 - 37n + 15$	
Possible factors	**Product**
$(n-1)(18n-15)$	Not an option
$(n-15)(18n-1)$	$18n^2 - 271n + 15$
$(n-3)(18n-5)$	$18n^2 - 59n + 15$
$(n-5)(18n-3)$	Not an option
$(2n-1)(9n-15)$	Not an option
$(2n-15)(9n-1)$	$18n^2 - 137n + 15$
$(2n-3)(9n-5)$	$18n^2 - 37n + 15$*
$(2n-5)(9n-3)$	Not an option
$(3n-1)(6n-15)$	Not an option
$(3n-15)(6n-1)$	Not an option
$(3n-3)(6n-5)$	Not an option
$(3n-5)(6n-3)$	Not an option

If the trinomial has no common factors, then neither factor can contain a common factor. That means this combination is not an option.

The correct factors are those whose product is the original trinomial. $(2n-3)(9n-5)$

Check by multiplying.

$$(2n-3)(9n-5)$$
$$18n^2 - 10n - 27n + 15$$
$$18n^2 - 37n + 15 \checkmark$$

Note:
Exercise:

Problem: Factor completely: $18x^2 - 3x - 10$.

Solution:

$(3x+2)(6x-5)$

Note:
Exercise:

Problem: Factor completely: $30y^2 - 53y - 21$.

Solution:

$(3y+1)(10y-21)$

Don’t forget to look for a GCF first.

Example:
Exercise:

Problem: Factor completely: $10y^4+55y^3+60y^2$.

Solution:
Solution

	$10y^4+55y^3+60y^2$
Notice the greatest common factor, and factor it first.	$15y^2\left(2y^2+11y+12\right)$
Factor the trinomial.	$5y^2(2y^2+11y+12)$ $y \cdot 2y$ $1 \cdot 12$, $2 \cdot 6$, $3 \cdot 4$

Consider all the combinations.

$2y^2+11y+12$	
Possible factors	**Product**
$(y+1)(2y+12)$	Not an option
$(y+12)(2y+1)$	$2y^2+25y+12$
$(y+2)(2y+6)$	Not an option
$(y+6)(2y+2)$	Not an option
$(y+3)(2y+4)$	Not an option
$(y+4)(2y+3)$	$2y^2+11y+12$*

If the trinomial has no common factors, then neither factor can contain a common factor. That means this combination is not an option.

The correct factors are those whose product is the original trinomial. Remember to include the factor $5y^2$.	$5y^2\,(y+4)\,(2y+3)$

Check by multiplying.

$$5y^2\,(y+4)\,(2y+3)$$
$$5y^2\,\left(2y^2+8y+3y+12\right)$$
$$10y^4+55y^3+60y^2\ \checkmark$$

Note:
Exercise:

Problem: Factor completely: $15n^3-85n^2+100n$.

Solution:

$5n\left(n-4\right)\left(3n-5\right)$

Note:
Exercise:

Problem: Factor completely: $56q^3+320q^2-96q$.

Solution:

$8q\left(q+6\right)\left(7q-2\right)$

Factor Trinomials using the "ac" Method

Another way to factor trinomials of the form ax^2+bx+c is the "ac" method. (The "ac" method is sometimes called the grouping method.) The "ac" method is actually an extension of the methods you used in the last section to factor trinomials with leading coefficient one. This method is very structured (that is step-by-step), and it always works!

Example:
How to Factor Trinomials Using the "ac" Method
Exercise:

Problem: Factor: $6x^2+7x+2$.

Solution:
Solution

Step 1. Factor any GCF.	Is there a greatest common factor? No.	$6x^2+7x+2$
Step 2. Find the product ac.	$a \cdot c$ $6 \cdot 2$ 12	$ax^2 + bx + c$ $6x^2+7x+2$
Step 3. Find two numbers m and n that: Multiply to ac $\quad m \cdot n = a \cdot c$ Add to b $\quad m+n=b$	Find two numbers that multiply to 12 and add to 7. Both factors must be positive. $3 \cdot 4 = 12 \quad 3+4=7$	

Step 4. Split the middle term using m, and n $\quad ax^2 + bx + c$ $\quad ax^2 + \underbrace{mx + nx}_{bx} + c$	Rewrite $7x$ as $3x + 4x$. Notice that $6x^2 + 3x + 4x + 2$ is equal to $6x^2 + 7x + 2$. We just split the middle term to get a more useful form.	$6x^2 + 7x + 2$ $\underbrace{6x^2 + 3x}\underbrace{+ 4x + 2}$
Step 5. Factor by grouping.		$3x(2x + 1) + 2(2x + 1)$ $(2x + 1)(3x + 2)$
Step 6. Check by multiplying.		$(2x + 1)(3x + 2)$ $6x^2 + 4x + 3x + 2$ $6x^2 + 7x + 2$ ✓

Note:
Exercise:

Problem: Factor: $6x^2 + 13x + 2$.

Solution:

$(x + 2)(6x + 1)$

Note:
Exercise:

Problem: Factor: $4y^2 + 8y + 3$.

Solution:

$(2y + 1)(2y + 3)$

Note:
Factor trinomials of the form using the “ac” method.

Factor any GCF.
Find the product ac.
Find two numbers *m* and *n* that: Multiply to ac $\quad m \cdot n = a \cdot c$
Add to b $\quad m + n = b$
Split the middle term using *m* and *n*:

$$ax^2 + bx + c$$
$$ax^2 + \overbrace{mx + nx}^{bx} + c$$

Factor by grouping.
Check by multiplying the factors.

When the third term of the trinomial is negative, the factors of the third term will have opposite signs.

Example:
Exercise:

Problem: Factor: $8u^2 - 17u - 21$.

Solution:
Solution

Is there a greatest common factor? No.		$\begin{array}{ccc} ax^2 & + bx & + c \\ 8u^2 & - 17u & - 21 \end{array}$
Find $a \cdot c$.	$a \cdot c$	
	$8(-21)$	
	-168	

Find two numbers that multiply to -168 and add to -17. The larger factor must be negative.

Factors of -168	**Sum of factors**
$1, -168$	$1 + (-168) = -167$
$2, -84$	$2 + (-84) = -82$
$3, -56$	$3 + (-56) = -53$
$4, -42$	$4 + (-42) = -38$
$6, -28$	$6 + (-28) = -22$

Factors of -168	Sum of factors
$7, -24$	$7 + (-24) = -17$*
$8, -21$	$8 + (-21) = -13$

Split the middle term using $7u$ and $-24u$.

$$8u^2 - 17u - 21$$
$$8u^2 + 7u - 24u - 21$$

Factor by grouping.

$$u\,(8u + 7) - 3\,(8u + 7)$$
$$(8u + 7)\,(u - 3)$$

Check by multiplying.

$$(8u + 7)\,(u - 3)$$
$$8u^2 - 24u + 7u - 21$$
$$8u^2 - 17u - 21\ \checkmark$$

Note:
Exercise:

Problem: Factor: $20h^2 + 13h - 15$.

Solution:

$(4n - 5)\,(5n + 3)$

Note:
Exercise:

Problem: Factor: $6g^2 + 19g - 20$.

Solution:

$(q + 4)\,(6q - 5)$

Example:
Exercise:

Problem: Factor: $2x^2 + 6x + 5$.

Solution:
Solution

Is there a greatest common factor? No.	$\begin{array}{ccccc} ax^2 & + & bx & + & c \\ 2x^2 & + & 6x & + & 5 \end{array}$
Find $a \cdot c$.	$a \cdot c$
	$2(5)$
	10

Find two numbers that multiply to 10 and add to 6.

Factors of 10	**Sum of factors**
$1, 10$	$1 + 10 = 11$
$2, 5$	$2 + 5 = 7$

There are no factors that multiply to 10 and add to 6. The polynomial is prime.

Note:
Exercise:

Problem: Factor: $10t^2 + 19t - 15$.

Solution:

$\left(2t + 5\right)\left(5t - 3\right)$

Note:
Exercise:

Problem: Factor: $3u^2 + 8u + 5$.

Solution:

$\left(u + 1\right)\left(3u + 5\right)$

Don’t forget to look for a common factor!

Example:
Exercise:

Problem: Factor: $10y^2 - 55y + 70$.

Solution:
Solution

Is there a greatest common factor? Yes. The GCF is 5.	$10y^2 - 55y + 70$
Factor it. Be careful to keep the factor of 5 all the way through the solution!	$5(2y^2 - 11y + 14)$
The trinomial inside the parentheses has a leading coefficient that is not 1.	$\begin{array}{l} ax^2 + bx + c \\ 5(2y^2 - 11y + 14) \end{array}$
Factor the trinomial.	$5(y - 2)(2y - 7)$
Check by mulitplying all three factors.	
$5\left(2y^2 - 2y - 4y + 14\right)$	
$5\left(2y^2 - 11y + 14\right)$	
$10y^2 - 55y + 70\checkmark$	

Note:
Exercise:

Problem: Factor: $16x^2 - 32x + 12$.

Solution:

$4\left(2x - 3\right)\left(2x - 1\right)$

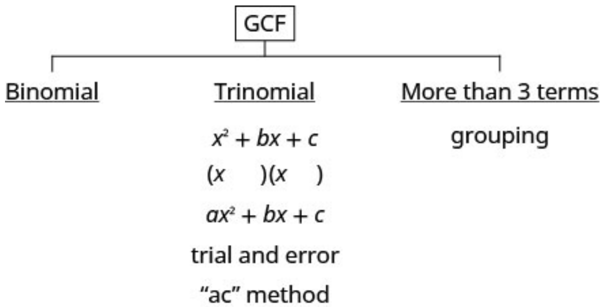
Note:
Exercise:

Problem: Factor: $18w^2 - 39w + 18$.

Solution:

$3\left(3w - 2\right)\left(2w - 3\right)$

We can now update the Preliminary Factoring Strategy, as shown in [link] and detailed in Choose a strategy to factor polynomials completely (updated), to include trinomials of the form $ax^2 + bx + c$. Remember, some polynomials are prime and so they cannot be factored.



Note:
Choose a strategy to factor polynomials completely (updated).

Is there a greatest common factor?

- Factor it.

Is the polynomial a binomial, trinomial, or are there more than three terms?

- If it is a binomial, right now we have no method to factor it.
- If it is a trinomial of the form $x^2 + bx + c$
  Undo FOIL $\left(x \quad\right)\left(x \quad\right)$.
- If it is a trinomial of the form $ax^2 + bx + c$
  Use Trial and Error or the “ac” method.
- If it has more than three terms
  Use the grouping method.

Check by multiplying the factors.

Note:
Access these online resources for additional instruction and practice with factoring trinomials of the form $ax^2 + bx + c$.

- Factoring Trinomials, a is not 1

Key Concepts

- **Factor Trinomials of the Form $ax^2 + bx + c$ using Trial and Error:** See [link].

  Write the trinomial in descending order of degrees.
  Find all the factor pairs of the first term.
  Find all the factor pairs of the third term.
  Test all the possible combinations of the factors until the correct product is found.
  Check by multiplying.

- **Factor Trinomials of the Form $ax^2 + bx + c$ Using the "ac" Method:** See [link].

  Factor any GCF.
  Find the product ac.
  Find two numbers *m* and *n* that: $\begin{array}{ll} \text{Multiply to } ac & m \cdot n = a \cdot c \\ \text{Add to } b & m + n = b \end{array}$
  Split the middle term using *m* and *n*:

  $$ax^2 + bx + c$$
  $$ax^2 + \overbrace{mx + nx}^{bx} + c$$

  Factor by grouping.
  Check by multiplying the factors.

- **Choose a strategy to factor polynomials completely (updated):**

  Is there a greatest common factor? Factor it.
  Is the polynomial a binomial, trinomial, or are there more than three terms?
  If it is a binomial, right now we have no method to factor it.
  If it is a trinomial of the form $x^2 + bx + c$ Undo FOIL $(x \quad)(x \quad)$.
  If it is a trinomial of the form $ax^2 + bx + c$ Use Trial and Error or the "ac" method.
  If it has more than three terms Use the grouping method.
  Check by multiplying the factors.

Practice Makes Perfect

Recognize a Preliminary Strategy to Factor Polynomials Completely

In the following exercises, identify the best method to use to factor each polynomial.

Exercise:

Problem:

ⓐ $10q^2 + 50$
ⓑ $a^2 - 5a - 14$
ⓒ $uv + 2u + 3v + 6$

Solution:

ⓐ factor the GCF, binomial ⓑ Undo FOIL ⓒ factor by grouping

Exercise:

Problem:

ⓐ $n^2 + 10n + 24$
ⓑ $8u^2 + 16$
ⓒ $pq + 5p + 2q + 10$

Exercise:

Problem:

ⓐ $x^2 + 4x - 21$
ⓑ $ab + 10b + 4a + 40$
ⓒ $6c^2 + 24$

Solution:

ⓐ undo FOIL ⓑ factor by grouping ⓒ factor the GCF, binomial

Exercise:

Problem:

ⓐ $20x^2 + 100$
ⓑ $uv + 6u + 4v + 24$
ⓒ $y^2 - 8y + 15$

Factor Trinomials of the form $ax^2 + bx + c$ with a GCF

In the following exercises, factor completely.

Exercise:

Problem: $5x^2 + 35x + 30$

Solution:

$5(x + 1)(x + 6)$

Exercise:

Problem: $12s^2 + 24s + 12$

Exercise:

Problem: $2z^2 - 2z - 24$

Solution:

$2(z - 4)(z + 3)$

Exercise:

Problem: $3u^2 - 12u - 36$

Exercise:

Problem: $7v^2 - 63v + 56$

Solution:

$7(v-1)(v-8)$

Exercise:

Problem: $5w^2 - 30w + 45$

Exercise:

Problem: $p^3 - 8p^2 - 20p$

Solution:

$p(p-10)(p+2)$

Exercise:

Problem: $q^3 - 5q^2 - 24q$

Exercise:

Problem: $3m^3 - 21m^2 + 30m$

Solution:

$3m(m-5)(m-2)$

Exercise:

Problem: $11n^3 - 55n^2 + 44n$

Exercise:

Problem: $5x^4 + 10x^3 - 75x^2$

Solution:

$5x^2(x-3)(x+5)$

Exercise:

Problem: $6y^4 + 12y^3 - 48y^2$

Factor Trinomials Using Trial and Error

In the following exercises, factor.

Exercise:

Problem: $2t^2 + 7t + 5$

Solution:

$(2t+5)(t+1)$

Exercise:

Problem: $5y^2+16y+11$

Exercise:

Problem: $11x^2+34x+3$

Solution:

$(11x+1)(x+3)$

Exercise:

Problem: $7b^2+50b+7$

Exercise:

Problem: $4w^2-5w+1$

Solution:

$(4w-1)(w-1)$

Exercise:

Problem: $5x^2-17x+6$

Exercise:

Problem: $6p^2-19p+10$

Solution:

$(3p-2)(2p-5)$

Exercise:

Problem: $21m^2-29m+10$

Exercise:

Problem: $4q^2-7q-2$

Solution:

$(4q+1)(q-2)$

Exercise:

Problem: $10y^2-53y-11$

Exercise:

Problem: $4p^2+17p-15$

Solution:

$(4p-3)(p+5)$

Exercise:

Problem: $6u^2+5u-14$

Exercise:

Problem: $16x^2-32x+16$

Solution:

$16(x-1)(x-1)$

Exercise:

Problem: $81a^2+153a-18$

Exercise:

Problem: $30q^3+140q^2+80q$

Solution:

$10q(3q+2)(q+4)$

Exercise:

Problem: $5y^3+30y^2-35y$

Factor Trinomials using the 'ac' Method

In the following exercises, factor.

Exercise:

Problem: $5n^2+21n+4$

Solution:

$(5n+1)(n+4)$

Exercise:

Problem: $8w^2+25w+3$

Exercise:

Problem: $9z^2+15z+4$

Solution:

$(3z+1)(3z+4)$

Exercise:

Problem: $3m^2 + 26m + 48$

Exercise:

Problem: $4k^2 - 16k + 15$

Solution:

$(2k - 3)(2k - 5)$

Exercise:

Problem: $4q^2 - 9q + 5$

Exercise:

Problem: $5s^2 - 9s + 4$

Solution:

$(5s - 4)(s - 1)$

Exercise:

Problem: $4r^2 - 20r + 25$

Exercise:

Problem: $6y^2 + y - 15$

Solution:

$(3y + 5)(2y - 3)$

Exercise:

Problem: $6p^2 + p - 22$

Exercise:

Problem: $2n^2 - 27n - 45$

Solution:

$(2n + 3)(n - 15)$

Exercise:

Problem: $12z^2 - 41z - 11$

Exercise:

Problem: $3x^2 + 5x + 4$

Solution:

prime

Exercise:

Problem: $4y^2 + 15y + 6$

Exercise:

Problem: $60y^2 + 290y - 50$

Solution:

$10(6y - 1)(y + 5)$

Exercise:

Problem: $6u^2 - 46u - 16$

Exercise:

Problem: $48z^3 - 102z^2 - 45z$

Solution:

$3z(8z + 3)(2z - 5)$

Exercise:

Problem: $90n^3 + 42n^2 - 216n$

Exercise:

Problem: $16s^2 + 40s + 24$

Solution:

$8\left(2s + 3\right)\left(s + 1\right)$

Exercise:

Problem: $24p^2 + 160p + 96$

Exercise:

Problem: $48y^2 + 12y - 36$

Solution:

$12\left(4y - 3\right)\left(y + 1\right)$

Exercise:

Problem: $30x^2 + 105x - 60$

Mixed Practice

In the following exercises, factor.

Exercise:

Problem: $12y^2 - 29y + 14$

Solution:

$(4y - 7)(3y - 2)$

Exercise:

Problem: $12x^2 + 36y - 24z$

Exercise:

Problem: $a^2 - a - 20$

Solution:

$(a - 5)(a + 4)$

Exercise:

Problem: $m^2 - m - 12$

Exercise:

Problem: $6n^2 + 5n - 4$

Solution:

$(2n - 1)(3n + 4)$

Exercise:

Problem: $12y^2 - 37y + 21$

Exercise:

Problem: $2p^2 + 4p + 3$

Solution:

prime

Exercise:

Problem: $3q^2 + 6q + 2$

Exercise:

Problem: $13z^2 + 39z - 26$

Solution:

$13\left(z^2 + 3z - 2\right)$

Exercise:

Problem: $5r^2 + 25r + 30$

Exercise:

Problem: $x^2 + 3x - 28$

Solution:

$(x + 7)\,(x - 4)$

Exercise:

Problem: $6u^2 + 7u - 5$

Exercise:

Problem: $3p^2 + 21p$

Solution:

$3p\,(p + 7)$

Exercise:

Problem: $7x^2 - 21x$

Exercise:

Problem: $6r^2 + 30r + 36$

Solution:

$6\,(r + 2)\,(r + 3)$

Exercise:

Problem: $18m^2 + 15m + 3$

Exercise:

Problem: $24n^2 + 20n + 4$

Solution:

$4\,(2n + 1)\,(3n + 1)$

Exercise:

Problem: $4a^2 + 5a + 2$

Exercise:

Problem: $x^2 + 2x - 24$

Solution:

$(x+6)(x-4)$

Exercise:

Problem: $2b^2-7b+4$

Everyday Math

Exercise:

Problem:

Height of a toy rocket The height of a toy rocket launched with an initial speed of 80 feet per second from the balcony of an apartment building is related to the number of seconds, t, since it is launched by the trinomial $-16t^2+80t+96$. Completely factor the trinomial.

Solution:

$-16(t-6)(t+1)$

Exercise:

Problem:

Height of a beach ball The height of a beach ball tossed up with an initial speed of 12 feet per second from a height of 4 feet is related to the number of seconds, t, since it is tossed by the trinomial $-16t^2+12t+4$. Completely factor the trinomial.

Writing Exercises

Exercise:

Problem:

List, in order, all the steps you take when using the “ac” method to factor a trinomial of the form ax^2+bx+c.

Solution:

Answers may vary.

Exercise:

Problem: How is the “ac” method similar to the “undo FOIL” method? How is it different?

Exercise:

Problem:

What are the questions, in order, that you ask yourself as you start to factor a polynomial? What do you need to do as a result of the answer to each question?

Solution:

Answers may vary.

Exercise:

Problem:

On your paper draw the chart that summarizes the factoring strategy. Try to do it without looking at the book. When you are done, look back at the book to finish it or verify it.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
recognize a preliminary strategy to factor polynomials completely.			
factor trinomials of the form $ax^2 + bx + c$ with a GCF.			
factor trinomials using trial and error.			
factor trinomials using the "ac" method.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Glossary

prime polynomials
: Polynomials that cannot be factored are prime polynomials.

Factor Special Products: ASE
By the end of this section, you will be able to:

- Factor perfect square trinomials
- Factor differences of squares
- Factor sums and differences of cubes
- Choose method to factor a polynomial completely

The strategy for factoring we developed in the last section will guide you as you factor most binomials, trinomials, and polynomials with more than three terms. We have seen that some binomials and trinomials result from special products—squaring binomials and multiplying conjugates. If you learn to recognize these kinds of polynomials, you can use the special products patterns to factor them much more quickly.

Factor Perfect Square Trinomials

Some trinomials are perfect squares. They result from multiplying a binomial times itself. You can square a binomial by using FOIL, but using the Binomial Squares pattern you saw in a previous chapter saves you a step. Let’s review the Binomial Squares pattern by squaring a binomial using FOIL.

$$(3x+4)^2$$
$$(3x+4)(3x+4)$$
$$\underset{F}{9x^2} + \underset{O}{12x} + \underset{I}{12x} + \underset{L}{16}$$
$$9x^2+24x+16$$

The first term is the square of the first term of the binomial and the last term is the square of the last. The middle term is twice the product of the two terms of the binomial.
Equation:

$$\begin{gathered}(3x)^2+2\,(3x\cdot 4)+4^2\\ 9x^2+24x+16\end{gathered}$$

The trinomial $9x^2$ + 24 +16 is called a perfect square trinomial. It is the square of the binomial $3x+4$.

We’ll repeat the Binomial Squares Pattern here to use as a reference in factoring.

Note:
Binomial Squares Pattern
If a and b are real numbers,
Equation:

$$(a+b)^2=a^2+2ab+b^2 \qquad (a-b)^2=a^2-2ab+b^2$$

When you square a binomial, the product is a perfect square trinomial. In this chapter, you are learning to factor—now, you will start with a perfect square trinomial and factor it into its prime factors.

You could factor this trinomial using the methods described in the last section, since it is of the form $ax^2 + bx + c$. But if you recognize that the first and last terms are squares and the trinomial fits the **perfect square trinomials pattern**, you will save yourself a lot of work.

Here is the pattern—the reverse of the binomial squares pattern.

Note:
Perfect Square Trinomials Pattern
If a and b are real numbers,
Equation:

$$a^2 + 2ab + b^2 = (a+b)^2 \qquad a^2 - 2ab + b^2 = (a-b)^2$$

To make use of this pattern, you have to recognize that a given trinomial fits it. Check first to see if the leading coefficient is a perfect square, a^2. Next check that the last term is a perfect square, b^2. Then check the middle term—is it twice the product, $2ab$? If everything checks, you can easily write the factors.

Example:
How to Factor Perfect Square Trinomials
Exercise:

Problem: Factor: $9x^2 + 12x + 4$.

Solution:
Solution

Step 1. Does the trinomial fit the perfect square trinomials pattern, $a^2 + 2ab + b^2$?		
• Is the first term a perfect square? Write it as a square, a^2.	Is $9x^2$ a perfect square? Yes—write it as $(3x)^2$.	$9x^2 + 12x + 4$ $(3x)^2$
• Is the last term a perfect square? Write it as a square, b^2.	Is 4 a perfect square? Yes—write it as $(2)^2$.	$(3x)^2 \qquad (2)^2$
• Check the middle term. Is it $2ab$?	Is $12x$ twice the product of $3x$ and 2?	$(3x)^2 \qquad (2)^2$
	Does it match? Yes, so we have a perfect square trinomial!	$2(3x)(2)$ $12x$

Step 2. Write the square of the binomial.	Write it as the square of a binomial.	$9x^2 + 12x + 4$ $a^2 + 2 \cdot a \cdot b + b^2$ $(3x)^2 + 2 \cdot 3x \cdot 2 + 2^2$ $(a + b)^2$ $(3x + 2)^2$
Step 3. Check. $(3x + 2)^2$ $(3x)^2 + 2 \cdot 3x \cdot 2 + 2^2$ $9x^2 + 12x + 4$ ✓		

Note:
Exercise:

Problem: Factor: $4x^2 + 12x + 9$.

Solution:

$(2x + 3)^2$

Note:
Exercise:

Problem: Factor: $9y^2 + 24y + 16$.

Solution:

$(3y + 4)^2$

The sign of the middle term determines which pattern we will use. When the middle term is negative, we use the pattern $a^2 - 2ab + b^2$, which factors to $(a - b)^2$.

The steps are summarized here.

Note:
Factor perfect square trinomials.

Step 1. Does the trinomial fit the pattern?	$a^2 + 2ab + b^2$	$a^2 - 2ab + b^2$
• Is the first term a perfect square? Write it as a square.	$(a)^2$	$(a)^2$
• Is the last term a perfect square? Write it as a square.	$(a)^2 \qquad (b)^2$	$(a)^2 \qquad (b)^2$
• Check the middle term. Is it $2ab$?	$(a)^2 \searrow_{2\cdot a\cdot b} \swarrow (b)^2$	$(a)^2 \searrow_{2\cdot a\cdot b} \swarrow (b)^2$
Step 2. Write the square of the binomial.	$(a+b)^2$	$(a-b)^2$
Step 3. Check by multiplying.		

We'll work one now where the middle term is negative.

Example:
Exercise:

Problem: Factor: $81y^2 - 72y + 16$.

Solution:
Solution

The first and last terms are squares. See if the middle term fits the pattern of a perfect square trinomial. The middle term is negative, so the binomial square would be $(a-b)^2$.

	$81y^2 - 72y + 16$
Are the first and last terms perfect squares?	$(9y)^2 \qquad (4)^2$
Check the middle term.	$(9y)^2 \searrow 2(9y)(4) \swarrow (4)^2$ $72y$
Does is match $(a-b)^2$? Yes.	$\overset{a^2}{(9y)^2} \overset{-2}{-2} \cdot \overset{a}{9y} \cdot \overset{b}{4} \overset{+}{+} \overset{b^2}{4^2}$
Write the square of a binomial.	

	$(9y-4)^2$
Check by mulitplying.	
$(9y-4)^2$	
$(9y)^2 - 2\cdot 9y \cdot 4 + 4^2$	
$81y^2 - 72y + 16\checkmark$	

Note:
Exercise:

Problem: Factor: $64y^2 - 80y + 25$.

Solution:

$(8y-5)^2$

Note:
Exercise:

Problem: Factor: $16z^2 - 72z + 81$.

Solution:

$(4z-9)^2$

The next example will be a perfect square trinomial with two variables.

Example:
Exercise:

Problem: Factor: $36x^2 + 84xy + 49y^2$.

Solution:
Solution

	$36x^2 + 84xy + 49y^2$
Test each term to verify the pattern.	$\underset{(6x)^2}{a^2} + \underset{2 \cdot 6x \cdot 7y}{2\ a\ b} + \underset{(7y)^2}{b^2}$
Factor.	$(6x + 7y)^2$
Check by mulitplying.	
$(6x + 7y)^2$	
$(6x)^2 + 2 \cdot 6x \cdot 7y + (7y)^2$	
$36x^2 + 84xy + 49y^2$✓	

Note:
Exercise:

Problem: Factor: $49x^2 + 84xy + 36y^2$.

Solution:

$(7x + 6y)^2$

Note:
Exercise:

Problem: Factor: $64m^2 + 112mn + 49n^2$.

Solution:

$(8m + 7n)^2$

Example:
Exercise:

Problem: Factor: $9x^2 + 50x + 25$.

Solution:
Solution

	$9x^2 + 50x + 25$
Are the first and last terms perfect squares?	$(3x)^2 \qquad (5)^2$
Check the middle term—is it $2ab$?	$(3x)^2 \searrow \underbrace{2(3x)(5)}_{30x} \swarrow (5)^2$
No! $30x \neq 50x$	This does not fit the pattern!
Factor using the “ac” method.	$9x^2 + 50x + 25$
Notice: $\begin{array}{c} ac \\ 9 \cdot 25 \\ 225 \end{array}$ and $\begin{array}{l} 5 \cdot 45 = 225 \\ 5 + 45 = 50 \end{array}$	
Split the middle term.	$9x^2 + 5x + 45x + 25$
Factor by grouping.	$x\,(9x + 5) + 5\,(9x + 5)$
	$(9x + 5)\,(x + 5)$

Check.

$$\begin{array}{l} (9x + 5)\,(x + 5) \\ 9x^2 + 45x + 5x + 25 \\ 9x^2 + 50x + 25\ \checkmark \end{array}$$

Note:
Exercise:

Problem: Factor: $16r^2 + 30rs + 9s^2$.

Solution:

$(8r + 3s)\,(2r + 3s)$

Note:
Exercise:

Problem: Factor: $9u^2 + 87u + 100$.

Solution:

$(3u + 4)\,(3u + 25)$

Remember the very first step in our Strategy for Factoring Polynomials? It was to ask “is there a greatest common factor?” and, if there was, you factor the GCF before going any further. Perfect square trinomials may have a GCF in all three terms and it should be factored out first. And, sometimes, once the GCF has been factored, you will recognize a perfect square trinomial.

Example:
Exercise:

Problem: Factor: $36x^2y - 48xy + 16y$.

Solution:
Solution

	$36x^2y - 48xy + 16y$
Is there a GCF? Yes, $4y$, so factor it out.	$4y\left(9x^2 - 12x + 4\right)$
Is this a perfect square trinomial?	
Verify the pattern.	$4y[\underbrace{(3x)^2}_{a^2} \underbrace{- 2}_{-2} \cdot \underbrace{3x}_{a} \cdot \underbrace{2}_{b} \underbrace{+}_{+} \underbrace{2^2}_{b^2}]$
Factor.	$4y(3x - 2)^2$
Remember: Keep the factor $4y$ in the final product.	
Check.	
$4y(3x - 2)^2$	
$4y\left[(3x)^2 - 2 \cdot 3x \cdot 2 + 2^2\right]$	
$4y(9x)^2 - 12x + 4$	
$36x^2y - 48xy + 16y$✓	

Note:
Exercise:

Problem: Factor: $8x^2y - 24xy + 18y$.

Solution:

$2y(2x-3)^2$

Note:
Exercise:

Problem: Factor: $27p^2q + 90pq + 75q$.

Solution:

$3q(3p+5)^2$

Factor Differences of Squares

The other special product you saw in the previous was the Product of Conjugates pattern. You used this to multiply two binomials that were conjugates. Here's an example:
Equation:

$$(3x-4)(3x+4)$$
$$9x^2-16$$

Remember, when you multiply conjugate binomials, the middle terms of the product add to 0. All you have left is a binomial, the difference of squares.

Multiplying conjugates is the only way to get a binomial from the product of two binomials.

Note:
Product of Conjugates Pattern
If *a* and *b* are real numbers
Equation:

$$(a-b)(a+b)=a^2-b^2$$

The product is called a difference of squares.

To factor, we will use the product pattern "in reverse" to factor the difference of squares. A **difference of squares** factors to a product of conjugates.

Note:
Difference of Squares Pattern
If a and b are real numbers,

$$a^2 - b^2 = (a-b)(a+b) \qquad \underbrace{a^2 \overset{\textit{difference}}{-} b^2}_{\textit{squares}} = \underbrace{(a-b)(a+b)}_{\textit{conjugates}}$$

Remember, “difference” refers to subtraction. So, to use this pattern you must make sure you have a binomial in which two squares are being subtracted.

Example:
How to Factor Differences of Squares
Exercise:

Problem: Factor: $x^2 - 4$.

Solution:
Solution

Step 1. Does the binomial fit the pattern?		$x^2 - 4$
• Is this a difference?	Yes	$x^2 - 4$
• Are the first and last terms perfect squares?	Yes	
Step 2. Write them as squares.	Write them as x^2 and 2^2.	$a^2 - b^2$ $(x)^2 - 2^2$
Step 3. Write the product of conjugates.		$(a-b)(a+b)$ $(x-2)(x+2)$
Step 4. Check.		$(x-2)(x+2)$ $x^2 - 4$ ✓

Note:
Exercise:

Problem: Factor: $h^2 - 81$.

Solution:

$(h-9)(h+9)$

Note:
Exercise:

Problem: Factor: $k^2 - 121$.

Solution:

$(k-11)(k+11)$

Note:
Factor differences of squares.

Step 1. Does the binomial fit the pattern?	$a^2 - b^2$
• Is this a difference?	$\_\_\_\_ - \_\_\_\_$
• Are the first and last terms perfect squares?	
Step 2. Write them as squares.	$(a)^2 - (b)^2$
Step 3. Write the product of conjugates.	$(a-b)(a+b)$
Step 4. Check by multiplying.	

It is important to remember that *sums of squares do not factor into a product of binomials*. There are no binomial factors that multiply together to get a sum of squares. After removing any GCF, the expression $a^2 + b^2$ is prime!

Don't forget that 1 is a perfect square. We'll need to use that fact in the next example.

Example:
Exercise:

Problem: Factor: $64y^2 - 1$.

Solution:
Solution

	$64y^2 - 1$
Is this a difference? Yes.	$64y^2 - 1$
Are the first and last terms perfect squares?	
Yes - write them as squares.	$\underset{(8y)^2}{a^2} \underset{-}{-} \underset{1^2}{b^2}$
Factor as the product of conjugates.	$\underset{(8y-1)}{(a-b)}\underset{(8y+1)}{(a+b)}$
Check by multiplying.	
$(8y-1)(8y+1)$	
$64y^2 - 1$✓	

Note:
Exercise:

Problem: Factor: $m^2 - 1$.

Solution:

$(m-1)(m+1)$

Note:
Exercise:

Problem: Factor: $81y^2 - 1$.

Solution:

$(9y-1)(9y+1)$

Example:
Exercise:

Problem: Factor: $121x^2 - 49y^2$.

Solution:
Solution

$$121x^2 - 49y^2$$

Is this a difference of squares? Yes. $(11x)^2 - (7y)^2$

Factor as the product of conjugates. $(11x - 7y)(11x + 7y)$

Check by multiplying.

$$(11x - 7y)(11x + 7y)$$
$$121x^2 - 49y^2 \checkmark$$

Note:
Exercise:

Problem: Factor: $196m^2 - 25n^2$.

Solution:

$(16m - 5n)(16m + 5n)$

Note:
Exercise:

Problem: Factor: $144p^2 - 9q^2$.

Solution:

$(12p - 3q)(12p + 3q)$

The binomial in the next example may look “backwards,” but it’s still the difference of squares.

Example:
Exercise:

Problem: Factor: $100 - h^2$.

Solution:
Solution

$$100 - h^2$$

Is this a difference of squares? Yes. $(10)^2 - (h)^2$

Factor as the product of conjugates. $(10 - h)(10 + h)$

Check by multiplying.

$$(10 - h)(10 + h)$$
$$100 - h^2 \checkmark$$

Be careful not to rewrite the original expression as $h^2 - 100$.

Factor $h^2 - 100$ on your own and then notice how the result differs from $(10 - h)(10 + h)$.

Note:
Exercise:

Problem: Factor: $144 - x^2$.

Solution:

$(12 - x)(12 + x)$

Note:
Exercise:

Problem: Factor: $169 - p^2$.

Solution:

$(13 - p)(13 + p)$

To completely factor the binomial in the next example, we'll factor a difference of squares twice!

Example:

Exercise:

Problem: Factor: $x^4 - y^4$.

Solution:
Solution

	$x^4 - y^4$
Is this a difference of squares? Yes.	$\left(x^2\right)^2 - \left(y^2\right)^2$
Factor it as the product of conjugates.	$\left(x^2 - y^2\right)\left(x^2 + y^2\right)$
Notice the first binomial is also a difference of squares!	$\left((x)^2 - (y)^2\right)\left(x^2 + y^2\right)$
Factor it as the product of conjugates. The last factor, the sum of squares, cannot be factored.	$(x - y)(x + y)\left(x^2 + y^2\right)$

Check by multiplying.

$$(x - y)(x + y)\left(x^2 + y^2\right)$$
$$\left[(x - y)(x + y)\right]\left(x^2 + y^2\right)$$
$$\left(x^2 - y^2\right)\left(x^2 + y^2\right)$$
$$x^4 - y^4\ \checkmark$$

Note:
Exercise:

Problem: Factor: $a^4 - b^4$.

Solution:

$\left(a^2 + b^2\right)(a + b)(a - b)$

Note:
Exercise:

Problem: Factor: $x^4 - 16$.

Solution:

$\left(x^2 + 4\right)(x + 2)(x - 2)$

As always, you should look for a common factor first whenever you have an expression to factor. Sometimes a common factor may “disguise” the difference of squares and you won’t recognize the perfect squares until you factor the GCF.

Example:
Exercise:

Problem: Factor: $8x^2y - 18y$.

Solution:
Solution

	$8x^2y - 98y$
Is there a GCF? Yes, $2y$—factor it out!	$2y\left(4x^2 - 49\right)$
Is the binomial a difference of squares? Yes.	$2y\left((2x)^2 - (7)^2\right)$
Factor as a product of conjugates.	$2y\left(2x - 7\right)\left(2x + 7\right)$

Check by multiplying.

$2y\left(2x - 7\right)\left(2x + 7\right)$
$2y\left[\left(2x - 7\right)\left(2x + 7\right)\right]$
$2y\left(4x^2 - 49\right)$
$8x^2y - 98y$ ✓

Note:
Exercise:

Problem: Factor: $7xy^2 - 175x$.

Solution:

$7x\left(y - 5\right)\left(y + 5\right)$

Note:
Exercise:

Problem: Factor: $45a^2b - 80b$.

Solution:

$5b\left(3a-4\right)\left(3a+4\right)$

Example:
Exercise:

Problem: Factor: $6x^2+96$.

Solution:
Solution

	$6x^2+96$
Is there a GCF? Yes, 6—factor it out!	$6\left(x^2+16\right)$
Is the binomial a difference of squares? No, it is a sum of squares. Sums of squares do not factor!	
Check by multiplying.	

$6\left(x^2+16\right)$
$6x^2+96\checkmark$

Note:
Exercise:

Problem: Factor: $8a^2+200$.

Solution:

$8\left(a^2+25\right)$

Note:
Exercise:

Problem: Factor: $36y^2+81$.

Solution:

$9\left(4y^2+9\right)$

Factor Sums and Differences of Cubes

There is another special pattern for factoring, one that we did not use when we multiplied polynomials. This is the pattern for the sum and difference of cubes. We will write these formulas first and then check them by multiplication.

Equation:

$$a^3 + b^3 = (a + b)\left(a^2 - ab + b^2\right)$$
$$a^3 - b^3 = (a - b)\left(a^2 + ab + b^2\right)$$

We'll check the first pattern and leave the second to you.

	$(a + b)(a^2 - ab + b^2)$
Distribute.	$a(a^2 - ab + b^2) + b(a^2 - ab + b^2)$
Multiply.	$a^3 - a^2b + ab^2 + a^2b - ab^2 + b^3$
Combine like terms.	$a^3 + b^3$

Note:
Sum and Difference of Cubes Pattern
Equation:

$$a^3 + b^3 = (a + b)\left(a^2 - ab + b^2\right)$$
$$a^3 - b^3 = (a - b)\left(a^2 + ab + b^2\right)$$

The two patterns look very similar, don't they? But notice the signs in the factors. The sign of the binomial factor matches the sign in the original binomial. And the sign of the middle term of the trinomial factor is the opposite of the sign in the original binomial. If you recognize the pattern of the signs, it may help you memorize the patterns.

$$a^3 \underbrace{\underbrace{+\, b^3 = (a\, +}_{\text{same sign}}\, b)\,(a^2\, -}_{\text{opposite signs}}\, ab + b^2)$$

$$a^3 \underbrace{\underbrace{-\, b^3 = (a\, -}_{\text{same sign}}\, b)\,(a^2\, +}_{\text{opposite signs}}\, ab + b^2)$$

The trinomial factor in the sum and difference of cubes pattern cannot be factored.

It can be very helpful if you learn to recognize the cubes of the integers from 1 to 10, just like you have learned to recognize squares. We have listed the cubes of the integers from 1 to 10 in [link].

$$a^2 - b^2 = (a-b)(a+b) \qquad \underbrace{a^2 \;\overset{\text{difference}}{-}\; b^2}_{\text{squares}} = \underbrace{(a-b)(a+b)}_{\text{conjugates}}$$

Example:
How to Factor the Sum or Difference of Cubes
Exercise:

Problem: Factor: $x^3 + 64$.

Solution:
Solution

Step 1. Does the binomial fit the sum or difference of cubes pattern?		$x^3 + 64$
• Is it a sum or difference?	This is a sum.	$x^3 + 64$
• Are the first and last terms perfect cubes?	Yes	
Step 2. Write the terms as cubes.	Write them as x^3 and 4^3	$\overset{a^3 \;+\; b^3}{x^3 + 4^3}$
Step 3. Use either the sum or difference of cubes pattern.	This is a sum of cubes.	$\left(\overset{a \;+\; b}{x + 4}\right)\left(\overset{a^2 \;-\; ab \;+\; b^2}{x^2 - 4x + 4^2}\right)$
Step 4. Simplify inside the parentheses.	Simplify 4^2.	$(x + 4)(x^2 - 4x + 16)$

Step 5. Check by multiplying the factors.		$$\begin{array}{r} x^2 - 4x + 16 \\ x + 4 \\ \hline 4x^2 - 16x + 64 \\ x^3 - 4x^2 + 16x \\ \hline x^3 + 64 \checkmark \end{array}$$

Note:
Exercise:

Problem: Factor: $x^3 + 27$.

Solution:

$(x + 3)\left(x^2 - 3x + 9\right)$

Note:
Exercise:

Problem: Factor: $y^3 + 8$.

Solution:

$(y + 2)\left(y^2 - 2y + 4\right)$

Note:
Factor the sum or difference of cubes.
To factor the sum or difference of cubes:

Does the binomial fit the sum or difference of cubes pattern?

- Is it a sum or difference?
- Are the first and last terms perfect cubes?

Write them as cubes.
Use either the sum or difference of cubes pattern.
Simplify inside the parentheses
Check by multiplying the factors.

Example:
Exercise:

Problem: Factor: $x^3 - 1000$.

Solution:
Solution

	$x^3 - 1000$
This binomial is a difference. The first and last terms are perfect cubes.	
Write the terms as cubes.	$\overset{a^3}{x^3} - \overset{b^3}{10^3}$
Use the difference of cubes pattern.	$(\overset{a}{x} - \overset{b}{10})(\overset{a^2}{x^2} + \overset{ab}{10 \cdot x} + \overset{b^2}{10^2})$
Simplify.	$(\overset{a}{x} - \overset{b}{10})(\overset{a^2}{x^2} + \overset{ab}{10x} + \overset{b^2}{100})$
Check by multiplying.	
$(x - 10)\ (x^2 + 10x + 100)$ $x^2 + 10x + 100$ $x - 100$ $x^3 + 10x^2 + 100x$ $-10x^2 - 100x - 1000$ ✓ $x^3 \qquad - 1000$	

Note:
Exercise:

Problem: Factor: $u^3 - 125$.

Solution:

$(u-5)\left(u^2+5u+25\right)$

Note:
Exercise:

Problem: Factor: v^3-343.

Solution:

$(v-7)\left(v^2+7v+49\right)$

Be careful to use the correct signs in the factors of the sum and difference of cubes.

Example:
Exercise:

Problem: Factor: $512-125p^3$.

Solution:
Solution

	$512-125p^3$
This binomial is a difference. The first and last terms are perfect cubes.	
Write the terms as cubes.	$\overset{a^3}{8^3} \overset{-}{-} \overset{b^3}{(5p)^3}$
Use the difference of cubes pattern.	$(\overset{a}{8} \overset{-}{-} \overset{b}{5p})(\overset{a^2}{8^2} \overset{+}{+} \overset{ab}{8 \cdot 5p} \overset{+}{+} \overset{b^2}{(5p)^2})$
Simplify.	

	$\left(\overset{a}{8} - \overset{b}{5p}\right)\left(\overset{a^2}{64} + \overset{ab}{40p} + \overset{b^2}{25p^2}\right)$
Check by multiplying.	We'll leave the check to you.

Note:
Exercise:

Problem: Factor: $64 - 27x^3$.

Solution:

$(4 - 3x)\left(16 - 12x + 9x^2\right)$

Note:
Exercise:

Problem: Factor: $27 - 8y^3$.

Solution:

$(3 - 2y)\left(9 - 6y + 4y^2\right)$

Example:
Exercise:

Problem: Factor: $27u^3 - 125v^3$.

Solution:
Solution

	$27u^3 - 125v^3$
This binomial is a difference. The first and last terms are	

perfect cubes.	
Write the terms as cubes.	$\overset{a^3}{(3u)^3} - \overset{b^3}{(5v)^3}$
Use the difference of cubes pattern.	$\left(\overset{a}{3u} - \overset{b}{5v}\right)\left(\overset{a^2}{(3u)^2} + \overset{ab}{3u \cdot 5v} + \overset{b^2}{(5v)^2}\right)$
Simplify.	$\left(\overset{a}{3u} - \overset{b}{5v}\right)\left(\overset{a^2}{9u^2} + \overset{ab}{15uv} + \overset{b^2}{25v^2}\right)$
Check by multiplying.	We'll leave the check to you.

Note:
Exercise:

Problem: Factor: $8x^3 - 27y^3$.

Solution:

$(2x - 3y)\left(4x^2 - 6xy + 9y^2\right)$

Note:
Exercise:

Problem: Factor: $1000m^3 - 125n^3$.

Solution:

$(10m - 5n)\left(100m^2 - 50mn + 25n^2\right)$

In the next example, we first factor out the GCF. Then we can recognize the sum of cubes.

Example:
Exercise:

Problem: Factor: $5m^3 + 40n^3$.

Solution:
Solution

	$5m^3 + 40n^3$
Factor the common factor.	$5(m^3 + 8n^3)$
This binomial is a sum. The first and last terms are perfect cubes.	
Write the terms as cubes.	$5\left(\overset{a^3}{m^3} \overset{+}{+} \overset{b^3}{(2n)^3}\right)$
Use the sum of cubes pattern.	$5\left(\overset{a}{m} \overset{+}{+} \overset{b}{2n}\right)\left(\overset{a^2}{m^2} \overset{-}{-} \overset{ab}{m \cdot 2n} \overset{+}{+} \overset{b^2}{(2n)^2}\right)$
Simplify.	$5\left(\overset{a}{m} \overset{+}{+} \overset{b}{2n}\right)\left(\overset{a^2}{m^2} \overset{-}{-} \overset{ab}{2m\,n} \overset{+}{+} \overset{b^2}{4n^2}\right)$

Check. To check, you may find it easier to multiply the sum of cubes factors first, then multiply that product by 5. We'll leave the multiplication for you.

$$5\left(m + 2n\right)\left(m^2 - 2mn + 4n^2\right)$$

Note:
Exercise:

Problem: Factor: $500p^3 + 4q^3$.

Solution:

$4(5p + q)\left(25p^2 - 5pq + q^2\right)$

Note:
Exercise:

Problem: Factor: $432c^3 + 686d^3$.

Solution:

$2\left(6c + 7d\right)\left(36c^2 - 42cd + 49d^2\right)$

Note:
Access these online resources for additional instruction and practice with factoring special products.

- Sum of Difference of Cubes
- Difference of Cubes Factoring

Key Concepts

- **Factor perfect square trinomials** See [link].

Step 1. Does the trinomial fit the pattern?	$a^2 + 2ab + b^2$	$a^2 - 2ab + b^2$
Is the first term a perfect square? Write it as a square.	$(a)^2$	$(a)^2$
Is the last term a perfect square? Write it as a square.	$(a)^2 \qquad (b)^2$	$(a)^2 \qquad (b)^2$
Check the middle term. Is it $2ab$?	$(a)^2 \searrow_{2\cdot a\cdot b} \swarrow (b)^2$	$(a)^2 \searrow_{2\cdot a\cdot b} \swarrow (b)^2$
Step 2. Write the square of the binomial.	$(a+b)^2$	$(a-b)^2$
Step 3. Check by multiplying.		

- **Factor differences of squares** See [link].

Step 1. Does the binomial fit the pattern?	$a^2 - b^2$
Is this a difference?	$\_\_\_\_ - \_\_\_\_$
Are the first and last terms perfect squares?	
Step 2. Write them as squares.	$(a)^2 - (b)^2$
Step 3. Write the product of conjugates.	$(a-b)(a+b)$
Step 4. Check by multiplying.	

- **Factor sum and difference of cubes** To factor the sum or difference of cubes: See [link].

  Does the binomial fit the sum or difference of cubes pattern? Is it a sum or difference? Are the first and last terms perfect cubes?
  Write them as cubes.
  Use either the sum or difference of cubes pattern.
  Simplify inside the parentheses
  Check by multiplying the factors.

Practice Makes Perfect

Factor Perfect Square Trinomials

In the following exercises, factor.

Exercise:

Problem: $16y^2 + 24y + 9$

Solution:

$(4y + 3)^2$

Exercise:

Problem: $25v^2 + 20v + 4$

Exercise:

Problem: $36s^2 + 84s + 49$

Solution:

$(6s + 7)^2$

Exercise:

Problem: $49s^2 + 154s + 121$

Exercise:

Problem: $100x^2 - 20x + 1$

Solution:

$(10x - 1)^2$

Exercise:

Problem: $64z^2 - 16z + 1$

Exercise:

Problem: $25n^2 - 120n + 144$

Solution:

$(5n - 12)^2$

Exercise:

Problem: $4p^2 - 52p + 169$

Exercise:

Problem: $49x^2 - 28xy + 4y^2$

Solution:

$(7x - 2y)^2$

Exercise:

Problem: $25r^2 - 60rs + 36s^2$

Exercise:

Problem: $25n^2 + 25n + 4$

Solution:

$(5n + 4)(5n + 1)$

Exercise:

Problem: $100y^2 - 52y + 1$

Exercise:

Problem: $64m^2 - 34m + 1$

Solution:

$(32m - 1)(2m - 1)$

Exercise:

Problem: $100x^2 - 25x + 1$

Exercise:

Problem: $10k^2 + 80k + 160$

Solution:

$10(k + 4)^2$

Exercise:

Problem: $64x^2 - 96x + 36$

Exercise:

Problem: $75u^3 - 30u^2v + 3uv^2$

Solution:

$3u(5u-v)^2$

Exercise:

Problem: $90p^3+300p^2q+250pq^2$

Factor Differences of Squares

In the following exercises, factor.

Exercise:

Problem: x^2-16

Solution:

$(x-4)\,(x+4)$

Exercise:

Problem: n^2-9

Exercise:

Problem: $25v^2-1$

Solution:

$(5v-1)\,(5v+1)$

Exercise:

Problem: $169q^2-1$

Exercise:

Problem: $121x^2-144y^2$

Solution:

$(11x-12y)\,(11x+12y)$

Exercise:

Problem: $49x^2-81y^2$

Exercise:

Problem: $169c^2-36d^2$

Solution:

$(13c-6d)\,(13c+6d)$

Exercise:

Problem: $36p^2 - 49q^2$

Exercise:

Problem: $4 - 49x^2$

Solution:

$(7x-2)(7x+2)$ $(2-7x)(2+7x)$

Exercise:

Problem: $121 - 25s^2$

Exercise:

Problem: $16z^4 - 1$

Solution:

$(2z-1)(2z+1)(4z^2+1)$

Exercise:

Problem: $m^4 - n^4$

Exercise:

Problem: $5q^2 - 45$

Solution:

$5(q-3)(q+3)$

Exercise:

Problem: $98r^3 - 72r$

Exercise:

Problem: $24p^2 + 54$

Solution:

$6\left(4p^2+9\right)$

Exercise:

Problem: $20b^2 + 140$

Factor Sums and Differences of Cubes

In the following exercises, factor.

Exercise:

Problem: $x^3 + 125$

Solution:

$(x + 5)(x^2 - 5x + 25)$

Exercise:

Problem: $n^3 + 512$

Exercise:

Problem: $z^3 - 27$

Solution:

$(z - 3)(z^2 + 3z + 9)$

Exercise:

Problem: $v^3 - 216$

Exercise:

Problem: $8 - 343t^3$

Solution:

$(2 - 7t)(4 + 14t + 49t^2)$

Exercise:

Problem: $125 - 27w^3$

Exercise:

Problem: $8y^3 - 125z^3$

Solution:

$(2y - 5z)(4y^2 + 10yz + 25z^2)$

Exercise:

Problem: $27x^3 - 64y^3$

Exercise:

Problem: $7k^3 + 56$

Solution:

$7(k+2)(k^2-2k+4)$

Exercise:

Problem: $6x^3-48y^3$

Exercise:

Problem: $2-16y^3$

Solution:

$2(1-2y)(1+2y+4y^2)$

Exercise:

Problem: $-2x^3-16y^3$

Mixed Practice

In the following exercises, factor.

Exercise:

Problem: $64a^2-25$

Solution:

$(8a-5)(8a+5)$

Exercise:

Problem: $121x^2-144$

Exercise:

Problem: $27q^2-3$

Solution:

$3(3q-1)(3q+1)$

Exercise:

Problem: $4p^2-100$

Exercise:

Problem: $16x^2-72x+81$

Solution:

$(4x-9)^2$

Exercise:

Problem: $36y^2+12y+1$

Exercise:

Problem: $8p^2+2$

Solution:

$2\left(4p^2+1\right)$

Exercise:

Problem: $81x^2+169$

Exercise:

Problem: $125-8y^3$

Solution:

$(5-2y)(25+10y+4y^2)$

Exercise:

Problem: $27u^3+1000$

Exercise:

Problem: $45n^2+60n+20$

Solution:

$5(3n+2)^2$

Exercise:

Problem: $48q^3-24q^2+3q$

Everyday Math

Exercise:

Problem:

Landscaping Sue and Alan are planning to put a 15 foot square swimming pool in their backyard. They will surround the pool with a tiled deck, the same width on all sides. If the width of the deck is w, the total area of the pool and deck is given by the trinomial $4w^2+60w+225$. Factor the trinomial.

Solution:

$(2w + 15)^2$

Exercise:

Problem:

Home repair The height a twelve foot ladder can reach up the side of a building if the ladder's base is b feet from the building is the square root of the binomial $144 - b^2$. Factor the binomial.

Writing Exercises

Exercise:

Problem:

Why was it important to practice using the binomial squares pattern in the chapter on multiplying polynomials?

Solution:

Answers may vary.

Exercise:

Problem: How do you recognize the binomial squares pattern?

Exercise:

Problem: Explain why $n^2 + 25 \neq (n + 5)^2$. Use algebra, words, or pictures.

Solution:

Answers may vary.

Exercise:

Problem: Maribel factored $y^2 - 30y + 81$ as $(y - 9)^2$. Was she right or wrong? How do you know?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
factor perfect square trinomials.			
factor differences of squares.			
factor sums and differences of cubes.			

ⓑ On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Glossary

perfect square trinomials pattern

If a and b are real numbers,

Equation:

$$a^2 + 2ab + b^2 = (a+b)^2 \qquad a^2 - 2ab + b^2 = (a-b)^2$$

difference of squares pattern

If a and b are real numbers,

$$a^2 - b^2 = (a-b)(a+b) \qquad \underbrace{a^2 \overset{\text{difference}}{-} b^2}_{\text{squares}} = \underbrace{(a-b)(a+b)}_{\text{conjugates}}$$

sum and difference of cubes pattern

Equation:

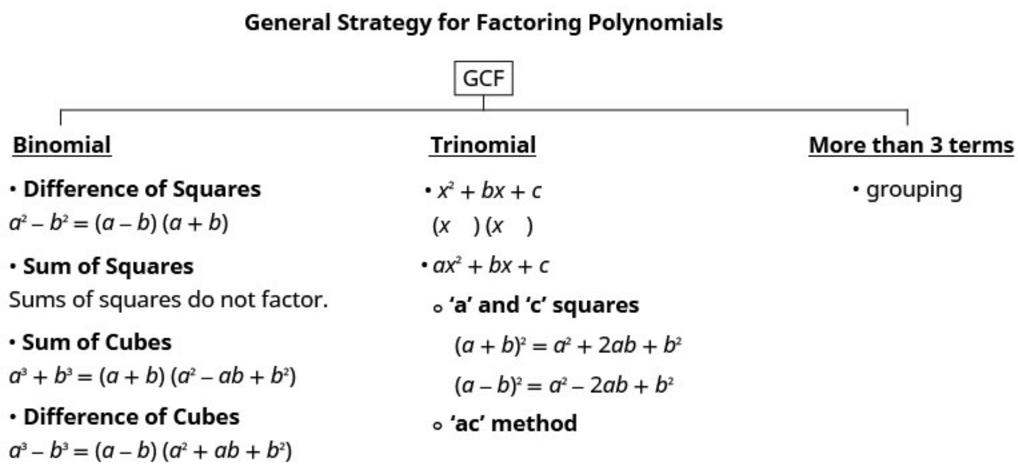
$$\begin{aligned} a^3 + b^3 &= (a+b)\left(a^2 - ab + b^2\right) \\ a^3 - b^3 &= (a-b)\left(a^2 + ab + b^2\right) \end{aligned}$$

General Strategy for Factoring Polynomials: ASE
By the end of this section, you will be able to:

- Recognize and use the appropriate method to factor a polynomial completely

Recognize and Use the Appropriate Method to Factor a Polynomial Completely

You have now become acquainted with all the methods of factoring that you will need in this course. The figure below summarizes all the factoring methods we have covered.



Note:
Factor polynomials.

Is there a greatest common factor?

- Factor it out.

Is the polynomial a binomial, trinomial, or are there more than three terms?

- If it is a binomial:
  Is it a sum?
  - Of squares? Sums of squares do not factor.
  - Of cubes? Use the sum of cubes pattern.

  Is it a difference?
  - Of squares? Factor as the product of conjugates.
  - Of cubes? Use the difference of cubes pattern.
- If it is a trinomial:
  Is it of the form $x^2 + bx + c$? Undo FOIL.
  Is it of the form $ax^2 + bx + c$?
  - If a and c are squares, check if it fits the trinomial square pattern.
  - Use the trial and error or “ac” method.

- If it has more than three terms: Use the grouping method.

Check.

- Is it factored completely?
- Do the factors multiply back to the original polynomial?

Remember, a polynomial is completely factored if, other than monomials, its factors are prime!

Example:
Exercise:

Problem: Factor completely: $4x^5 + 12x^4$.

Solution:
Solution

Is there a GCF?	Yes, $4x^4$.	$4x^5 + 12x^4$
	Factor out the GCF.	$4x^4\left(x+3\right)$
In the parentheses, is it a binomial, a trinomial, or are there more than three terms?	Binomial.	
Is it a sum?		Yes.
Of squares? Of cubes?		No.
Check.		
Is the expression factored completely?		Yes.
Multiply.		

$$4x^4\left(x+3\right)$$
$$4x^4 \cdot x + 4x^4 \cdot 3$$
$$4x^5 + 12x^4 \checkmark$$

Note:
Exercise:

Problem: Factor completely: $3a^4 + 18a^3$.

Solution:

$3a^3\left(a+6\right)$

Note:
Exercise:

Problem: Factor completely: $45b^6 + 27b^5$.

Solution:

$9b^5\,(5b + 3)$

Example:
Exercise:

Problem: Factor completely: $12x^2 - 11x + 2$.

Solution:
Solution

		$12x^2 - 11x + 2$
Is there a GCF?	No.	
Is it a binomial, trinomial, or are there more than three terms?	Trinomial.	
Are a and c perfect squares?	No, $a = 12$, not a perfect square.	
Use trial and error or the "ac" method. We will use trial and error here.		$12x^2 - 11x + 2$ $1x, 12x$ $\quad -1, -2$ $2x, 6x$ $3x, 4x$

$12x^2 - 11x + 2$	
Possible factors	**Product**
$(x - 1)(12x - 2)$	Not an option
$(x - 2)(12x - 1)$	$12x^2 - 25x + 2$
$(2x - 1)(6x - 2)$	Not an option
$(2x - 2)(6x - 1)$	Not an option
$(3x - 1)(4x - 2)$	Not an option
$(3x - 2)(4x - 1)$	$12x^2 - 11x + 2$

If the trinomial has no common factors, then neither factor can contain a common factor. That means each of these combinations is not an option.

Check.

$$(3x - 2)\,(4x - 1)$$

$$12x^2 - 3x - 8x + 2$$

$12x^2 - 11x + 2$ ✓

Note:
Exercise:

Problem: Factor completely: $10a^2 - 17a + 6$.

Solution:

$(5a - 6)(2a - 1)$

Note:
Exercise:

Problem: Factor completely: $8x^2 - 18x + 9$.

Solution:

$(2x - 3)(4x - 3)$

Example:
Exercise:

Problem: Factor completely: $g^3 + 25g$.

Solution:
Solution

Is there a GCF?	Yes, g.	$g^3 + 25g$
Factor out the GCF.		$g\left(g^2 + 25\right)$
In the parentheses, is it a binomial, trinomial, or are there more than three terms?	Binomial.	
Is it a sum ? Of squares?	Yes.	Sums of squares are prime.
Check.		

Is the expression factored completely? Yes.

Multiply.

$g\left(g^2 + 25\right)$

$g^3 + 25g$ ✓

Note:
Exercise:

Problem: Factor completely: $x^3 + 36x$.

Solution:

$x\left(x^2 + 36\right)$

Note:
Exercise:

Problem: Factor completely: $27y^2 + 48$.

Solution:

$3\left(9y^2 + 16\right)$

Example:
Exercise:

Problem: Factor completely: $12y^2 - 75$.

Solution:
Solution

Is there a GCF?	Yes, 3.	$12y^2 - 75$
Factor out the GCF.		$3\left(4y^2 - 25\right)$
In the parentheses, is it a binomial, trinomial, or are there more than three terms?	Binomial.	
Is it a sum?	No.	
Is it a difference? Of squares or cubes?	Yes, squares.	$3\left((2y)^2 - (5)^2\right)$
Write as a product of conjugates.		$3(2y - 5)(2y + 5)$
Check.		
Is the expression factored completely?	Yes.	
Neither binomial is a difference of squares.		
Multiply.		

$3(2y - 5)(2y + 5)$

$3\left(4y^2 - 25\right)$

$12y^2 - 75\ \checkmark$

Note:
Exercise:

Problem: Factor completely: $16x^3 - 36x$.

Solution:

$4x\,(2x-3)\,(2x+3)$

Note:
Exercise:

Problem: Factor completely: $27y^2 - 48$.

Solution:

$3\,(3y-4)\,(3y+4)$

Example:
Exercise:

Problem: Factor completely: $4a^2 - 12ab + 9b^2$.

Solution:
Solution

Is there a GCF?	No.	$4a^2 - 12ab + 9b^2$
Is it a binomial, trinomial, or are there more terms?		
Trinomial with $a \neq 1$. But the first term is a perfect square.		
Is the last term a perfect square?	Yes.	$(2a)^2 - 12ab + (3b)^2$
Does it fit the pattern, $a^2 - 2ab + b^2$?	Yes.	$(2a)^2 - 12ab + (3b)^2$ $-2(2a)(3b)$ $12ab$
Write it as a square.		$(2a - 3b)^2$

Check your answer.		
Is the expression factored completely?		
Yes.		
The binomial is not a difference of squares.		
Multiply.		
$(2a-3b)^2$		
$(2a)^2-2\cdot 2a\cdot 3b+(3b)^2$		
$4a^2-12ab+9b^2$✓		

Note:
Exercise:

Problem: Factor completely: $4x^2+20xy+25y^2$.

Solution:

$(2x+5y)^2$

Note:
Exercise:

Problem: Factor completely: $9m^2+42mn+49n^2$.

Solution:

$(3m+7n)^2$

Example:
Exercise:

Problem: Factor completely: $6y^2-18y-60$.

Solution:
Solution

Is there a GCF?	Yes, 6.	$6y^2 - 1$
Factor out the GCF.	Trinomial with leading coefficient 1.	$6\left(y^2 - \right.$
In the parentheses, is it a binomial, trinomial, or are there more terms?		
"Undo" FOIL.	$6\left(y\quad\right)\left(y\quad\right)$	$6\left(y + 2\right.$
Check your answer.		
Is the expression factored completely?		Ye
Neither binomial is a difference of squares.		
Multiply.		

$$6\left(y+2\right)\left(y-5\right)$$
$$6\left(y^2-5y+2y-10\right)$$
$$6\left(y^2-3y-10\right)$$
$$6y^2-18y-60\ \checkmark$$

Note:
Exercise:

Problem: Factor completely: $8y^2+16y-24$.

Solution:

$8\left(y-1\right)\left(y+3\right)$

Note:
Exercise:

Factor completely: $5u^2-15u-270$.
Problem:

Solution:

$5\left(u-9\right)\left(u+6\right)$

Example:
Exercise:

Problem: Factor completely: $24x^3+81$.

Solution:
Solution

Is there a GCF?	Yes, 3.	$24x^3+81$
Factor it out.		$3\left(8x^3+27\right)$
In the parentheses, is it a binomial, trinomial, or are there more than three terms?	Binomial.	
Is it a sum or difference?	Sum.	
Of squares or cubes?	Sum of cubes.	$3\left(\overset{a^3}{(2x)^3} \overset{+}{+} \overset{b^3}{(3)^3}\right)$
Write it using the sum of cubes pattern.		$3\left(\overset{a}{2x} \overset{+}{+} \overset{b}{3}\right)\left(\overset{a^2}{(2x)^2} \overset{-}{-} \overset{ab}{2x \cdot 3} \overset{+}{+} \overset{b^2}{3^2}\right)$
Is the expression factored completely?	Yes.	$3\left(2x+3\right)\left(4x^2-6x+9\right)$
Check by multiplying.		We leave the check to you.

Note:
Exercise:

Problem: Factor completely: $250m^3+432$.

Solution:

$2\left(5m+6\right)\left(25m^2-30m+36\right)$

Note:
Exercise:

Problem: Factor completely: $81q^3+192$.

Solution:

$81\left(q+2\right)\left(q^2-2q+4\right)$

Example:
Exercise:

Problem: Factor completely: $2x^4-32$.

Solution:

Solution

Is there a GCF?	Yes, 2.	$2x^4 - 32$
Factor it out.		$2\left(x^4 - 16\right)$
In the parentheses, is it a binomial, trinomial, or are there more than three terms?	Binomial.	
Is it a sum or difference?	Yes.	
Of squares or cubes?	Difference of squares.	$2\left(\left(x^2\right)^2 - (\right.$
Write it as a product of conjugates.		$2\left(x^2 - 4\right)\left(x^2\right.$
The first binomial is again a difference of squares.		$2\left(\left(x\right)^2 - \left(2\right)^2\right)$
Write it as a product of conjugates.		$2\left(x - 2\right)\left(x + 2\right)$
Is the expression factored completely?	Yes.	

None of these binomials is a difference of squares.

Check your answer.

Multiply.

$$2(x-2)(x+2)(x^2+4)$$
$$2(x^2-4)(x^2+4)$$
$$2(x^4-16)$$
$$2x^4-32\checkmark$$

Note:
Exercise:

Problem: Factor completely: $4a^4 - 64$.

Solution:

$4\left(a^2 + 4\right)\left(a - 2\right)\left(a + 2\right)$

Note:
Exercise:

Problem: Factor completely: $7y^4 - 7$.

Solution:

$7\left(y^2 + 1\right)\left(y - 1\right)\left(y + 1\right)$

Example:

Exercise:

Problem: Factor completely: $3x^2 + 6bx - 3ax - 6ab$.

Solution:
Solution

Is there a GCF?	Yes, 3.	$3x^2 + 6bx - 3ax - 6ab$
Factor out the GCF.		$3\left(x^2 + 2bx - ax - 2ab\right)$
In the parentheses, is it a binomial, trinomial, or are there more terms?	More than 3 terms.	
Use grouping.		$3\left[x\left(x+2b\right) - a\left(x+2b\right)\right]$ $3\left(x+2b\right)\left(x-a\right)$
Check your answer.		

Is the expression factored completely? Yes.
Multiply.

$3\left(x+2b\right)\left(x-a\right)$

$3\left(x^2 - ax + 2bx - 2ab\right)$

$3x^2 - 3ax + 6bx - 6ab$ ✓

Note:
Exercise:

Problem: Factor completely: $6x^2 - 12xc + 6bx - 12bc$.

Solution:

$6\left(x+b\right)\left(x-2c\right)$

Note:
Exercise:

Problem: Factor completely: $16x^2 + 24xy - 4x - 6y$.

Solution:

$2(4x-1)(x+3y)$

Example:
Exercise:

Problem: Factor completely: $10x^2 - 34x - 24$.

Solution:
Solution

Is there a GCF?	Yes, 2.	$10x^2 - 34x - 24$
Factor out the GCF.		$2\left(5x^2 - 17x - 12\right)$
In the parentheses, is it a binomial, trinomial, or are there more than three terms?	Trinomial with $a \neq 1$.	
Use trial and error or the "ac" method.		$2(5x^2 - 17x{-}12)$
		$2\left(5x+3\right)\left(x-4\right)$
Check your answer. Is the expression factored completely? Yes.		

Multiply.

$2\left(5x+3\right)\left(x-4\right)$
$2\left(5x^2 - 20x + 3x - 12\right)$
$2\left(5x^2 - 17x - 12\right)$
$10x^2 - 34x - 24\,\checkmark$

Note:
Exercise:

Problem: Factor completely: $4p^2 - 16p + 12$.

Solution:

$4\left(p-1\right)\left(p-3\right)$

Note:
Exercise:

Problem: Factor completely: $6q^2 - 9q - 6$.

Solution:

$3\left(q-2\right)\left(2q+1\right)$

Key Concepts

- **General Strategy for Factoring Polynomials See [link].**
- **How to Factor Polynomials**

Is there a greatest common factor? Factor it out.
Is the polynomial a binomial, trinomial, or are there more than three terms?

- If it is a binomial:
  Is it a sum?
  - Of squares? Sums of squares do not factor.
  - Of cubes? Use the sum of cubes pattern.

  Is it a difference?
  - Of squares? Factor as the product of conjugates.
  - Of cubes? Use the difference of cubes pattern.
- If it is a trinomial:
  Is it of the form $x^2 + bx + c$? Undo FOIL.
  Is it of the form $ax^2 + bx + c$?
  - If ‘a’ and ‘c’ are squares, check if it fits the trinomial square pattern.
  - Use the trial and error or ‘ac’ method.
- If it has more than three terms:
  Use the grouping method.

Check. Is it factored completely? Do the factors multiply back to the original polynomial?

Practice Makes Perfect

Recognize and Use the Appropriate Method to Factor a Polynomial Completely

In the following exercises, factor completely.
Exercise:

Problem: $10x^4 + 35x^3$

Solution:

$5x^3(2x + 7)$

Exercise:

Problem: $18p^6 + 24p^3$

Exercise:

Problem: $y^2 + 10y - 39$

Solution:

$(y - 3)(y + 13)$

Exercise:

Problem: $b^2 - 17b + 60$

Exercise:

Problem: $2n^2 + 13n - 7$

Solution:

$(2n - 1)(n + 7)$

Exercise:

Problem: $8x^2 - 9x - 3$

Exercise:

Problem: $a^5 + 9a^3$

Solution:

$a^3(a^2 + 9)$

Exercise:

Problem: $75m^3 + 12m$

Exercise:

Problem: $121r^2 - s^2$

Solution:

$(11r - s)(11r + s)$

Exercise:

Problem: $49b^2 - 36a^2$

Exercise:

Problem: $8m^2 - 32$

Solution:

$8(m - 2)(m + 2)$

Exercise:

Problem: $36q^2 - 100$

Exercise:

Problem: $25w^2 - 60w + 36$

Solution:

$(5w - 6)^2$

Exercise:

Problem: $49b^2 - 112b + 64$

Exercise:

Problem: $m^2 + 14mn + 49n^2$

Solution:

$(m + 7n)^2$

Exercise:

Problem: $64x^2 + 16xy + y^2$

Exercise:

Problem: $7b^2 + 7b - 42$

Solution:

$7(b + 3)(b - 2)$

Exercise:

Problem: $3n^2 + 30n + 72$

Exercise:

Problem: $3x^3 - 81$

Solution:

$3\left(x - 3\right)\left(x^2 + 3x + 9\right)$

Exercise:

Problem: $5t^3 - 40$

Exercise:

Problem: $k^4 - 16$

Solution:

$(k - 2)(k + 2)(k^2 + 4)$

Exercise:

Problem: $m^4 - 81$

Exercise:

Problem: $15pq - 15p + 12q - 12$

Solution:

$3(5p + 4)(q - 1)$

Exercise:

Problem: $12ab - 6a + 10b - 5$

Exercise:

Problem: $4x^2 + 40x + 84$

Solution:

$4\left(x + 3\right)\left(x + 7\right)$

Exercise:

Problem: $5q^2 - 15q - 90$

Exercise:

Problem: $u^5 + u^2$

Solution:

$u^2\left(u + 1\right)\left(u^2 - u + 1\right)$

Exercise:

Problem: $5n^3 + 320$

Exercise:

Problem: $4c^2 + 20cd + 81d^2$

Solution:

prime

Exercise:

Problem: $25x^2 + 35xy + 49y^2$

Exercise:

Problem: $10m^4 - 6250$

Solution:

$10\left(m - 5\right)\left(m + 5\right)\left(m^2 + 25\right)$

Exercise:

Problem: $3v^4 - 768$

Everyday Math

Exercise:

Problem:

Watermelon drop A springtime tradition at the University of California San Diego is the Watermelon Drop, where a watermelon is dropped from the seventh story of Urey Hall.

ⓐ The binomial $-16t^2+80$ gives the height of the watermelon t seconds after it is dropped. Factor the greatest common factor from this binomial.
ⓑ If the watermelon is thrown down with initial velocity 8 feet per second, its height after t seconds is given by the trinomial $-16t^2-8t+80$. Completely factor this trinomial.

Solution:

ⓐ $-16\left(t^2-5\right)$ ⓑ $-8\left(2t+5\right)\left(t-2\right)$

Exercise:

Problem:

Pumpkin drop A fall tradition at the University of California San Diego is the Pumpkin Drop, where a pumpkin is dropped from the eleventh story of Tioga Hall.

ⓐ The binomial $-16t^2+128$ gives the height of the pumpkin t seconds after it is dropped. Factor the greatest common factor from this binomial.
ⓑ If the pumpkin is thrown down with initial velocity 32 feet per second, its height after t seconds is given by the trinomial $-16t^2-32t+128$. Completely factor this trinomial.

Writing Exercises

Exercise:

Problem:

The difference of squares y^4-625 can be factored as $\left(y^2-25\right)\left(y^2+25\right)$. But it is not *completely* factored. What more must be done to completely factor it?

Exercise:

Problem:

Of all the factoring methods covered in this chapter (GCF, grouping, undo FOIL, ‘ac’ method, special products) which is the easiest for you? Which is the hardest? Explain your answers.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don’t get it!
recognize and use the appropriate method to factor a polynomial completely.			

ⓑ Overall, after looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

Quadratic Equations: ASE
By the end of this section, you will be able to:

- Solve quadratic equations by using the Zero Product Property
- Solve quadratic equations factoring
- Solve applications modeled by quadratic equations

We have already solved linear equations, equations of the form $ax+by=c$. In linear equations, the variables have no exponents. Quadratic equations are equations in which the variable is squared. Listed below are some examples of quadratic equations:
Equation:

$$x^2+5x+6=0 \qquad 3y^2+4y=10 \qquad 64u^2-81=0 \qquad n\left(n+1\right)=42$$

The last equation doesn't appear to have the variable squared, but when we simplify the expression on the left we will get n^2+n.

The general form of a quadratic equation is $ax^2+bx+c=0$, with $a\ne 0$.

Note:
Quadratic Equation
An equation of the form $ax^2+bx+c=0$ is called a quadratic equation.
Equation:

$$a, b, \text{ and } c \text{ are real numbers and } a\ne 0$$

To solve quadratic equations we need methods different than the ones we used in solving linear equations. We will look at one method here and then several others in a later chapter.

Solve Quadratic Equations Using the Zero Product Property

We will first solve some quadratic equations by using the Zero Product Property. The **Zero Product Property** says that if the product of two quantities is zero, it must be that at least one of the quantities is zero. The only way to get a product equal to zero is to multiply by zero itself.

Note:
Zero Product Property
If $a\cdot b=0$, then either $a=0$ or $b=0$ or both.

We will now use the Zero Product Property, to solve a quadratic equation.

Example:
How to Use the Zero Product Property to Solve a Quadratic Equation
Exercise:

Problem: Solve: $(x+1)(x-4)=0$.

Solution:
Solution

Step 1. Set each factor equal to zero.	The product equals zero, so at least one factor must equal zero.	$(x+1)(x-4)=0$ $x+1=0 \quad \text{or} \quad x-4=0$
Step 2. Solve the linear equations.	Solve each equation.	$x=-1 \quad \text{or} \quad x=4$
Step 3. Check.	Substitute each solution separately into the original equation.	$x=-1$ $(x+1)(x-4)=0$ $(-1+1)(-1-4)\stackrel{?}{=}0$ $(0)(-5)\stackrel{?}{=}0$ $0=0\checkmark$ $x=4$ $(x+1)(x-4)=0$ $(4+1)(4-4)\stackrel{?}{=}0$ $(5)(0)\stackrel{?}{=}0$ $0=0\checkmark$

Note:
Exercise:

Problem: Solve: $(x-3)(x+5)=0$.

Solution:

$x=3, x=-5$

Note:
Exercise:

Problem: Solve: $(y-6)(y+9)=0$.

Solution:

$y=6, y=-9$

We usually will do a little more work than we did in this last example to solve the linear equations that result from using the Zero Product Property.

Example:
Exercise:

Problem: Solve: $(5n-2)(6n-1)=0$.

Solution:
Solution

	$(5n-2)(6n-1)=0$	
Use the Zero Product Property to set each factor to 0.	$5n-2=0$	$6n-1=0$
Solve the equations.	$n=\frac{2}{5}$	$n=\frac{1}{6}$
Check your answers.		
$n=\frac{2}{5}$ $(5n-2)(6n-1)=0$ $\left(5\cdot\frac{2}{5}-2\right)\left(6\cdot\frac{2}{5}-1\right)\stackrel{?}{=}0$ $(2-2)\left(\frac{12}{5}-\frac{5}{5}\right)\stackrel{?}{=}0$ $(0)\left(\frac{7}{5}\right)\stackrel{?}{=}0$ $0=0\checkmark$ $n=\frac{1}{6}$ $(5n-2)(6n-1)=0$ $\left(5\cdot\frac{1}{6}-2\right)\left(6\cdot\frac{1}{6}-1\right)\stackrel{?}{=}0$ $\left(\frac{5}{6}-\frac{12}{6}\right)(1-1)\stackrel{?}{=}0$ $\left(-\frac{7}{6}\right)(0)\stackrel{?}{=}0$ $0=0\checkmark$		

Note:
Exercise:

Problem: Solve: $(3m-2)(2m+1)=0$.

Solution:

$m=\frac{2}{3}, m=-\frac{1}{2}$

Note:
Exercise:

Problem: Solve: $(4p+3)\,(4p-3)=0.$

Solution:

$p=-\frac{3}{4}, p=\frac{3}{4}$

Notice when we checked the solutions that each of them made just one factor equal to zero. But the product was zero for both solutions.

Example:
Exercise:

Problem: Solve: $3p\,(10p+7)=0.$

Solution:
Solution

	$3p\,(10p+7)=0$	
Use the Zero Product Property to set each factor to 0.	$3p=0$	$10p+7=0$
Solve the equations.	$p=0$	$10p=-7$
		$p=-\frac{7}{10}$
Check your answers.		
$p=0$ $3p(10p+7)=0$ $3\cdot 0(10\cdot 0+7)\stackrel{?}{=}0$ $0(0+7)\stackrel{?}{=}0$ $0(7)\stackrel{?}{=}0$ $0=0\checkmark$ $p=-\frac{7}{10}$ $3p(10p+7)=0$ $3\left(-\frac{7}{10}\right)10\left(-\frac{7}{10}\right)+7\stackrel{?}{=}0$ $\left(-\frac{21}{10}\right)(-7+7)\stackrel{?}{=}0$ $\left(-\frac{21}{10}\right)(0)\stackrel{?}{=}0$ $0=0\checkmark$		

Note:
Exercise:

Problem: Solve: $2u\left(5u-1\right)=0$.

Solution:

$u=0, u=\frac{1}{5}$

Note:
Exercise:

Problem: Solve: $w\left(2w+3\right)=0$.

Solution:

$w=0, w=-\frac{3}{2}$

It may appear that there is only one factor in the next example. Remember, however, that $\left(y-8\right)^2$ means $\left(y-8\right)\left(y-8\right)$.

Example:
Exercise:

Problem: Solve: $\left(y-8\right)^2=0$.

Solution:
Solution

	$(y-8)^2=0$	
Rewrite the left side as a product.	$(y-8)(y-8)=0$	
Use the Zero Product Property and set each factor to 0.	$y-8=0$	$y-8=0$
Solve the equations.	$y=8$	$y=8$
When a solution repeats, we call it a double root.		

Check your answer.		
$y = 8$ $(y - 8)^2 = 0$ $(8 - 8)^2 \stackrel{?}{=} 0$ $(0)^2 \stackrel{?}{=} 0$ $0 = 0 ✓$		

Note:
Exercise:

Problem: Solve: $(x + 1)^2 = 0$.

Solution:

$x = 1$

Note:
Exercise:

Problem: Solve: $(v - 2)^2 = 0$.

Solution:

$v = 2$

Solve Quadratic Equations by Factoring

Each of the equations we have solved in this section so far had one side in factored form. In order to use the Zero Product Property, the quadratic equation must be factored, with zero on one side. So we be sure to start with the quadratic equation in standard form, $ax^2 + bx + c = 0$. Then we factor the expression on the left.

Example:
How to Solve a Quadratic Equation by Factoring
Exercise:

Problem: Solve: $x^2 + 2x - 8 = 0$.

Solution:
Solution

Step 1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$.	The equation is already in standard form.	$x^2 + 2x - 8 = 0$
Step 2. Factor the quadratic expression.	Factor $x^2 + 2x - 8$ $(x + 4)(x - 2)$	$(x + 4)(x - 2) = 0$
Step 3. Use the Zero Product Property.	Set each factor equal to zero.	$x + 4 = 0$ or $x - 2 = 0$
Step 4. Solve the linear equations.	We have two linear equations.	$x = -4$ or $x = 2$
Step 5. Check.	Substitute each solution separately into the original equation.	$x^2 + 2x - 8 = 0$ $x = -4$ $(-4)^2 - 2(-4) - 8 \stackrel{?}{=} 0$ $16 + (-8) - 8 \stackrel{?}{=} 0$ $0 = 0$ ✓ $x^2 + 2x - 8 = 0$ $x = 2$ $2^2 - 2(2) - 8 \stackrel{?}{=} 0$ $4 + 4 - 8 \stackrel{?}{=} 0$ $0 = 0$ ✓

Note:
Exercise:

Problem: Solve: $x^2 - x - 12 = 0$.

Solution:

$x = 4, x = -3$

Note:
Exercise:

Problem: Solve: $b^2 + 9b + 14 = 0$.

Solution:

$b = -2, b = -7$

Note:
Solve a quadratic equation by factoring.

Write the quadratic equation in standard form, $ax^2 + bx + c = 0$.
Factor the quadratic expression.
Use the Zero Product Property.
Solve the linear equations.
Check.

Before we factor, we must make sure the quadratic equation is in standard form.

Example:
Exercise:

Problem: Solve: $2y^2 = 13y + 45$.

Solution:
Solution

	$2y^2 = 13y + 45$	
Write the quadratic equation in standard form.	$2y^2 - 13y - 45 = 0$	
Factor the quadratic expression.	$(2y + 5)(y - 9) = 0$	
Use the Zero Product Property to set each factor to 0.	$2y + 5 = 0$	$y - 9 = 0$
Solve each equation.	$y = -\frac{5}{2}$	$y = 9$
Check your answers.		
$y = -\frac{5}{2}$ $2y^2 = 13y + 45$ $2\left(-\frac{5}{2}\right)^2 \stackrel{?}{=} 13\left(-\frac{5}{2}\right) + 45$ $2\left(\frac{25}{4}\right)^2 \stackrel{?}{=} \left(-\frac{65}{2}\right) + \frac{90}{2}$ $\frac{25}{2} = \frac{25}{2}$ ✓ $y = 9$ $2y^2 = 13y + 45$ $2(9)^2 \stackrel{?}{=} 13(9) + 45$ $2(81)^2 \stackrel{?}{=} 117 + 45$ $162 = 162$ ✓		

Note:
Exercise:

Problem: Solve: $3c^2 = 10c - 8$.

Solution:

$c = 0, c = \frac{4}{3}$

Note:
Exercise:

Problem: Solve: $2d^2 - 5d = 3$.

Solution:

$d = 3, d = -\frac{1}{2}$

Example:
Exercise:

Problem: Solve: $5x^2 - 13x = 7x$.

Solution:
Solution

	$5x^2 - 13x = 7x$	
Write the quadratic equation in standard form.	$5x^2 - 20x = 0$	
Factor the left side of the equation.	$5x(x-4) = 0$	
Use the Zero Product Property to set each factor to 0.	$5x = 0$	$x - 4 = 0$
Solve each equation.	$x = 0$	$x = 4$
Check your answers.		

$x = 0$ $5x^2 - 13x = 7x$ $5(0)^2 - 13(0) \stackrel{?}{=} 7(0)$ $0 - 0 \stackrel{?}{=} 0$ $0 = 0$ ✓ $x = 4$ $5x^2 - 13x = 7x$ $5(4)^2 - 13(4) \stackrel{?}{=} 7(4)$ $5(16) - 52 \stackrel{?}{=} 28$ $28 = 28$ ✓		

Note:
Exercise:

Problem: Solve: $6a^2 + 9a = 3a$.

Solution:

$a = 0, a = -1$

Note:
Exercise:

Problem: Solve: $45b^2 - 2b = -17b$.

Solution:

$b = 0, b = -\frac{1}{3}$

Solving quadratic equations by factoring will make use of all the factoring techniques you have learned in this chapter! Do you recognize the special product pattern in the next example?

Example:
Exercise:

Problem: Solve: $144q^2 = 25$.

Solution:
Solution

	$144q^2 = 25$	

Write the quadratic equation in standard form.	$144q^2 - 25 = 0$	
Factor. It is a difference of squares.	$(12q - 5)(12q + 5) = 0$	
Use the Zero Product Property to set each factor to 0.	$12q–5 = 0$	$12q + 5 = 0$
Solve each equation.	$12q = 5$ $q = \frac{5}{12}$	$12q = –5$ $q = -\frac{5}{12}$
Check your answers.		

Note:
Exercise:

Problem: Solve: $25p^2 = 49$.

Solution:

$p = \frac{7}{5}$, p $= -\frac{7}{5}$

Note:
Exercise:

Problem: Solve: $36x^2 = 121$.

Solution:

$x = \frac{11}{6}$, $x = -\frac{11}{6}$

The left side in the next example is factored, but the right side is not zero. In order to use the Zero Product Property, one side of the equation must be zero. We'll multiply the factors and then write the equation in standard form.

Example:
Exercise:

Problem: Solve: $(3x - 8)(x - 1) = 3x$.

Solution:
Solution

	$(3x-8)(x-1) = 3x$
Multiply the binomials.	$3x^2-11x+8 = 3x$
Write the quadratic equation in standard form.	$3x^2-14x+8 = 0$
Factor the trinomial.	$(3x-2)(x-4) = 0$
Use the Zero Product Property to set each factor to 0.	$3x-2 = 0 \quad x-4 = 0$
Solve each equation.	$3x = 2 \quad x = 4$
	$x = \frac{2}{3}$
Check your answers.	The check is left to you!

Note:
Exercise:

Problem: Solve: $(2m+1)(m+3) = 12m$.

Solution:

$m = 1, m = \frac{3}{2}$

Note:
Exercise:

Problem: Solve: $(k+1)(k-1) = 8$.

Solution:

$k = 3, k = -3$

The Zero Product Property also applies to the product of three or more factors. If the product is zero, at least one of the factors must be zero. We can solve some equations of degree more than two by using the Zero Product Property, just like we solved quadratic equations.

Example:
Exercise:

Problem: Solve: $9m^3 + 100m = 60m^2$.

Solution:
Solution

	$9m^3 + 100m = 60m^2$
Bring all the terms to one side so that the other side is zero.	$9m^3 - 60m^2 + 100m = 0$
Factor the greatest common factor first.	$m(9m^2 - 60m + 100) = 0$
Factor the trinomial.	$m(3m - 10)(3m - 10) = 0$
Use the Zero Product Property to set each factor to 0.	$m = 0 \quad 3m - 10 = 0 \quad 3m - 10 = 0$
Solve each equation.	$m = 0 \quad m = \frac{10}{3} \quad m = \frac{10}{3}$
Check your answers.	The check is left to you.

Note:
Exercise:

Problem: Solve: $8x^3 = 24x^2 - 18x$.

Solution:

$x = 0, x = \frac{3}{2}$

Note:
Exercise:

Problem: Solve: $16y^2 = 32y^3 + 2y$.

Solution:

$y = 0, y = \frac{1}{4}$

When we factor the quadratic equation in the next example we will get three factors. However the first factor is a constant. We know that factor cannot equal 0.

Example:
Exercise:

Problem: Solve: $4x^2 = 16x + 84$.

Solution:
Solution

	$4x^2 = 16x + 84$
Write the quadratic equation in standard form.	$4x^2 - 16x - 84 = 0$
Factor the greatest common factor first.	$4\left(x^2 - 4x - 21\right) = 0$
Factor the trinomial.	$4\left(x - 7\right)\left(x + 3\right) = 0$
Use the Zero Product Property to set each factor to 0.	$4 \neq 0 \quad x - 7 = 0 \quad x + 3 =$
Solve each equation.	$4 \neq 0 \quad x = 7 \quad x =$
Check your answers.	The check is left to you.

Note:
Exercise:

Problem: Solve: $18a^2 - 30 = -33a$.

Solution:

$a = -\frac{5}{2}, a = \frac{2}{3}$

Note:
Exercise:

Problem: Solve: $123b = -6 - 60b^2$.

Solution:

$b = 2, b = \frac{1}{20}$

Solve Applications Modeled by Quadratic Equations

The problem solving strategy we used earlier for applications that translate to linear equations will work just as well for applications that translate to quadratic equations. We will copy the problem solving strategy here so we can use it for reference.

Note:
Use a problem-solving strategy to solve word problems.

Read the problem. Make sure all the words and ideas are understood.
Identify what we are looking for.
Name what we are looking for. Choose a variable to represent that quantity.
Translate into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.
Solve the equation using good algebra techniques.
Check the answer in the problem and make sure it makes sense.
Answer the question with a complete sentence.

We will start with a number problem to get practice translating words into a quadratic equation.

Example:
Exercise:

Problem: The product of two consecutive integers is 132. Find the integers.

Solution:
Solution

Step 1. Read the problem.

Step 2. Identify what we are looking for. We are looking for two consecutive integers

Step 3. Name what we are looking for. Let $n =$ the first integer
$n+1 =$ the next consecutive integer

Step 4. Translate into an equation. Restate the problem in a sentence. The product of the two consecutive integer

The first integer times the next integer is 1

Translate to an equation. $n(n+1) = 132$

Step 5. Solve the equation. $n^2+n = 132$

Bring all the terms to one side. $n^2+n-132 = 0$

Factor the trinomial. $(n-11)(n+12) = 0$

Use the zero product property. $n-11 = 0$ $n+12 =$

Solve the equations. $n = 11$ $n =$

There are two values for n that are solutions to this problem. So there are two sets of consecutive integers that will work.

If the first integer is $n = 11$	If the first integer is $n = -12$
then the next integer is $n + 1$	then the next integer is $n + 1$
$11 + 1$	$-12 + 1$
12	-11

Step 6. Check the answer.

The consecutive integers are $11, 12$ and $-11, -12$. The product $11 \cdot 12 = 132$ and the product $-11\left(-12\right) = 132$. Both pairs of consecutive integers are solutions.

Step 7. Answer the question. The consecutive integers are $11, 12$ and $-11, -12$.

Note:
Exercise:

Problem: The product of two consecutive integers is 240. Find the integers.

Solution:

$-15, -16$ and $15, 16$

Note:
Exercise:

Problem: The product of two consecutive integers is 420. Find the integers.

Solution:

$-21, -20$ and $20, 21$

Were you surprised by the pair of negative integers that is one of the solutions to the previous example? The product of the two positive integers and the product of the two negative integers both give 132.

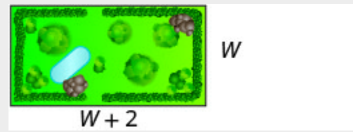
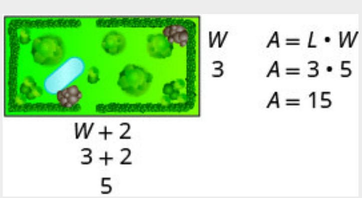
In some applications, negative solutions will result from the algebra, but will not be realistic for the situation.

Example:
Exercise:

Problem:

A rectangular garden has an area 15 square feet. The length of the garden is two feet more than the width. Find the length and width of the garden.

Solution:
Solution

Step 1. Read the problem. In problems involving geometric figures, a sketch can help you visualize the situation.		
Step 2. Identify what you are looking for.	We are looking for the length and width.	
Step 3. Name what you are looking for. The length is two feet more than width.	Let W = the width of the garden. $W + 2$ = the length of the garden	
Step 4. Translate into an equation. Restate the important information in a sentence.	The area of the rectangular garden is 15 square feet.	
Use the formula for the area of a rectangle.	$A = L \cdot W$	
Substitute in the variables.	$15 = (W + 2)W$	
Step 5. Solve the equation. Distribute first.	$15 = W^2 + 2W$	
Get zero on one side.	$0 = W^2 + 2W - 15$	
Factor the trinomial.	$0 = (W + 5)(W - 3)$	
Use the Zero Product Property.	$0 = W + 5$	$0 = W - 3$
Solve each equation.	$-5 = W$	$3 = W$
Since W is the width of the garden, it does not make sense for it to be negative. We eliminate that value for W.	$\cancel{-5 = W}$ $W = 3$	$3 = W$ Width is 3 feet.
Find the value of the length.	$W + 2 = \text{length}$	
	$3 + 2$	
	5	Length is 5 feet.
Step 6. Check the answer. Does the answer make sense?		
		
	Yes, this makes sense.	

Step 7. Answer the question.	The width of the garden is 3 feet and the length is 5 feet.

Note:
Exercise:

Problem:

A rectangular sign has area 30 square feet. The length of the sign is one foot more than the width. Find the length and width of the sign.

Solution:

5 feet and 6 feet

Note:
Exercise:

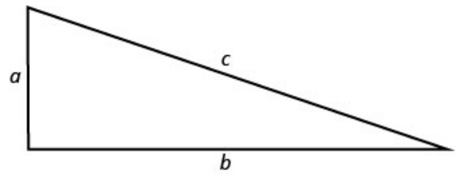
Problem:

A rectangular patio has area 180 square feet. The width of the patio is three feet less than the length. Find the length and width of the patio.

Solution:

12 feet and 15 feet

Recall the Pythagorean Theorem: $a^2 + b^2 = c^2$. It gives the relation between the legs and the hypotenuse of a right triangle.

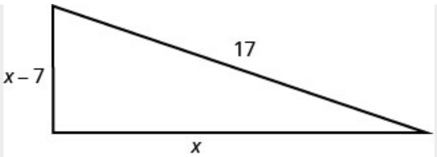


We will use this formula to in the next example.

Example:
Exercise:

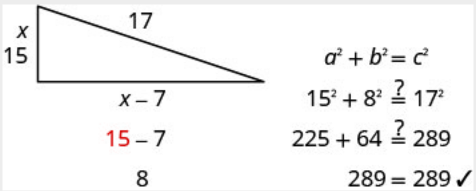
Problem:

Justine wants to put a deck in the corner of her backyard in the shape of a right triangle, as shown below. The hypotenuse will be 17 feet long. The length of one side will be 7 feet less than the length of the other side. Find the lengths of the sides of the deck.



Solution:
Solution

Step 1. Read the problem.			
Step 2. Identify what you are looking for.	We are looking for the lengths of the sides of the deck.		
Step 3. Name what you are looking for. One side is 7 less than the other.	Let x = length of a side of the deck $x - 7$ = length of other side		
Step 4. Translate into an equation. Since this is a right triangle we can use the Pythagorean Theorem.	$a^2 + b^2 = c^2$		
Substitute in the variables.	$x^2 + (x-7)^2 = 17^2$		
Step 5. Solve the equation.	$x^2 + x^2 - 14x + 49 = 289$		
Simplify.	$2x^2 - 14x + 49 = 289$		
It is a quadratic equation, so get zero on one side.	$2x^2 - 14x - 240 = 0$		
Factor the greatest common factor.	$2(x^2 - 7x - 120) = 0$		
Factor the trinomial.	$2(x-15)(x+8) = 0$		
Use the Zero Product Property.	$2 \neq 0$	$x - 15 = 0$	$x + 8 = 0$
Solve.	$2 \neq 0$	$x = 15$	$x = -8$
Since x is a side of the triangle, $x = -8$ does not make sense.	$2 \neq 0$	$x = 15$	$\cancel{x = -8}$
Find the length of the other side.			
If the length of one side is		$x = 15$	
then the length of the other side is		$x - 7$	

		15 – 7	
	8 is the length of the other side.		
Step 6. Check the answer. Do these numbers make sense?			
			
Step 7. Answer the question.	The sides of the deck are 8, 15, and 17 feet.		

Note:
Exercise:

Problem:

A boat’s sail is a right triangle. The length of one side of the sail is 7 feet more than the other side. The hypotenuse is 13. Find the lengths of the two sides of the sail.

Solution:

5 feet and 12 feet

Note:
Exercise:

Problem:

A meditation garden is in the shape of a right triangle, with one leg 7 feet. The length of the hypotenuse is one more than the length of one of the other legs. Find the lengths of the hypotenuse and the other leg.

Solution:

24 feet and 25 feet

Key Concepts

- **Zero Product Property** If $a \cdot b = 0$, then either $a = 0$ or $b = 0$ or both. See [link].
- **Solve a quadratic equation by factoring** To solve a quadratic equation by factoring: See [link].

  Write the quadratic equation in standard form,$ax^2 + bx + c = 0$.

Factor the quadratic expression.
Use the Zero Product Property.
Solve the linear equations.
Check.

- **Use a problem solving strategy to solve word problems** See [link].

Readthe problem. Make sure all the words and ideas are understood.
Identifywhat we are looking for.
Namewhat we are looking for. Choose a variable to represent that quantity.
Translateinto an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.
Solvethe equation using good algebra techniques.
Checkthe answer in the problem and make sure it makes sense.
Answerthe question with a complete sentence.

Section Exercises

Practice Makes Perfect

Use the Zero Product Property

In the following exercises, solve.
Exercise:

Problem: $(x-3)(x+7)=0$

Solution:

$x=3, x=-7$

Exercise:

Problem: $(y-11)(y+1)=0$

Exercise:

Problem: $(3a-10)(2a-7)=0$

Solution:

$a=10/3, a=7/2$

Exercise:

Problem: $(5b+1)(6b+1)=0$

Exercise:

Problem: $6m(12m-5)=0$

Solution:

$m=0, m=5/12$

Exercise:

Problem: $2x\left(6x-3\right)=0$

Exercise:

Problem: $\left(y-3\right)^{2}=0$

Solution:

$y=3$

Exercise:

Problem: $\left(b+10\right)^{2}=0$

Exercise:

Problem: $\left(2x-1\right)^{2}=0$

Solution:

$x=1/2$

Exercise:

Problem: $\left(3y+5\right)^{2}=0$

Solve Quadratic Equations by Factoring

In the following exercises, solve.

Exercise:

Problem: $x^{2}+7x+12=0$

Solution:

$x=3,x=4\;x=-3,x=-4$

Exercise:

Problem: $y^{2}-8y+15=0$

Exercise:

Problem: $5a^{2}-26a=24$

Solution:

$a=-5/4,a=6\;a=-4/5,a=6$

Exercise:

Problem: $4b^{2}+7b=-3$

Exercise:

Problem: $4m^{2}=17m-15$

Solution:

$m = 5/4, m = 3$

Exercise:

Problem: $n^2 = 5 - 6n$ $n^2 = 5n - 6$

Exercise:

Problem: $7a^2 + 14a = 7a$

Solution:

$a = -1, a = 0$

Exercise:

Problem: $12b^2 - 15b = -9b$

Exercise:

Problem: $49m^2 = 144$

Solution:

$m = 12/7, m = -12/7$

Exercise:

Problem: $625 = x^2$

Exercise:

Problem: $(y - 3)(y + 2) = 4y$

Solution:

$y = -1, y = 6$

Exercise:

Problem: $(p - 5)(p + 3) = -7$

Exercise:

Problem: $(2x + 1)(x - 3) = -4x$

Solution:

$x = 3/2, x = -1$

Exercise:

Problem: $(x + 6)(x - 3) = -8$

Exercise:

Problem: $16p^3 = 24p^2 - 9p$

Solution:

$p = 0, p = ¾$

Exercise:

Problem: $m^3 - 2m^2 = -m$

Exercise:

Problem: $20x^2 - 60x = -45$

Solution:

$x = -2/3\ x = 3/2$

Exercise:

Problem: $3y^2 - 18y = -27$

Solve Applications Modeled by Quadratic Equations

In the following exercises, solve.

Exercise:

Problem: The product of two consecutive integers is 56. Find the integers.

Solution:

$7 \text{ and } 8; -8 \text{ and } -7$

Exercise:

Problem: The product of two consecutive integers is 42. Find the integers.

Exercise:

Problem:

The area of a rectangular carpet is 28 square feet. The length is three feet more than the width. Find the length and the width of the carpet.

Solution:

$4 \text{ feet and } 7 \text{ feet}$

Exercise:

Problem:

A rectangular retaining wall has area 15 square feet. The height of the wall is two feet less than its length. Find the height and the length of the wall.

Exercise:

Problem:

A pennant is shaped like a right triangle, with hypotenuse 10 feet. The length of one side of the pennant is two feet longer than the length of the other side. Find the length of the two sides of the pennant.

Solution:

$6\text{ feet and }8\text{ feet}$

Exercise:

Problem:

A reflecting pool is shaped like a right triangle, with one leg along the wall of a building. The hypotenuse is 9 feet longer than the side along the building. The third side is 7 feet longer than the side along the building. Find the lengths of all three sides of the reflecting pool.

Mixed Practice

In the following exercises, solve.

Exercise:

Problem: $\left(x+8\right)\left(x-3\right)=0$

Solution:

$x=-8,x=3$

Exercise:

Problem: $\left(3y-5\right)\left(y+7\right)=0$

Exercise:

Problem: $p^2+12p+11=0$

Solution:

$p=-1,p=-11$

Exercise:

Problem: $q^2-12q-13=0$

Exercise:

Problem: $m^2=6m+16$

Solution:

$m=-2,m=8$

Exercise:

Problem: $4n^2+19n=5$

Exercise:

Problem: $a^3 - a^2 - 42a = 0$

Solution:

$a = 0, a = -6, a = 7$

Exercise:

Problem: $4b^2 - 60b + 224 = 0$

Exercise:

Problem: The product of two consecutive integers is 110. Find the integers.

Solution:

$10 \text{ and } 11; -11 \text{ and } -10$

Exercise:

Problem:

The length of one leg of a right triangle is three more than the other leg. If the hypotenuse is 15, find the lengths of the two legs.

Everyday Math

Exercise:

Problem:

Area of a patio If each side of a square patio is increased by 4 feet, the area of the patio would be 196 square feet. Solve the equation $(s+4)^2 = 196$ for s to find the length of a side of the patio.

Solution:

10 feet

Exercise:

Problem:

Watermelon drop A watermelon is dropped from the tenth story of a building. Solve the equation $-16t^2 + 144 = 0$ for t to find the number of seconds it takes the watermelon to reach the ground.

Writing Exercises

Exercise:

Problem:

Explain how you solve a quadratic equation. How many answers do you expect to get for a quadratic equation?

Solution:

Answers may vary.

Exercise:

Problem:

Give an example of a quadratic equation that has a GCF and none of the solutions to the equation is zero.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve quadratic equations by using the Zero Product Property.			
solve quadratic equations by factoring.			
solve applications modeled by quadratic equations.			

ⓑ Overall, after looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

Chapter Review Exercises

7.1 Greatest Common Factor and Factor by Grouping

Find the Greatest Common Factor of Two or More Expressions

In the following exercises, find the greatest common factor.

Exercise:

Problem: $42, 60$

Solution:

6

Exercise:

Problem: $450, 420$

Exercise:

Problem: $90, 150, 105$

Solution:

15

Exercise:

Problem: $60, 294, 630$

Factor the Greatest Common Factor from a Polynomial

In the following exercises, factor the greatest common factor from each polynomial.

Exercise:

Problem: $24x-42$

Solution:

$6(4x-7)$

Exercise:

Problem: $35y+84$

Exercise:

Problem: $15m^4+6m^2n$

Solution:

$3m^2(5m^2+2n)$

Exercise:

Problem: $24pt^4+16t^7$

Factor by Grouping

In the following exercises, factor by grouping.

Exercise:

Problem: $ax-ay+bx-by$

Solution:

$(a+b)\,(x-y)$

Exercise:

Problem: $x^2y-xy^2+2x-2y$

Exercise:

Problem: $x^2+7x-3x-21$

Solution:

$(x-3)\,(x+7)$

Exercise:

Problem: $4x^2-16x+3x-12$

Exercise:

Problem: m^3+m^2+m+1

Solution:

$(m^2 + 1)(m + 1)$

Exercise:

Problem: $5x - 5y - y + x$

7.2 Factor Trinomials of the form $x^2 + bx + c$

Factor Trinomials of the Form $x^2 + bx + c$

In the following exercises, factor each trinomial of the form $x^2 + bx + c$.

Exercise:

Problem: $u^2 + 17u + 72$

Solution:

$(u + 8)(u + 9)$

Exercise:

Problem: $a^2 + 14a + 33$

Exercise:

Problem: $k^2 - 16k + 60$

Solution:

$(k - 6)(k - 10)$

Exercise:

Problem: $r^2 - 11r + 28$

Exercise:

Problem: $y^2 + 6y - 7$

Solution:

$(y + 7)(y - 1)$

Exercise:

Problem: $m^2 + 3m - 54$

Exercise:

Problem: $s^2 - 2s - 8$

Solution:

$(s - 4)(s + 2)$

Exercise:

Problem: $x^2 - 3x - 10$

Factor Trinomials of the Form $x^2 + bxy + cy^2$

In the following examples, factor each trinomial of the form $x^2 + bxy + cy^2$.

Exercise:

Problem: $x^2 + 12xy + 35y^2$

Solution:

$(x + 5y)(x + 7y)$

Exercise:

Problem: $u^2 + 14uv + 48v^2$

Exercise:

Problem: $a^2 + 4ab - 21b^2$

Solution:

$(a + 7b)(a - 3b)$

Exercise:

Problem: $p^2 - 5pq - 36q^2$

7.3 Factoring Trinomials of the form $ax^2 + bx + c$

Recognize a Preliminary Strategy to Factor Polynomials Completely

In the following exercises, identify the best method to use to factor each polynomial.

Exercise:

Problem: $y^2 - 17y + 42$

Solution:

Undo FOIL

Exercise:

Problem: $12r^2 + 32r + 5$

Exercise:

Problem: $8a^3 + 72a$

Solution:

Factor the GCF

Exercise:

Problem: $4m - mn - 3n + 12$

Factor Trinomials of the Form $ax^2 + bx + c$ with a GCF

In the following exercises, factor completely.

Exercise:

Problem: $6x^2 + 42x + 60$

Solution:

$6\left(x+2\right)\left(x+5\right)$

Exercise:

Problem: $8a^2 + 32a + 24$

Exercise:

Problem: $3n^4 - 12n^3 - 96n^2$

Solution:

$3n^2\left(n-8\right)\left(n+4\right)$

Exercise:

Problem: $5y^4 + 25y^2 - 70y$

Factor Trinomials Using the "ac" Method

In the following exercises, factor.

Exercise:

Problem: $2x^2 + 9x + 4$

Solution:

$\left(x+4\right)\left(2x+1\right)$

Exercise:

Problem: $3y^2 + 17y + 10$

Exercise:

Problem: $18a^2 - 9a + 1$

Solution:

$\left(3a-1\right)\left(6a-1\right)$

Exercise:

Problem: $8u^2 - 14u + 3$

Exercise:

Problem: $15p^2 + 2p - 8$

Solution:

$(5p + 4)(3p - 2)$

Exercise:

Problem: $15x^2 + 6x - 2$

Exercise:

Problem: $40s^2 - s - 6$

Solution:

$(5s - 2)(8s + 3)$

Exercise:

Problem: $20n^2 - 7n - 3$

Factor Trinomials with a GCF Using the “ac” Method

In the following exercises, factor.

Exercise:

Problem: $3x^2 + 3x - 36$

Solution:

$3(x + 4)(x - 3)$

Exercise:

Problem: $4x^2 + 4x - 8$

Exercise:

Problem: $60y^2 - 85y - 25$

Solution:

$5(4y + 1)(3y - 5)$

Exercise:

Problem: $18a^2 - 57a - 21$

7.4 Factoring Special Products

Factor Perfect Square Trinomials

In the following exercises, factor.

Exercise:

Problem: $25x^2 + 30x + 9$

Solution:

$(5x + 3)^2$

Exercise:

Problem: $16y^2 + 72y + 81$

Exercise:

Problem: $36a^2 - 84ab + 49b^2$

Solution:

$(6a - 7b)^2$

Exercise:

Problem: $64r^2 - 176rs + 121s^2$

Exercise:

Problem: $40x^2 + 360x + 810$

Solution:

$10(2x + 9)^2$

Exercise:

Problem: $75u^2 + 180u + 108$

Exercise:

Problem: $2y^3 - 16y^2 + 32y$

Solution:

$2y(y - 4)^2$

Exercise:

Problem: $5k^3 - 70k^2 + 245k$

Factor Differences of Squares

In the following exercises, factor.

Exercise:

Problem: $81r^2 - 25$

Solution:

$(9r-5)(9r+5)$

Exercise:

Problem: $49a^2-144$

Exercise:

Problem: $169m^2-n^2$

Solution:

$(13m+n)(13m-n)$

Exercise:

Problem: $64x^2-y^2$

Exercise:

Problem: $25p^2-1$

Solution:

$(5p-1)(5p+1)$

Exercise:

Problem: $1-16s^2$

Exercise:

Problem: $9-121y^2$

Solution:

$(3+11y)(3-11y)$

Exercise:

Problem: $100k^2-81$

Exercise:

Problem: $20x^2-125$

Solution:

$5(2x-5)(2x+5)$

Exercise:

Problem: $18y^2-98$

Exercise:

Problem: $49u^3 - 9u$

Solution:

$u\,(7u + 3)\,(7u - 3)$

Exercise:

Problem: $169n^3 - n$

Factor Sums and Differences of Cubes

In the following exercises, factor.

Exercise:

Problem: $a^3 - 125$

Solution:

$(a - 5)\left(a^2 + 5a + 25\right)$

Exercise:

Problem: $b^3 - 216$

Exercise:

Problem: $2m^3 + 54$

Solution:

$2\,(m + 3)\left(m^2 - 3m + 9\right)$

Exercise:

Problem: $81x^3 + 3$

[7.5 General Strategy for Factoring Polynomials]

Recognize and Use the Appropriate Method to Factor a Polynomial Completely

In the following exercises, factor completely.

Exercise:

Problem: $24x^3 + 44x^2$

Solution:

$4x^2\,(6x + 11)$

Exercise:

Problem: $24a^4 - 9a^3$

Exercise:

Problem: $16n^2 - 56mn + 49m^2$

Solution:

$(4n - 7m)^2$

Exercise:

Problem: $6a^2 - 25a - 9$

Exercise:

Problem: $5r^2 + 22r - 48$

Solution:

$(r + 6)\,(5r - 8)$

Exercise:

Problem: $5u^4 - 45u^2$

Exercise:

Problem: $n^4 - 81$

Solution:

$\left(n^2 + 9\right)(n + 3)\,(n - 3)$

Exercise:

Problem: $64j^2 + 225$

Exercise:

Problem: $5x^2 + 5x - 60$

Solution:

$5\,(x - 3)\,(x + 4)$

Exercise:

Problem: $b^3 - 64$

Exercise:

Problem: $m^3 + 125$

Solution:

$(m + 5)(m^2 - 5m + 25)$

Exercise:

Problem: $2b^2 - 2bc + 5cb - 5c^2$

7.6 Quadratic Equations

Use the Zero Product Property

In the following exercises, solve.

Exercise:

Problem: $(a-3)(a+7)=0$

Solution:

$a=3\; a=-7$

Exercise:

Problem: $(b-3)(b+10)=0$

Exercise:

Problem: $3m(2m-5)(m+6)=0$

Solution:

$m=0\; m=-3\; m=\frac{5}{2}$

Exercise:

Problem: $7n(3n+8)(n-5)=0$

Solve Quadratic Equations by Factoring

In the following exercises, solve.

Exercise:

Problem: $x^2+9x+20=0$

Solution:

$x=-4, x=-5$

Exercise:

Problem: $y^2-y-72=0$

Exercise:

Problem: $2p^2-11p=40$

Solution:

$p=-\frac{5}{2}, p=8$

Exercise:

Problem: $q^3 + 3q^2 + 2q = 0$

Exercise:

Problem: $144m^2 - 25 = 0$

Solution:

$m = \frac{5}{12}, m = -\frac{5}{12}$

Exercise:

Problem: $4n^2 = 36$

Solve Applications Modeled by Quadratic Equations

In the following exercises, solve.

Exercise:

Problem: The product of two consecutive numbers is 462. Find the numbers.

Solution:

$-21, -22$ $21, 22$

Exercise:

Problem:

The area of a rectangular shaped patio 400 square feet. The length of the patio is 9 feet more than its width. Find the length and width.

Practice Test

In the following exercises, find the Greatest Common Factor in each expression.

Exercise:

Problem: $14y - 42$

Solution:

$7\left(y - 6\right)$

Exercise:

Problem: $-6x^2 - 30x$

Exercise:

Problem: $80a^2 + 120a^3$

Solution:

$40a^2(2 + 3a)$

Exercise:

Problem: $5m\left(m-1\right)+3\left(m-1\right)$

In the following exercises, factor completely.

Exercise:

Problem: $x^2+13x+36$

Solution:

$(x+7)(x+6)$

Exercise:

Problem: $p^2+pq-12q^2$

Exercise:

Problem: $3a^3-6a^2-72a$

Solution:

$3a(a^2-2a-14)$

Exercise:

Problem: $s^2-25s+84$

Exercise:

Problem: $5n^2+30n+45$

Solution:

$5\left(n+1\right)\left(n+5\right)$

Exercise:

Problem: $64y^2-49$

Exercise:

Problem: $xy-8y+7x-56$

Solution:

$\left(x-8\right)\left(y+7\right)$

Exercise:

Problem: $40r^2+810$

Exercise:

Problem: $9s^2-12s+4$

Solution:

$(3s-2)^2$

Exercise:

Problem: $n^2+12n+36$

Exercise:

Problem: $100-a^2$

Solution:

$(10-a)(10+a)$

Exercise:

Problem: $6x^2-11x-10$

Exercise:

Problem: $3x^2-75y^2$

Solution:

$3(x+5y)(x-5y)$

Exercise:

Problem: $c^3-1000d^3$

Exercise:

Problem: $ab-3b-2a+6$

Solution:

$(a-3)(b-2)$

Exercise:

Problem: $6u^2+3u-18$

Exercise:

Problem: $8m^2+22m+5$

Solution:

$(4m+1)(2m+5)$

In the following exercises, solve.

Exercise:

Problem: $x^2+9x+20=0$

Exercise:

Problem: $y^2 = y + 132$

Solution:

$\text{y} = -11, \text{y} = 12$

Exercise:

Problem: $5a^2 + 26a = 24$

Exercise:

Problem: $9b^2 - 9 = 0$

Solution:

$b = 1, b = -1$

Exercise:

Problem: $16 - m^2 = 0$

Exercise:

Problem: $4n^2 + 19 + 21 = 0$

Solution:

$n = -\frac{7}{4}, n = -3$

Exercise:

Problem: $(x - 3)(x + 2) = 6$

Exercise:

Problem: The product of two consecutive integers is 156. Find the integers.

Solution:

$12 \text{ and } 13; -13 \text{ and } -12$

Exercise:

Problem:

The area of a rectangular place mat is 168 square inches. Its length is two inches longer than the width. Find the length and width of the placemat.

Glossary

quadratic equations
: are equations in which the variable is squared.

Zero Product Property

The Zero Product Property states that, if the product of two quantities is zero, at least one of the quantities is zero.

Introduction
class="introduction"

Square roots are used to determine the time it would take for a stone falling from the edge of this cliff to hit the land below.

Suppose a stone falls from the edge of a cliff. The number of feet the stone has dropped after seconds can be found by multiplying 16 times the square of . But to calculate the number of seconds it would take the stone to hit the land below, we need to use a square root. In this chapter, we will

introduce and apply the properties of square roots, and extend these concepts to higher order roots and rational exponents.

Simplify and Use Square Roots: ASE
By the end of this section, you will be able to:

- Simplify expressions with square roots
- Estimate square roots
- Approximate square roots
- Simplify variable expressions with square roots

Simplify Expressions with Square Roots

Remember that when a number n is multiplied by itself, we write n^2 and read it “n squared.” For example, 15^2 reads as “15 squared,” and 225 is called the square of 15, since $15^2 = 225$.

Note:
Square of a Number
If $n^2 = m$, then m is the square of n.

Sometimes we will need to look at the relationship between numbers and their squares in reverse. Because 225 is the square of 15, we can also say that 15 is a square root of 225. A number whose square is m is called a *square root* of m.

Note:
Square Root of a Number
If $n^2 = m$, then n is a square root of m.

Notice $(-15)^2 = 225$ also, so -15 is also a square root of 225. Therefore, both 15 and -15 are square roots of 225.

So, every positive number has two square roots—one positive and one negative. What if we only wanted the positive square root of a positive number? The *radical sign*, $\sqrt{m}$, denotes the positive square root. The positive square root is also called the principal square root.

We also use the radical sign for the square root of zero. Because $0^2 = 0$, $\sqrt{0} = 0$. Notice that zero has only one square root.

Note:
Square Root Notation

$\sqrt{m}$ is read as "the square root of m."
If $m = n^2$, then $\sqrt{m} = n$, for $n \geq 0$.
The square root of m, $\sqrt{m}$, is the positive number whose square is m.

Since 15 is the positive square root of 225, we write $\sqrt{225} = 15$. Fill in [link] to make a table of square roots you can refer to as you work this chapter.

$\sqrt{1}$	$\sqrt{4}$	$\sqrt{9}$	$\sqrt{16}$	$\sqrt{25}$	$\sqrt{36}$	$\sqrt{49}$	$\sqrt{64}$	$\sqrt{81}$	$\sqrt{100}$	$\sqrt{121}$	$\sqrt{144}$	$\sqrt{169}$	$\sqrt{196}$	$\sqrt{225}$
														15

We know that every positive number has two square roots and the radical sign indicates the positive one. We write $\sqrt{225} = 15$. If we want to find the negative square root of a number, we place a negative in front of the radical sign. For example, $-\sqrt{225} = -15$.

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt{36}$ ⓑ $\sqrt{196}$ ⓒ $-\sqrt{81}$ ⓓ $-\sqrt{289}$.

Solution:
Solution

ⓐ

	$\sqrt{36}$
Since $6^2 = 36$	6

ⓑ

	$\sqrt{196}$
Since $14^2 = 196$	14

ⓒ

	$-\sqrt{81}$
The negative is in front of the radical sign.	-9

ⓓ

	$-\sqrt{289}$
The negative is in front of the radical sign.	-17

Note:
Exercise:

Problem: Simplify: ⓐ $-\sqrt{49}$ ⓑ $\sqrt{225}$.

Solution:

ⓐ -7 ⓑ 15

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{64}$ ⓑ $-\sqrt{121}$.

Solution:

ⓐ 8 ⓑ -11

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt{-169}$ ⓑ $-\sqrt{64}$.

Solution:
Solution

ⓐ

	$\sqrt{-169}$
There is no real number whose square is -169.	$\sqrt{-169}$ is not a real number.

ⓑ

	$-\sqrt{64}$
The negative is in front of the radical.	-8

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{-196}$ ⓑ $-\sqrt{81}$.

Solution:

ⓐ not a real number ⓑ -9

Note:
Exercise:

Problem: Simplify: ⓐ $-\sqrt{49}$ ⓑ $\sqrt{-121}$.

Solution:

ⓐ -7 ⓑ not a real number

When using the order of operations to simplify an expression that has square roots, we treat the radical as a grouping symbol.

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt{25}+\sqrt{144}$ ⓑ $\sqrt{25+144}$.

Solution:
Solution

ⓐ

$$\sqrt{25}+\sqrt{144}$$

Use the order of operations.	$5+12$
Simplify.	17

ⓑ

$$\sqrt{25+144}$$

Simplify under the radical sign.	$\sqrt{169}$
Simplify.	13

Notice the different answers in parts ⓐ and ⓑ!

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{9}+\sqrt{16}$ ⓑ $\sqrt{9+16}$.

Solution:

ⓐ 7 ⓑ 5

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{64+225}$ ⓑ $\sqrt{64}+\sqrt{225}$.

Solution:

ⓐ 17 ⓑ 23

Estimate Square Roots

So far we have only considered square roots of perfect square numbers. The square roots of other numbers are not whole numbers. Look at [link] below.

Number	Square Root
4	$\sqrt{4}$ = 2
5	$\sqrt{5}$
6	$\sqrt{6}$
7	$\sqrt{7}$
8	$\sqrt{8}$
9	$\sqrt{9}$ = 3

The square roots of numbers between 4 and 9 must be between the two consecutive whole numbers 2 and 3, and they are not whole numbers. Based on the pattern in the table above, we could say that $\sqrt{5}$ must be between 2 and 3. Using inequality symbols, we write:

Equation:

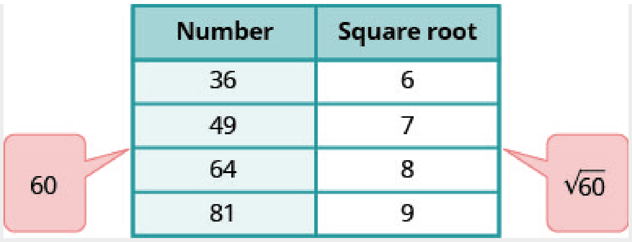
$$2 < \sqrt{5} < 3$$

Example:
Exercise:

Problem: Estimate $\sqrt{60}$ between two consecutive whole numbers.

Solution:
Solution

Think of the perfect square numbers closest to 60. Make a small table of these perfect squares and their squares roots.



Number	Square root
36	6
49	7
64	8
81	9

Locate 60 between two consecutive perfect squares.	$49 < 60 < 64$
$\sqrt{60}$ is between their square roots.	$7 < \sqrt{60} < 8$

Note:
Exercise:

Problem: Estimate the square root $\sqrt{38}$ between two consecutive whole numbers.

Solution:

$6 < \sqrt{38} < 7$

Note:
Exercise:

Problem: Estimate the square root $\sqrt{84}$ between two consecutive whole numbers.

Solution:

$9 < \sqrt{84} < 10$

Approximate Square Roots

There are mathematical methods to approximate square roots, but nowadays most people use a calculator to find them. Find the $\sqrt{x}$ key on your calculator. You will use this key to approximate square roots.

When you use your calculator to find the square root of a number that is not a perfect square, the answer that you see is not the exact square root. It is an approximation, accurate to the number of digits shown on your calculator's display. The symbol for an approximation is $\approx$ and it is read 'approximately.'

Suppose your calculator has a 10-digit display. You would see that
Equation:

$$\sqrt{5} \approx 2.236067978$$

If we wanted to round $\sqrt{5}$ to two decimal places, we would say
Equation:

$$\sqrt{5} \approx 2.24$$

How do we know these values are approximations and not the exact values? Look at what happens when we square them:
Equation:

$$\begin{aligned} (2.236067978)^2 &= 5.000000002 \\ (2.24)^2 &= 5.0176 \end{aligned}$$

Their squares are close to 5, but are not exactly equal to 5.

Using the square root key on a calculator and then rounding to two decimal places, we can find:
Equation:

$$\begin{aligned} \sqrt{4} &= 2 \\ \sqrt{5} &\approx 2.24 \\ \sqrt{6} &\approx 2.45 \\ \sqrt{7} &\approx 2.65 \\ \sqrt{8} &\approx 2.83 \\ \sqrt{9} &= 3 \end{aligned}$$

Example:
Exercise:

Problem: Round $\sqrt{17}$ to two decimal places.

Solution:
Solution

	$\sqrt{17}$
Use the calculator square root key.	$4.123105626\ldots$
Round to two decimal places.	4.12
	$\sqrt{17} \approx 4.12$

Note:
Exercise:

Problem: Round $\sqrt{11}$ to two decimal places.

Solution:

≈ 3.32

Note:
Exercise:

Problem: Round $\sqrt{13}$ to two decimal places.

Solution:

≈ 3.61

Simplify Variable Expressions with Square Roots

What if we have to find a square root of an expression with a variable? Consider $\sqrt{9x^2}$. Can you think of an expression whose square is $9x^2$?
Equation:

$$\begin{aligned}(?)^2 &= 9x^2 \\ (3x)^2 &= 9x^2, \qquad \text{so } \sqrt{9x^2} = 3x\end{aligned}$$

When we use the radical sign to take the square root of a variable expression, we should specify that $x \geq 0$ to make sure we get the *principal square root*.

However, in this chapter we will assume that each variable in a square-root expression represents a non-negative number and so we will not write $x \geq 0$ next to every radical.

What about square roots of higher powers of variables? Think about the Power Property of Exponents we used in Chapter 6.
Equation:

$$(a^m)^n = a^{m \cdot n}$$

If we square a^m, the exponent will become $2m$.
Equation:

$$(a^m)^2 = a^{2m}$$

How does this help us take square roots? Let's look at a few:
Equation:

$$\begin{aligned}\sqrt{25u^8} &= 5u^4 \quad \text{because } \left(5u^4\right)^2 = 25u^8 \\ \sqrt{16r^{20}} &= 4r^{10} \quad \text{because } \left(4r^{10}\right)^2 = 16r^{20} \\ \sqrt{196q^{36}} &= 14q^{18} \quad \text{because } \left(14q^{18}\right)^2 = 196q^{36}\end{aligned}$$

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt{x^6}$ ⓑ $\sqrt{y^{16}}$.

Solution:
Solution

ⓐ

	$\sqrt{x^6}$
Since $\left(x^3\right)^2 = x^6$.	x^3

ⓑ

	$\sqrt{y^{16}}$
Since $\left(y^8\right)^2 = y^{16}$.	y^8

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{y^8}$ ⓑ $\sqrt{z^{12}}$.

Solution:

ⓐ y^4 ⓑ z^6

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{m^4}$ ⓑ $\sqrt{b^{10}}$.

Solution:

ⓐ m^2 ⓑ b^5

Example:
Exercise:

Problem: Simplify: $\sqrt{16n^2}$.

Solution:
Solution

	$\sqrt{16n^2}$
Since $(4n)^2 = 16n^2$.	$4n$

Note:
Exercise:

Problem: Simplify: $\sqrt{64x^2}$.

Solution:

$8x$

Note:
Exercise:

Problem: Simplify: $\sqrt{169y^2}$.

Solution:

$13y$

Example:
Exercise:

Problem: Simplify: $-\sqrt{81c^2}$.

Solution:
Solution

	$-\sqrt{81c^2}$
Since $(9c)^2 = 81c^2$.	$-9c$

Note:
Exercise:

Problem: Simplify: $-\sqrt{121y^2}$.

Solution:

$-11y$

Note:
Exercise:

Problem: Simplify: $-\sqrt{100p^2}$.

Solution:

$-10p$

Example:
Exercise:

Problem: Simplify: $\sqrt{36x^2y^2}$.

Solution:
Solution

	$\sqrt{36x^2y^2}$
Since $(6xy)^2 = 36x^2y^2$.	$6xy$

Note:
Exercise:

Problem: Simplify: $\sqrt{100a^2b^2}$.

Solution:

$10ab$

Note:
Exercise:

Problem: Simplify: $\sqrt{225m^2n^2}$.

Solution:

$15mn$

Example:
Exercise:

Problem: Simplify: $\sqrt{64p^{64}}$.

Solution:
Solution

	$\sqrt{64p^{64}}$
Since $\left(8p^{32}\right)^2 = 64p^{64}$.	$8p^{32}$

Note:
Exercise:

Problem: Simplify: $\sqrt{49x^{30}}$.

Solution:

$7x^{15}$

Note:
Exercise:

Problem: Simplify: $\sqrt{81w^{36}}$.

Solution:

$9w^{18}$

Example:
Exercise:

Problem: Simplify: $\sqrt{121a^6b^8}$

Solution:
Solution

	$\sqrt{121a^6b^8}$
Since $\left(11a^3b^4\right)^2 = 121a^6b^8$.	$11a^3b^4$

Note:
Exercise:

Problem: Simplify: $\sqrt{169x^{10}y^{14}}$.

Solution:

$13x^5y^7$

Note:
Exercise:

Problem: Simplify: $\sqrt{144p^{12}q^{20}}$.

Solution:

$12p^6q^{10}$

Note:
Access this online resource for additional instruction and practice with square roots.

- Square Roots

Key Concepts

- Note that the square root of a negative number is not a real number.
- Every positive number has two square roots, one positive and one negative. The positive square root of a positive number is the principal square root.
- We can estimate square roots using nearby perfect squares.
- We can approximate square roots using a calculator.
- When we use the radical sign to take the square root of a variable expression, we should specify that $x \geq 0$ to make sure we get the principal square root.

Practice Makes Perfect

Simplify Expressions with Square Roots

In the following exercises, simplify.

Exercise:

Problem: $\sqrt{36}$

Solution:

6

Exercise:

Problem: $\sqrt{4}$

Exercise:

Problem: $\sqrt{64}$

Solution:

8

Exercise:

Problem: $\sqrt{169}$

Exercise:

Problem: $\sqrt{9}$

Solution:

3

Exercise:

Problem: $\sqrt{16}$

Exercise:

Problem: $\sqrt{100}$

Solution:

10

Exercise:

Problem: $\sqrt{144}$

Exercise:

Problem: $-\sqrt{4}$

Solution:

-2

Exercise:

Problem: $-\sqrt{100}$

Exercise:

Problem: $-\sqrt{1}$

Solution:

-1

Exercise:

Problem: $-\sqrt{121}$

Exercise:

Problem: $\sqrt{-121}$

Solution:

not a real number

Exercise:

Problem: $\sqrt{-36}$

Exercise:

Problem: $\sqrt{-9}$

Solution:

not a real number

Exercise:

Problem: $\sqrt{-49}$

Exercise:

Problem: $\sqrt{9+16}$

Solution:

5

Exercise:

Problem: $\sqrt{25+144}$

Exercise:

Problem: $\sqrt{9}+\sqrt{16}$

Solution:

7

Exercise:

Problem: $\sqrt{25}+\sqrt{144}$

Estimate Square Roots

In the following exercises, estimate each square root between two consecutive whole numbers.

Exercise:

Problem: $\sqrt{70}$

Solution:

$8 < \sqrt{70} < 9$

Exercise:

Problem: $\sqrt{55}$

Exercise:

Problem: $\sqrt{200}$

Solution:

$14 < \sqrt{200} < 15$

Exercise:

Problem: $\sqrt{172}$

Approximate Square Roots

In the following exercises, approximate each square root and round to two decimal places.

Exercise:

Problem: $\sqrt{19}$

Solution:

4.36

Exercise:

Problem: $\sqrt{21}$

Exercise:

Problem: $\sqrt{53}$

Solution:

7.28

Exercise:

Problem: $\sqrt{47}$

Simplify Variable Expressions with Square Roots

In the following exercises, simplify.

Exercise:

Problem: $\sqrt{y^2}$

Solution:

y

Exercise:

Problem: $\sqrt{b^2}$

Exercise:

Problem: $\sqrt{a^{14}}$

Solution:

a^7

Exercise:

Problem: $\sqrt{w^{24}}$

Exercise:

Problem: $\sqrt{49x^2}$

Solution:

$7x$

Exercise:

Problem: $\sqrt{100y^2}$

Exercise:

Problem: $\sqrt{121m^{20}}$

Solution:

$11m^{10}$

Exercise:

Problem: $\sqrt{25h^{44}}$

Exercise:

Problem: $\sqrt{81x^{36}}$

Solution:

$9x^{18}$

Exercise:

Problem: $\sqrt{144z^{84}}$

Exercise:

Problem: $-\sqrt{81x^{18}}$

Solution:

$-9x^{9}$

Exercise:

Problem: $-\sqrt{100m^{32}}$

Exercise:

Problem: $-\sqrt{64a^{2}}$

Solution:

$-8a$

Exercise:

Problem: $-\sqrt{25x^{2}}$

Exercise:

Problem: $\sqrt{144x^2y^2}$

Solution:

$12xy$

Exercise:

Problem: $\sqrt{196a^2b^2}$

Exercise:

Problem: $\sqrt{169w^8y^{10}}$

Solution:

$13w^4y^5$

Exercise:

Problem: $\sqrt{81p^{24}q^6}$

Exercise:

Problem: $\sqrt{9c^8d^{12}}$

Solution:

$3c^4d^6$

Exercise:

Problem: $\sqrt{36r^6s^{20}}$

Everyday Math

Exercise:

Problem:

Decorating Denise wants to have a square accent of designer tiles in her new shower. She can afford to buy 625 square centimeters of the designer tiles. How long can a side of the accent be?

Solution:

25 centimeters

Exercise:

Problem:

Decorating Morris wants to have a square mosaic inlaid in his new patio. His budget allows for 2025 square inch tiles. How long can a side of the mosaic be?

Writing Exercises

Exercise:

Problem: Why is there no real number equal to $\sqrt{-64}$?

Solution:

Answers will vary.

Exercise:

Problem: What is the difference between 9^2 and $\sqrt{9}$?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
simplify expressions with square roots.			
estimate square roots.			
approximate square roots.			
simplify variable expressions with square roots.			

ⓑ On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Glossary

square of a number

- If $n^2 = m$, then m is the square of n

square root of a number

- If $n^2 = m$, then n is a square root of m

square root notation

- If $m = n^2$, then $\sqrt{m} = n$. We read $\sqrt{m}$ as ‘the square root of m.’

Simplify Square Roots: ASE
By the end of this section, you will be able to:

- Use the Product Property to simplify square roots
- Use the Quotient Property to simplify square roots

With computers and calculators it is rare that we need to simplify a square root just to obtain a numerical approximation. But simplifying a square root can also result in an easier to understand and use expression so it is still worthwhile.

A square root is considered *simplified* if its radicand contains no perfect square factors.

Note:
Simplified Square Root
$\sqrt{a}$ is considered simplified if a has no perfect square factors.

So $\sqrt{31}$ is simplified. But $\sqrt{32}$ is not simplified, because 16 is a perfect square factor of 32.

Use the Product Property to Simplify Square Roots

The properties we will use to simplify expressions with square roots are similar to the properties of exponents. We know that $(ab)^m = a^m b^m$. The corresponding property of square roots says that $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$.

Note:
Product Property of Square Roots
If a, b are non-negative real numbers, then $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$.

We use the Product Property of Square Roots to remove all perfect square factors from a radical. We will show how to do this in [link].

Example:
How To Use the Product Property to Simplify a Square Root
Exercise:

Problem: Simplify: $\sqrt{50}$.

Solution:
Solution

Step 1. Find the largest perfect square factor of the radicand. Rewrite the radicand as a product using the perfect square factor.	25 is the largest perfect square factor of 50. $50 = 25 \cdot 2$ Always write the perfect square factor first.	$\sqrt{50}$ $\sqrt{25 \cdot 2}$
Step 2. Use the product rule to rewrite the radical as the product of two radicals.		$\sqrt{25} \cdot \sqrt{2}$
Step 3. Simplify the square root of the perfect square.		$5\sqrt{2}$

Note:
Exercise:

Problem: Simplify: $\sqrt{48}$.

Solution:

$4\sqrt{3}$

Note:
Exercise:

Problem: Simplify: $\sqrt{45}$.

Solution:

$3\sqrt{5}$

Notice in the previous example that the simplified form of $\sqrt{50}$ is $5\sqrt{2}$, which is the product of an integer and a square root. We always write the integer in front of the square root.

Note:
Simplify a square root using the product property.

Find the largest perfect square factor of the radicand. Rewrite the radicand as a product using the perfect-square factor.
Use the product rule to rewrite the radical as the product of two radicals.
Simplify the square root of the perfect square.

Example:
Exercise:

Problem: Simplify: $\sqrt{500}$.

Solution:
Solution

	$\sqrt{500}$
Rewrite the radicand as a product using the largest perfect square factor.	$\sqrt{100 \cdot 5}$
Rewrite the radical as the product of two radicals.	$\sqrt{100} \cdot \sqrt{5}$
Simplify.	$10\sqrt{5}$

Note:
Exercise:

Problem: Simplify: $\sqrt{288}$.

Solution:

$12\sqrt{2}$

Note:
Exercise:

Problem: Simplify: $\sqrt{432}$.

Solution:

$12\sqrt{3}$

We could use the simplified form $10\sqrt{5}$ to estimate $\sqrt{500}$. We know 5 is between 2 and 3, and $\sqrt{500}$ is $10\sqrt{5}$. So $\sqrt{500}$ is between 20 and 30.

The next example is much like the previous examples, but with variables.

Example:
Exercise:

Problem: Simplify: $\sqrt{x^3}$.

Solution:
Solution

	$\sqrt{x^3}$
Rewrite the radicand as a product using the largest perfect square factor.	$\sqrt{x^2 \cdot x}$
Rewrite the radical as the product of two radicals.	$\sqrt{x^2} \cdot \sqrt{x}$
Simplify.	$x\sqrt{x}$

Note:
Exercise:

Problem: Simplify: $\sqrt{b^5}$.

Solution:

$b^2\sqrt{b}$

Note:
Exercise:

Problem: Simplify: $\sqrt{p^9}$.

Solution:

$p^4\sqrt{p}$

We follow the same procedure when there is a coefficient in the radical, too.

Example:
Exercise:

Problem: Simplify: $\sqrt{25y^5}$.

Solution:
Solution

	$\sqrt{25y^5}$
Rewrite the radicand as a product using the largest perfect square factor.	$\sqrt{25y^4 \cdot y}$
Rewrite the radical as the product of two radicals.	$\sqrt{25y^4} \cdot \sqrt{y}$
Simplify.	$5y^2\sqrt{y}$

Note:
Exercise:

Problem: Simplify: $\sqrt{16x^7}$.

Solution:

$4x^3\sqrt{x}$

Note:
Exercise:

Problem: Simplify: $\sqrt{49v^9}$.

Solution:

$7v^4\sqrt{v}$

In the next example both the constant and the variable have perfect square factors.

Example:
Exercise:

Problem: Simplify: $\sqrt{72n^7}$.

Solution:
Solution

	$\sqrt{72n^7}$
Rewrite the radicand as a product using the largest perfect square factor.	$\sqrt{36n^6 \cdot 2n}$
Rewrite the radical as the product of two radicals.	$\sqrt{36n^6} \cdot \sqrt{2n}$
Simplify.	$6n^3\sqrt{2n}$

Note:
Exercise:

Problem: Simplify: $\sqrt{32y^5}$.

Solution:

$4y^2\sqrt{2y}$

Note:
Exercise:

Problem: Simplify: $\sqrt{75a^9}$.

Solution:

$5a^4\sqrt{3a}$

Example:
Exercise:

Problem: Simplify: $\sqrt{63u^3v^5}$.

Solution:
Solution

	$\sqrt{63u^3v^5}$
Rewrite the radicand as a product using the largest perfect square factor.	$\sqrt{9u^2v^4 \cdot 7uv}$
Rewrite the radical as the product of two radicals.	$\sqrt{9u^2v^4} \cdot \sqrt{7uv}$
Simplify.	$3uv^2\sqrt{7uv}$

Note:
Exercise:

Problem: Simplify: $\sqrt{98a^7b^5}$.

Solution:

$7a^3b^2\sqrt{2ab}$

Note:
Exercise:

Problem: Simplify: $\sqrt{180m^9n^{11}}$.

Solution:

$6m^4n^5\sqrt{5mn}$

We have seen how to use the Order of Operations to simplify some expressions with radicals. To simplify $\sqrt{25}+\sqrt{144}$ we must simplify each square root separately first, then add to get the sum of 17.

The expression $\sqrt{17}+\sqrt{7}$ cannot be simplified—to begin we'd need to simplify each square root, but neither 17 nor 7 contains a perfect square factor.

In the next example, we have the sum of an integer and a square root. We simplify the square root but cannot add the resulting expression to the integer.

Example:
Exercise:

Problem: Simplify: $3+\sqrt{32}$.

Solution:
Solution

	$3+\sqrt{32}$
Rewrite the radicand as a product using the largest perfect square factor.	$3+\sqrt{16\cdot 2}$
Rewrite the radical as the product of two radicals.	$3+\sqrt{16}\cdot\sqrt{2}$
Simplify.	$3+4\sqrt{2}$

The terms are not like and so we cannot add them. Trying to add an integer and a radical is like trying to add an integer and a variable—

they are not like terms!

Note:
Exercise:

Problem: Simplify: $5 + \sqrt{75}$.

Solution:

$5 + 5\sqrt{3}$

Note:
Exercise:

Problem: Simplify: $2 + \sqrt{98}$.

Solution:

$2 + 7\sqrt{2}$

The next example includes a fraction with a radical in the numerator. Remember that in order to simplify a fraction you need a common factor in the numerator and denominator.

Example:
Exercise:

Problem: Simplify: $\frac{4-\sqrt{48}}{2}$.

Solution:
Solution

	$\frac{4-\sqrt{48}}{2}$
Rewrite the radicand as a product using the largest perfect square factor.	$\frac{4-\sqrt{16\cdot 3}}{2}$
Rewrite the radical as the product of two radicals.	$\frac{4-\sqrt{16}\cdot\sqrt{3}}{2}$
Simplify.	$\frac{4-4\sqrt{3}}{2}$
Factor the common factor from the numerator.	$\frac{4\left(1-\sqrt{3}\right)}{2}$
Remove the common factor, 2, from the numerator and denominator.	$\frac{\cancel{2}\cdot 2\left(1-\sqrt{3}\right)}{\cancel{2}}$
Simplify.	$2\left(1-\sqrt{3}\right)$

Note:
Exercise:

Problem: Simplify: $\frac{10-\sqrt{75}}{5}$.

Solution:

$2-\sqrt{3}$

Note:
Exercise:

Problem: Simplify: $\frac{6-\sqrt{45}}{3}$.

Solution:

$2-\sqrt{5}$

Use the Quotient Property to Simplify Square Roots

Whenever you have to simplify a square root, the first step you should take is to determine whether the radicand is a perfect square. A *perfect square fraction* is a fraction in which both the numerator and the denominator are perfect squares.

Example:
Exercise:

Problem: Simplify: $\sqrt{\frac{9}{64}}$.

Solution:
Solution

	$\sqrt{\frac{9}{64}}$
Since $\left(\frac{3}{8}\right)^2=\frac{9}{64}$	$\frac{3}{8}$

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{25}{16}}$.

Solution:

$\frac{5}{4}$

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{49}{81}}$.

Solution:

$\frac{7}{9}$

If the numerator and denominator have any common factors, remove them. You may find a perfect square fraction!

Example:
Exercise:

Problem: Simplify: $\sqrt{\frac{45}{80}}$.

Solution:
Solution

$\sqrt{\frac{45}{80}}$

Simplify inside the radical first. Rewrite showing the common factors of the numerator and denominator. $\sqrt{\frac{5 \cdot 9}{5 \cdot 16}}$

Simplify the fraction by removing common factors. $\sqrt{\frac{9}{16}}$

Simplify. $\left(\frac{3}{4}\right)^2 = \frac{9}{16}$ $\frac{3}{4}$

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{75}{48}}$.

Solution:

$\frac{5}{4}$

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{98}{162}}$.

Solution:

$\frac{7}{9}$

In the last example, our first step was to simplify the fraction under the radical by removing common factors. In the next example we will use the Quotient Property to simplify under the radical. We divide the like bases by subtracting their exponents, $\frac{a^m}{a^n} = a^{m-n}, a \neq 0$.

Example:
Exercise:

Problem: Simplify: $\sqrt{\frac{m^6}{m^4}}$.

Solution:
Solution

	$\sqrt{\frac{m^6}{m^4}}$
Simplify the fraction inside the radical first. Divide the like bases by subtracting the exponents.	$\sqrt{m^2}$
Simplify.	m

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{a^8}{a^6}}$.

Solution:

a

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{x^{14}}{x^{10}}}$.

Solution:

x^2

Example:
Exercise:

Problem: Simplify: $\sqrt{\frac{48p^7}{3p^3}}$.

Solution:
Solution

	$\sqrt{\frac{48p^7}{3p^3}}$
Simplify the fraction inside the radical first.	$\sqrt{16p^4}$
Simplify.	$4p^2$

Note:

Exercise:

Problem: Simplify: $\sqrt{\frac{75x^5}{3x}}$.

Solution:

$5x^2$

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{72z^{12}}{2z^{10}}}$.

Solution:

$6z$

Remember the Quotient to a Power Property? It said we could raise a fraction to a power by raising the numerator and denominator to the power separately.

Equation:

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$$

We can use a similar property to simplify a square root of a fraction. After removing all common factors from the numerator and denominator, if the fraction is not a perfect square we simplify the numerator and denominator separately.

Note:
Quotient Property of Square Roots
If a, b are non-negative real numbers and $b \neq 0$, then
Equation:

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

Example:
Exercise:

Problem: Simplify: $\sqrt{\frac{21}{64}}$.

Solution:
Solution

	$\sqrt{\frac{21}{64}}$
We cannot simplify the fraction inside the radical. Rewrite using the quotient property.	$\frac{\sqrt{21}}{\sqrt{64}}$
Simplify the square root of 64. The numerator cannot be simplified.	$\frac{\sqrt{21}}{8}$

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{19}{49}}$.

Solution:

$\frac{\sqrt{19}}{7}$

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{28}{81}}$.

Solution:

$\frac{2\sqrt{7}}{9}$

Example:
How to Use the Quotient Property to Simplify a Square Root
Exercise:

Problem: Simplify: $\sqrt{\frac{27m^3}{196}}$.

Solution:
Solution

Step 1. Simplify the fraction in the radicand, if possible.	$\frac{27m^3}{196}$ cannot be simplified.	$\sqrt{\frac{27m^3}{196}}$

Step 2. Use the Quotient Property to rewrite the radical as the quotient of two radicals.	We rewrite $\sqrt{\frac{27m^3}{196}}$ as the quotient of $\sqrt{27m^3}$ and $\sqrt{196}$.	$\frac{\sqrt{27m^3}}{\sqrt{196}}$

Step 3. Simplify the radicals in the numerator and the denominator.	$9m^2$ and 196 are perfect squares.	$\frac{\sqrt{9m^2} \cdot \sqrt{3m}}{\sqrt{196}}$ $\frac{3m\sqrt{3m}}{14}$

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{24p^3}{49}}$.

Solution:

$\frac{2p\sqrt{6p}}{7}$

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{48x^5}{100}}$.

Solution:

$\frac{2x^2\sqrt{3x}}{5}$

Note:
Simplify a square root using the quotient property.

Simplify the fraction in the radicand, if possible.
Use the Quotient Property to rewrite the radical as the quotient of two radicals.
Simplify the radicals in the numerator and the denominator.

Example:
Exercise:

Problem: Simplify: $\sqrt{\frac{45x^5}{y^4}}$.

Solution:
Solution

	$\sqrt{\frac{45x^5}{y^4}}$
We cannot simplify the fraction in the radicand. Rewrite using the Quotient Property.	$\frac{\sqrt{45x^5}}{\sqrt{y^4}}$
Simplify the radicals in the numerator and the denominator.	$\frac{\sqrt{9x^4}\cdot\sqrt{5x}}{y^2}$
Simplify.	$\frac{3x^2\sqrt{5x}}{y^2}$

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{80m^3}{n^6}}$.

Solution:

$\frac{4m\sqrt{5m}}{n^3}$

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{54u^7}{v^8}}$.

Solution:

$\frac{3u^3\sqrt{6u}}{v^4}$

Be sure to simplify the fraction in the radicand first, if possible.

Example:
Exercise:

Problem: Simplify: $\sqrt{\frac{81d^9}{25d^4}}$.

Solution:
Solution

	$\sqrt{\frac{81d^9}{25d^4}}$
Simplify the fraction in the radicand.	$\sqrt{\frac{81d^5}{25}}$
Rewrite using the Quotient Property.	$\frac{\sqrt{81d^5}}{\sqrt{25}}$
Simplify the radicals in the numerator and the denominator.	$\frac{\sqrt{81d^4}\cdot\sqrt{d}}{5}$
Simplify.	$\frac{9d^2\sqrt{d}}{5}$

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{64x^7}{9x^3}}$.

Solution:

$\frac{8x^2}{3}$

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{16a^9}{100a^5}}$.

Solution:

$\frac{2a^2}{5}$

Example:
Exercise:

Problem: Simplify: $\sqrt{\frac{18p^5q^7}{32pq^2}}$.

Solution:
Solution

	$\sqrt{\frac{18p^5q^7}{32pq^2}}$
Simplify the fraction in the radicand, if possible.	$\sqrt{\frac{9p^4q^5}{16}}$
Rewrite using the Quotient Property.	$\frac{\sqrt{9p^4q^5}}{\sqrt{16}}$
Simplify the radicals in the numerator and the denominator.	$\frac{\sqrt{9p^4q^4}\cdot\sqrt{q}}{4}$
Simplify.	$\frac{3p^2q^2\sqrt{q}}{4}$

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{50x^5y^3}{72x^4y}}$.

Solution:

$\frac{5y\sqrt{x}}{6}$

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{48m^7n^2}{125m^5n^9}}$.

Solution:

$\frac{4m\sqrt{3}}{5n^3\sqrt{5n}}$

Key Concepts

- **Simplified Square Root** $\sqrt{a}$ is considered simplified if a has no perfect-square factors.
- **Product Property of Square Roots** If a, b are non-negative real numbers, then
  Equation:

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$$

- **Simplify a Square Root Using the Product Property** To simplify a square root using the Product Property:

  Find the largest perfect square factor of the radicand. Rewrite the radicand as a product using the perfect square factor.
  Use the product rule to rewrite the radical as the product of two radicals.
  Simplify the square root of the perfect square.

- **Quotient Property of Square Roots** If a, b are non-negative real numbers and $b \neq 0$, then
  Equation:

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

- **Simplify a Square Root Using the Quotient Property** To simplify a square root using the Quotient Property:

  Simplify the fraction in the radicand, if possible.
  Use the Quotient Rule to rewrite the radical as the quotient of two radicals.
  Simplify the radicals in the numerator and the denominator.

Practice Makes Perfect

Use the Product Property to Simplify Square Roots

In the following exercises, simplify.
Exercise:

Problem: $\sqrt{27}$

Solution:

$3\sqrt{3}$

Exercise:

Problem: $\sqrt{80}$

Exercise:

Problem: $\sqrt{125}$

Solution:

$5\sqrt{5}$

Exercise:

Problem: $\sqrt{96}$

Exercise:

Problem: $\sqrt{200}$

Solution:

$10\sqrt{2}$

Exercise:

Problem: $\sqrt{147}$

Exercise:

Problem: $\sqrt{450}$

Solution:

$15\sqrt{2}$

Exercise:

Problem: $\sqrt{252}$

Exercise:

Problem: $\sqrt{800}$

Solution:

$20\sqrt{2}$

Exercise:

Problem: $\sqrt{288}$

Exercise:

Problem: $\sqrt{675}$

Solution:

$15\sqrt{3}$

Exercise:

Problem: $\sqrt{1250}$

Exercise:

Problem: $\sqrt{x^7}$

Solution:

$x^3\sqrt{x}$

Exercise:

Problem: $\sqrt{y^{11}}$

Exercise:

Problem: $\sqrt{p^3}$

Solution:

$p\sqrt{p}$

Exercise:

Problem: $\sqrt{q^5}$

Exercise:

Problem: $\sqrt{m^{13}}$

Solution:

$m^6\sqrt{m}$

Exercise:

Problem: $\sqrt{n^{21}}$

Exercise:

Problem: $\sqrt{r^{25}}$

Solution:

$r^{12}\sqrt{r}$

Exercise:

Problem: $\sqrt{s^{33}}$

Exercise:

Problem: $\sqrt{49n^{17}}$

Solution:

$7n^8\sqrt{n}$

Exercise:

Problem: $\sqrt{25m^9}$

Exercise:

Problem: $\sqrt{81r^{15}}$

Solution:

$9r^7\sqrt{r}$

Exercise:

Problem: $\sqrt{100s^{19}}$

Exercise:

Problem: $\sqrt{98m^5}$

Solution:

$7m^2\sqrt{2m}$

Exercise:

Problem: $\sqrt{32n^{11}}$

Exercise:

Problem: $\sqrt{125r^{13}}$

Solution:

$5r^6\sqrt{5r}$

Exercise:

Problem: $\sqrt{80s^{15}}$

Exercise:

Problem: $\sqrt{200p^{13}}$

Solution:

$10p^{6}\sqrt{2p}$

Exercise:

Problem: $\sqrt{128q^{3}}$

Exercise:

Problem: $\sqrt{242m^{23}}$

Solution:

$11m^{11}\sqrt{2m}$

Exercise:

Problem: $\sqrt{175n^{13}}$

Exercise:

Problem: $\sqrt{147m^{7}n^{11}}$

Solution:

$7m^{3}n^{5}\sqrt{3mn}$

Exercise:

Problem: $\sqrt{48m^7n^5}$

Exercise:

Problem: $\sqrt{75r^{13}s^9}$

Solution:

$5r^6s^4\sqrt{3rs}$ 70)

Exercise:

Problem: $\sqrt{96r^3s^3}$

Exercise:

Problem: $\sqrt{300p^9q^{11}}$

Solution:

$10p^4q^5\sqrt{3pq}$

Exercise:

Problem: $\sqrt{192q^3r^7}$

Exercise:

Problem: $\sqrt{242m^{13}n^{21}}$

Solution:

$11m^6n^{10}\sqrt{2mn}$

Exercise:

Problem: $\sqrt{150m^9n^3}$

Exercise:

Problem: $5+\sqrt{12}$

Solution:

$5+2\sqrt{3}$

Exercise:

Problem: $8+\sqrt{96}$

Exercise:

Problem: $1+\sqrt{45}$

Solution:

$1+3\sqrt{5}$

Exercise:

Problem: $3+\sqrt{125}$

Exercise:

Problem: $\frac{10-\sqrt{24}}{2}$

Solution:

$5-2\sqrt{6}$

Exercise:

Problem: $\frac{8-\sqrt{80}}{4}$

Exercise:

Problem: $\frac{3+\sqrt{90}}{3}$

Solution:

$1+\sqrt{10}$

Exercise:

Problem: $\frac{15+\sqrt{75}}{5}$

Use the Quotient Property to Simplify Square Roots

In the following exercises, simplify.

Exercise:

Problem: $\sqrt{\frac{49}{64}}$

Solution:

$\frac{7}{8}$

Exercise:

Problem: $\sqrt{\frac{100}{36}}$

Exercise:

Problem: $\sqrt{\frac{121}{16}}$

Solution:

$\frac{11}{4}$

Exercise:

Problem: $\sqrt{\frac{144}{169}}$

Exercise:

Problem: $\sqrt{\frac{72}{98}}$

Solution:

$\frac{6}{7}$

Exercise:

Problem: $\sqrt{\frac{75}{12}}$

Exercise:

Problem: $\sqrt{\frac{45}{125}}$

Solution:

$\frac{3}{5}$

Exercise:

Problem: $\sqrt{\frac{300}{243}}$

Exercise:

Problem: $\sqrt{\frac{x^{10}}{x^6}}$

Solution:

x^2

Exercise:

Problem: $\sqrt{\frac{p^{20}}{p^{10}}}$

Exercise:

Problem: $\sqrt{\frac{y^4}{y^8}}$

Solution:

$\frac{1}{y^2}$

Exercise:

Problem: $\sqrt{\frac{q^8}{q^{14}}}$

Exercise:

Problem: $\sqrt{\frac{200x^7}{2x^3}}$

Solution:

$10x^2$

Exercise:

Problem: $\sqrt{\frac{98y^{11}}{2y^5}}$

Exercise:

Problem: $\sqrt{\frac{96p^9}{6p}}$

Solution:

$4p^4$

Exercise:

Problem: $\sqrt{\frac{108q^{10}}{3q^2}}$

Exercise:

Problem: $\sqrt{\frac{36}{35}}$

Solution:

$\frac{6}{\sqrt{35}}$

Exercise:

Problem: $\sqrt{\frac{144}{65}}$

Exercise:

Problem: $\sqrt{\frac{20}{81}}$

Solution:

$\frac{2\sqrt{5}}{9}$

Exercise:

Problem: $\sqrt{\frac{21}{196}}$

Exercise:

Problem: $\sqrt{\frac{96x^7}{121}}$

Solution:

$\frac{4x^3\sqrt{6x}}{11}$

Exercise:

Problem: $\sqrt{\frac{108y^4}{49}}$

Exercise:

Problem: $\sqrt{\frac{300m^5}{64}}$

Solution:

$\frac{10m^2\sqrt{3m}}{8}$

Exercise:

Problem: $\sqrt{\frac{125n^7}{169}}$

Exercise:

Problem: $\sqrt{\frac{98r^5}{100}}$

Solution:

$\frac{7r^2\sqrt{2r}}{10}$

Exercise:

Problem: $\sqrt{\frac{180s^{10}}{144}}$

Exercise:

Problem: $\sqrt{\frac{28q^6}{225}}$

Solution:

$\frac{2q^3\sqrt{7}}{15}$

Exercise:

Problem: $\sqrt{\frac{150r^3}{256}}$

Exercise:

Problem: $\sqrt{\frac{75r^9}{s^8}}$

Solution:

$\frac{5r^4\sqrt{3r}}{s^4}$

Exercise:

Problem: $\sqrt{\frac{72x^5}{y^6}}$

Exercise:

Problem: $\sqrt{\frac{28p^7}{q^2}}$

Solution:

$$\frac{4p^3\sqrt{7p}}{q}$$

Exercise:

Problem: $\sqrt{\frac{45r^3}{s^{10}}}$

Exercise:

Problem: $\sqrt{\frac{100x^5}{36x^3}}$

Solution:

$$\frac{5x}{3}$$

Exercise:

Problem: $\sqrt{\frac{49r^{12}}{16r^6}}$

Exercise:

Problem: $\sqrt{\frac{121p^5}{81p^2}}$

Solution:

$$\frac{11p\sqrt{p}}{9}$$

Exercise:

Problem: $\sqrt{\frac{25r^8}{64r}}$

Exercise:

Problem: $\sqrt{\frac{32x^5y^3}{18x^3y}}$

Solution:

$\frac{4xy}{3}$

Exercise:

Problem: $\sqrt{\frac{75r^6s^8}{48rs^4}}$

Exercise:

Problem: $\sqrt{\frac{27p^2q}{108p^5q^3}}$

Solution:

$\frac{1}{2pq\sqrt{p}}$

Exercise:

Problem: $\sqrt{\frac{50r^5s^2}{128r^2s^5}}$

Everyday Math

Exercise:

Problem:

ⓐ Elliott decides to construct a square garden that will take up 288 square feet of his yard. Simplify $\sqrt{288}$ to determine the length and the width of his garden. Round to the nearest tenth of a foot.

ⓑ Suppose Elliott decides to reduce the size of his square garden so that he can create a 5-foot-wide walking path on the north and east sides of the garden. Simplify $\sqrt{288} - 5$ to determine the length and width of the new garden. Round to the nearest tenth of a foot.

Solution:

ⓐ 17.0 feet ⓑ 15.0 feet

Exercise:

Problem:

ⓐ Melissa accidentally drops a pair of sunglasses from the top of a roller coaster, 64 feet above the ground. Simplify $\sqrt{\frac{64}{16}}$ to determine the number of seconds it takes for the sunglasses to reach the ground.
ⓑ Suppose the sunglasses in the previous example were dropped from a height of 144 feet. Simplify $\sqrt{\frac{144}{16}}$ to determine the number of seconds it takes for the sunglasses to reach the ground.

Writing Exercises

Exercise:

Problem: Explain why $\sqrt{x^4} = x^2$. Then explain why $\sqrt{x^{16}} = x^8$.

Solution:

Answers will vary.

Exercise:

Problem: Explain why $7+\sqrt{9}$ is not equal to $\sqrt{7+9}$.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
use the Product Property to simplify square roots.			
use the Quotient Property to simplify square roots.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

Add and Subtract Square Roots: ASE
By the end of this section, you will be able to:

- Add and subtract like square roots
- Add and subtract square roots that need simplification

We know that we must follow the order of operations to simplify expressions with square roots. The radical is a grouping symbol, so we work inside the radical first. We simplify $\sqrt{2+7}$ in this way:

	$\sqrt{2+7}$
Add inside the radical.	$\sqrt{9}$
Simplify.	3

So if we have to add $\sqrt{2}+\sqrt{7}$, we must not combine them into one radical.
Equation:

$$\sqrt{2}+\sqrt{7}\neq\sqrt{2+7}$$

Trying to add square roots with different radicands is like trying to add unlike terms.

But, just like we can add	$x+x,$	we can add	$\sqrt{3}+\sqrt{3}.$
	$x+x=2x$		$\sqrt{3}+\sqrt{3}=2\sqrt{3}$

Adding square roots with the same radicand is just like adding like terms. We call square roots with the same radicand like square roots to remind us they work the same as like terms.

Note:
Like Square Roots
Square roots with the same radicand are called **like square roots**.

We add and subtract like square roots in the same way we add and subtract like terms. We know that $3x+8x$ is $11x$. Similarly we add $3\sqrt{x}+8\sqrt{x}$ and the result is $11\sqrt{x}$.

Add and Subtract Like Square Roots

Think about adding like terms with variables as you do the next few examples. When you have like radicands, you just add or subtract the coefficients. When the radicands are not like, you cannot combine the terms.

Example:
Exercise:

Problem: Simplify: $2\sqrt{2} - 7\sqrt{2}$.

Solution:
Solution

	$2\sqrt{2} - 7\sqrt{2}$
Since the radicals are like, we subtract the coefficients.	$-5\sqrt{2}$

Note:
Exercise:

Problem: Simplify: $8\sqrt{2} - 9\sqrt{2}$.

Solution:

$-\sqrt{2}$

Note:
Exercise:

Problem: Simplify: $5\sqrt{3} - 9\sqrt{3}$.

Solution:

$-4\sqrt{3}$

Example:
Exercise:

Problem: Simplify: $3\sqrt{y}+4\sqrt{y}$.

Solution:
Solution

	$3\sqrt{y}+4\sqrt{y}$
Since the radicals are like, we add the coefficients.	$7\sqrt{y}$

Note:
Exercise:

Problem: Simplify: $2\sqrt{x}+7\sqrt{x}$.

Solution:

$9\sqrt{x}$

Note:
Exercise:

Problem: Simplify: $5\sqrt{u}+3\sqrt{u}$.

Solution:

$8\sqrt{u}$

Example:
Exercise:

Problem: Simplify: $4\sqrt{x}-2\sqrt{y}$.

Solution:
Solution

Since the radicals are not like, we cannot subtract them. We leave the expression as is.

$$4\sqrt{x} - 2\sqrt{y}$$

$$4\sqrt{x} - 2\sqrt{y}$$

Note:
Exercise:

Problem: Simplify: $7\sqrt{p} - 6\sqrt{q}$.

Solution:

$7\sqrt{p} - 6\sqrt{q}$

Note:
Exercise:

Problem: Simplify: $6\sqrt{a} - 3\sqrt{b}$.

Solution:

$6\sqrt{a} - 3\sqrt{b}$

Example:
Exercise:

Problem: Simplify: $5\sqrt{13} + 4\sqrt{13} + 2\sqrt{13}$.

Solution:
Solution

Since the radicals are like, we add the coefficients.	$5\sqrt{13}+4\sqrt{13}+2\sqrt{13}$ $11\sqrt{13}$

Note:
Exercise:

Problem: Simplify: $4\sqrt{11}+2\sqrt{11}+3\sqrt{11}$.

Solution:

$9\sqrt{11}$

Note:
Exercise:

Problem: Simplify: $6\sqrt{10}+2\sqrt{10}+3\sqrt{10}$.

Solution:

$11\sqrt{10}$

Example:
Exercise:

Problem: Simplify: $2\sqrt{6}-6\sqrt{6}+3\sqrt{3}$.

Solution:
Solution

	$2\sqrt{6}-6\sqrt{6}+3\sqrt{3}$
Since the first two radicals are like, we subtract their coefficients.	$-4\sqrt{6}+3\sqrt{3}$

Note:
Exercise:

Problem: Simplify: $5\sqrt{5} - 4\sqrt{5} + 2\sqrt{6}$.

Solution:

$\sqrt{5} + 2\sqrt{6}$

Note:
Exercise:

Problem: Simplify: $3\sqrt{7} - 8\sqrt{7} + 2\sqrt{5}$.

Solution:

$-5\sqrt{7} + 2\sqrt{5}$

Example:
Exercise:

Problem: Simplify: $2\sqrt{5n} - 6\sqrt{5n} + 4\sqrt{5n}$.

Solution:
Solution

	$2\sqrt{5n} - 6\sqrt{5n} + 4\sqrt{5n}$
Since the radicals are like, we combine them.	$0\sqrt{5n}$
Simplify.	0

Note:

Exercise:

Problem: Simplify: $\sqrt{7x} - 7\sqrt{7x} + 4\sqrt{7x}$.

Solution:

$-2\sqrt{7x}$

Note:
Exercise:

Problem: Simplify: $4\sqrt{3y} - 7\sqrt{3y} + 2\sqrt{3y}$.

Solution:

$-\sqrt{3y}$

When radicals contain more than one variable, as long as all the variables and their exponents are identical, the radicals are like.

Example:
Exercise:

Problem: Simplify: $\sqrt{3xy} + 5\sqrt{3xy} - 4\sqrt{3xy}$.

Solution:
Solution

	$\sqrt{3xy} + 5\sqrt{3xy} - 4\sqrt{3xy}$
Since the radicals are like, we combine them.	$2\sqrt{3xy}$

Note:

Exercise:

Problem: Simplify: $\sqrt{5xy} + 4\sqrt{5xy} - 7\sqrt{5xy}$.

Solution:

$-2\sqrt{5xy}$

Note:
Exercise:

Problem: Simplify: $3\sqrt{7mn} + \sqrt{7mn} - 4\sqrt{7mn}$.

Solution:

0

Add and Subtract Square Roots that Need Simplification

Remember that we always simplify square roots by removing the largest perfect-square factor. Sometimes when we have to add or subtract square roots that do not appear to have like radicals, we find like radicals after simplifying the square roots.

Example:
Exercise:

Problem: Simplify: $\sqrt{20} + 3\sqrt{5}$.

Solution:
Solution

	$\sqrt{20}+3\sqrt{5}$
Simplify the radicals, when possible.	$\sqrt{4}\cdot\sqrt{5}+3\sqrt{5}$ $2\sqrt{5}+3\sqrt{5}$
Combine the like radicals.	$5\sqrt{5}$

Note:
Exercise:

Problem: Simplify: $\sqrt{18}+6\sqrt{2}$.

Solution:

$9\sqrt{2}$

Note:
Exercise:

Problem: Simplify: $\sqrt{27}+4\sqrt{3}$.

Solution:

$7\sqrt{3}$

Example:
Exercise:

Problem: Simplify: $\sqrt{48}-\sqrt{75}$.

Solution:
Solution

	$\sqrt{48}-\sqrt{75}$
Simplify the radicals.	$\sqrt{16}\cdot\sqrt{3}-\sqrt{25}\cdot\sqrt{3}$ $4\sqrt{3}-5\sqrt{3}$
Combine the like radicals.	$-\sqrt{3}$

Note:
Exercise:

Problem: Simplify: $\sqrt{32}-\sqrt{18}$.

Solution:

$\sqrt{2}$

Note:
Exercise:

Problem: Simplify: $\sqrt{20}-\sqrt{45}$.

Solution:

$-\sqrt{5}$

Just like we use the Associative Property of Multiplication to simplify $5\left(3x\right)$ and get $15x$, we can simplify $5\left(3\sqrt{x}\right)$ and get $15\sqrt{x}$. We will use the Associative Property to do this in the next example.

Example:
Exercise:

Problem: Simplify: $5\sqrt{18} - 2\sqrt{8}$.

Solution:
Solution

	$5\sqrt{18} - 2\sqrt{8}$
Simplify the radicals.	$5 \cdot \sqrt{9} \cdot \sqrt{2} - 2 \cdot \sqrt{4} \cdot \sqrt{2}$
	$5 \cdot 3 \cdot \sqrt{2} - 2 \cdot 2 \cdot \sqrt{2}$
	$15\sqrt{2} - 4\sqrt{2}$
Combine the like radicals.	$11\sqrt{2}$

Note:
Exercise:

Problem: Simplify: $4\sqrt{27} - 3\sqrt{12}$.

Solution:

$6\sqrt{3}$

Note:
Exercise:

Problem: Simplify: $3\sqrt{20} - 7\sqrt{45}$.

Solution:

$-15\sqrt{5}$

Example:
Exercise:

Problem: Simplify: $\frac{3}{4}\sqrt{192} - \frac{5}{6}\sqrt{108}$.

Solution:
Solution

	$\frac{3}{4}\sqrt{192} - \frac{5}{6}\sqrt{108}$
Simplify the radicals.	$\frac{3}{4}\sqrt{64} \cdot \sqrt{3} - \frac{5}{6}\sqrt{36} \cdot \sqrt{3}$
	$\frac{3}{4} \cdot 8 \cdot \sqrt{3} - \frac{5}{6} \cdot 6 \cdot \sqrt{3}$
	$6\sqrt{3} - 5\sqrt{3}$
Combine the like radicals.	$\sqrt{3}$

Note:
Exercise:

Problem: Simplify: $\frac{2}{3}\sqrt{108} - \frac{5}{7}\sqrt{147}$.

Solution:

$-\sqrt{3}$

Note:
Exercise:

Problem: Simplify: $\frac{3}{5}\sqrt{200} - \frac{3}{4}\sqrt{128}$.

Solution:

0

Example:
Exercise:

Problem: Simplify: $\frac{2}{3}\sqrt{48} - \frac{3}{4}\sqrt{12}$.

Solution:
Solution

	$\frac{2}{3}\sqrt{48} - \frac{3}{4}\sqrt{12}$
Simplify the radicals.	$\frac{2}{3}\sqrt{16} \cdot \sqrt{3} - \frac{3}{4}\sqrt{4} \cdot \sqrt{3}$
	$\frac{2}{3} \cdot 4 \cdot \sqrt{3} - \frac{3}{4} \cdot 2 \cdot \sqrt{3}$
	$\frac{8}{3}\sqrt{3} - \frac{3}{2}\sqrt{3}$
Find a common denominator to subtract the coefficients of the like radicals.	$\frac{16}{6}\sqrt{3} - \frac{9}{6}\sqrt{3}$
Simplify.	$\frac{7}{6}\sqrt{3}$

Note:
Exercise:

Problem: Simplify: $\frac{2}{5}\sqrt{32} - \frac{1}{3}\sqrt{8}$.

Solution:

$\frac{14}{15}\sqrt{2}$

Note:
Exercise:

Problem: Simplify: $\frac{1}{3}\sqrt{80} - \frac{1}{4}\sqrt{125}$.

Solution:

$\frac{1}{12}\sqrt{5}$

In the next example, we will remove constant and variable factors from the square roots.

Example:
Exercise:

Problem: Simplify: $\sqrt{18n^5} - \sqrt{32n^5}$.

Solution:
Solution

	$\sqrt{18n^5} - \sqrt{32n^5}$
Simplify the radicals.	$\sqrt{9n^4} \cdot \sqrt{2n} - \sqrt{16n^4} \cdot \sqrt{2n}$
	$3n^2\sqrt{2n} - 4n^2\sqrt{2n}$
Combine the like radicals.	$-n^2\sqrt{2n}$

Note:
Exercise:

Problem: Simplify: $\sqrt{32m^7} - \sqrt{50m^7}$.

Solution:

$-m^3\sqrt{2m}$

Note:
Exercise:

Problem: Simplify: $\sqrt{27p^3} - \sqrt{48p^3}$.

Solution:

$-p\sqrt{3p}$

Example:
Exercise:

Problem: Simplify: $9\sqrt{50m^2} - 6\sqrt{48m^2}$.

Solution:
Solution

	$9\sqrt{50m^2} - 6\sqrt{48m^2}$
Simplify the radicals.	$9\sqrt{25m^2} \cdot \sqrt{2} - 6\sqrt{16m^2} \cdot \sqrt{3}$
	$9 \cdot 5m \cdot \sqrt{2} - 6 \cdot 4m \cdot \sqrt{3}$
	$45m\sqrt{2} - 24m\sqrt{3}$
The radicals are not like and so cannot be combined.	

Note:
Exercise:

Problem: Simplify: $5\sqrt{32x^2} - 3\sqrt{48x^2}$.

Solution:

$20x\sqrt{2} - 12x\sqrt{3}$

Note:
Exercise:

Problem: Simplify: $7\sqrt{48y^2} - 4\sqrt{72y^2}$.

Solution:

$28y\sqrt{3} - 24y\sqrt{2}$

Example:
Exercise:

Problem: Simplify: $2\sqrt{8x^2} - 5x\sqrt{32} + 5\sqrt{18x^2}$.

Solution:
Solution

	$2\sqrt{8x^2} - 5x\sqrt{32} + 5\sqrt{18x^2}$
Simplify the radicals.	$2\sqrt{4x^2}\cdot\sqrt{2} - 5x\sqrt{16}\cdot\sqrt{2} + 5\sqrt{9x^2}\cdot\sqrt{2}$
	$2\cdot 2x\cdot\sqrt{2} - 5x\cdot 4\cdot\sqrt{2} + 5\cdot 3x\cdot\sqrt{2}$
	$4x\sqrt{2} - 20x\sqrt{2} + 15x\sqrt{2}$
Combine the like radicals.	$-x\sqrt{2}$

Note:
Exercise:

Problem: Simplify: $3\sqrt{12x^2} - 2x\sqrt{48} + 4\sqrt{27x^2}$.

Solution:

$10x\sqrt{3}$

Note:
Exercise:

Problem: Simplify: $3\sqrt{18x^2} - 6x\sqrt{32} + 2\sqrt{50x^2}$.

Solution:

$-5x\sqrt{2}$

Note:
Access this online resource for additional instruction and practice with the adding and subtracting square roots.

- Adding/Subtracting Square Roots

Key Concepts

- To add or subtract like square roots, add or subtract the coefficients and keep the like square root.
- Sometimes when we have to add or subtract square roots that do not appear to have like radicals, we find like radicals after simplifying the square roots.

Practice Makes Perfect

Add and Subtract Like Square Roots

In the following exercises, simplify.
Exercise:

Problem: $8\sqrt{2} - 5\sqrt{2}$

Solution:

$3\sqrt{2}$

Exercise:

Problem: $7\sqrt{2}-3\sqrt{2}$

Exercise:

Problem: $3\sqrt{5}+6\sqrt{5}$

Solution:

$9\sqrt{5}$

Exercise:

Problem: $4\sqrt{5}+8\sqrt{5}$

Exercise:

Problem: $9\sqrt{7}-10\sqrt{7}$

Solution:

$-\sqrt{7}$

Exercise:

Problem: $11\sqrt{7}-12\sqrt{7}$

Exercise:

Problem: $7\sqrt{y}+2\sqrt{y}$

Solution:

$9\sqrt{y}$

Exercise:

Problem: $9\sqrt{n}+3\sqrt{n}$

Exercise:

Problem: $\sqrt{a}-4\sqrt{a}$

Solution:

$-3\sqrt{a}$

Exercise:

Problem: $\sqrt{b}-6\sqrt{b}$

Exercise:

Problem: $5\sqrt{c}+2\sqrt{c}$

Solution:

$7\sqrt{c}$

Exercise:

Problem: $7\sqrt{d}+2\sqrt{d}$

Exercise:

Problem: $8\sqrt{a}-2\sqrt{b}$

Solution:

$6\sqrt{b}$

Exercise:

Problem: $5\sqrt{c}-3\sqrt{d}$

Exercise:

Problem: $5\sqrt{m}+\sqrt{n}$

Solution:

$5\sqrt{m}+\sqrt{n}$

Exercise:

Problem: $\sqrt{n}+3\sqrt{p}$

Exercise:

Problem: $8\sqrt{7}+2\sqrt{7}+3\sqrt{7}$

Solution:

$13\sqrt{7}$

Exercise:

Problem: $6\sqrt{5}+3\sqrt{5}+\sqrt{5}$

Exercise:

Problem: $3\sqrt{11}+2\sqrt{11}-8\sqrt{11}$

Solution:

$-3\sqrt{11}$

Exercise:

Problem: $2\sqrt{15}+5\sqrt{15}-9\sqrt{15}$

Exercise:

Problem: $3\sqrt{3}-8\sqrt{3}+7\sqrt{5}$

Solution:

$-5\sqrt{3}+7\sqrt{5}$

Exercise:

Problem: $5\sqrt{7}-8\sqrt{7}+6\sqrt{3}$

Exercise:

Problem: $6\sqrt{2}+2\sqrt{2}-3\sqrt{5}$

Solution:

$8\sqrt{2}-3\sqrt{5}$

Exercise:

Problem: $7\sqrt{5}+\sqrt{5}-8\sqrt{10}$

Exercise:

Problem: $3\sqrt{2a}-4\sqrt{2a}+5\sqrt{2a}$

Solution:

$4\sqrt{2a}$

Exercise:

Problem: $\sqrt{11b}-5\sqrt{11b}+3\sqrt{11b}$

Exercise:

Problem: $8\sqrt{3c}+2\sqrt{3c}-9\sqrt{3c}$

Solution:

$\sqrt{3c}$

Exercise:

Problem: $3\sqrt{5d}+8\sqrt{5d}-11\sqrt{5d}$

Exercise:

Problem: $5\sqrt{3ab}+\sqrt{3ab}-2\sqrt{3ab}$

Solution:

$4\sqrt{3ab}$

Exercise:

Problem: $8\sqrt{11cd}+5\sqrt{11cd}-9\sqrt{11cd}$

Exercise:

Problem: $2\sqrt{pq}-5\sqrt{pq}+4\sqrt{pq}$

Solution:

$\sqrt{pq}$

Exercise:

Problem: $11\sqrt{2rs} - 9\sqrt{2rs} + 3\sqrt{2rs}$

Add and Subtract Square Roots that Need Simplification

In the following exercises, simplify.

Exercise:

Problem: $\sqrt{50} + 4\sqrt{2}$

Solution:

$9\sqrt{2}$

Exercise:

Problem: $\sqrt{48} + 2\sqrt{3}$

Exercise:

Problem: $\sqrt{80} - 3\sqrt{5}$

Solution:

$\sqrt{5}$

Exercise:

Problem: $\sqrt{28} - 4\sqrt{7}$

Exercise:

Problem: $\sqrt{27} - \sqrt{75}$

Solution:

$-2\sqrt{3}$

Exercise:

Problem: $\sqrt{72} - \sqrt{98}$

Exercise:

Problem: $\sqrt{48} + \sqrt{27}$

Solution:

$7\sqrt{3}$

Exercise:

Problem: $\sqrt{45} + \sqrt{80}$

Exercise:

Problem: $2\sqrt{50} - 3\sqrt{72}$

Solution:

$-8\sqrt{2}$

Exercise:

Problem: $3\sqrt{98} - \sqrt{128}$

Exercise:

Problem: $2\sqrt{12} + 3\sqrt{48}$

Solution:

$16\sqrt{3}$

Exercise:

Problem: $4\sqrt{75} + 2\sqrt{108}$

Exercise:

Problem: $\frac{2}{3}\sqrt{72}+\frac{1}{5}\sqrt{50}$

Solution:

$3\sqrt{2}$

Exercise:

Problem: $\frac{2}{5}\sqrt{75}+\frac{3}{4}\sqrt{48}$

Exercise:

Problem: $\frac{1}{2}\sqrt{20}-\frac{2}{3}\sqrt{45}$

Solution:

$-\sqrt{5}$

Exercise:

Problem: $\frac{2}{3}\sqrt{54}-\frac{3}{4}\sqrt{96}$

Exercise:

Problem: $\frac{1}{6}\sqrt{27}-\frac{3}{8}\sqrt{48}$

Solution:

$-\sqrt{3}$

Exercise:

Problem: $\frac{1}{8}\sqrt{32}-\frac{1}{10}\sqrt{50}$

Exercise:

Problem: $\frac{1}{4}\sqrt{98}-\frac{1}{3}\sqrt{128}$

Solution:

$-\frac{3}{4}\sqrt{2}$

Exercise:

Problem: $\frac{1}{3}\sqrt{24}+\frac{1}{4}\sqrt{54}$

Exercise:

Problem: $\sqrt{72a^5}-\sqrt{50a^5}$

Solution:

$-a^2\sqrt{2a}$

Exercise:

Problem: $\sqrt{48b^5}-\sqrt{75b^5}$

Exercise:

Problem: $\sqrt{80c^7}-\sqrt{20c^7}$

Solution:

$2c^3\sqrt{5c}$

Exercise:

Problem: $\sqrt{96d^9}-\sqrt{24d^9}$

Exercise:

Problem: $9\sqrt{80p^4}-6\sqrt{98p^4}$

Solution:

$36p^2\sqrt{5}-42p^2\sqrt{2}$

Exercise:

Problem: $8\sqrt{72q^6}-3\sqrt{75q^6}$

Exercise:

Problem: $2\sqrt{50r^8} + 4\sqrt{54r^8}$

Solution:

$10r^4\sqrt{2} + 12r^4\sqrt{6}$

Exercise:

Problem: $5\sqrt{27s^6} + 2\sqrt{20s^6}$

Exercise:

Problem: $3\sqrt{20x^2} - 4\sqrt{45x^2} + 5x\sqrt{80}$

Solution:

$14x\sqrt{5}$

Exercise:

Problem: $2\sqrt{28x^2} - \sqrt{63x^2} + 6x\sqrt{7}$

Exercise:

Problem: $3\sqrt{128y^2} + 4y\sqrt{162} - 8\sqrt{98y^2}$

Solution:

$-12y\sqrt{2}$

Exercise:

Problem: $3\sqrt{75y^2} + 8y\sqrt{48} - \sqrt{300y^2}$

Mixed Practice

Exercise:

Problem: $2\sqrt{8} + 6\sqrt{8} - 5\sqrt{8}$

Solution:

$3\sqrt{8}$

Exercise:

Problem: $\frac{2}{3}\sqrt{27} + \frac{3}{4}\sqrt{48}$

Exercise:

Problem: $\sqrt{175k^4} - \sqrt{63k^4}$

Solution:

$-2k^2\sqrt{7}$

Exercise:

Problem: $\frac{5}{6}\sqrt{162} + \frac{3}{16}\sqrt{128}$

Exercise:

Problem: $2\sqrt{363} - 2\sqrt{300}$

Solution:

$2\sqrt{3}$

Exercise:

Problem: $\sqrt{150} + 4\sqrt{6}$

Exercise:

Problem: $9\sqrt{2} - 8\sqrt{2}$

Solution:

$\sqrt{2}$

Exercise:

Problem: $5\sqrt{x} - 8\sqrt{y}$

Exercise:

Problem: $8\sqrt{13} - 4\sqrt{13} - 3\sqrt{13}$

Solution:

$\sqrt{13}$

Exercise:

Problem: $5\sqrt{12c^4} - 3\sqrt{27c^6}$

Exercise:

Problem: $\sqrt{80a^5} - \sqrt{45a^5}$

Solution:

$a^2\sqrt{5a}$

Exercise:

Problem: $\frac{3}{5}\sqrt{75} - \frac{1}{4}\sqrt{48}$

Exercise:

Problem: $21\sqrt{19} - 2\sqrt{19}$

Solution:

$19\sqrt{19}$

Exercise:

Problem: $\sqrt{500} + \sqrt{405}$

Exercise:

Problem: $\frac{5}{6}\sqrt{27} + \frac{5}{8}\sqrt{48}$

Solution:

$5\sqrt{3}$

Exercise:

Problem: $11\sqrt{11}-10\sqrt{11}$

Exercise:

Problem: $\sqrt{75}-\sqrt{108}$

Solution:

$-\sqrt{3}$

Exercise:

Problem: $2\sqrt{98}-4\sqrt{72}$

Exercise:

Problem: $4\sqrt{24x^2}-\sqrt{54x^2}+3x\sqrt{6}$

Solution:

$8x\sqrt{6}$

Exercise:

Problem: $8\sqrt{80y^6}-6\sqrt{48y^6}$

Everyday Math

Exercise:

Problem:

A decorator decides to use square tiles as an accent strip in the design of a new shower, but she wants to rotate the tiles to look like diamonds. She will use 9 large tiles that measure 8 inches on a side and 8 small tiles that measure 2 inches on a side. $9\left(8\sqrt{2}\right)+8\left(2\sqrt{2}\right)$. Determine the width of the accent strip by simplifying the expression $9\left(8\sqrt{2}\right)+8\left(2\sqrt{2}\right)$. (Round to the nearest tenth of an inch.)

Solution:

124.5 inches

Exercise:

Problem:

Suzy wants to use square tiles on the border of a spa she is installing in her backyard. She will use large tiles that have area of 12 square inches, medium tiles that have area of 8 square inches, and small tiles that have area of 4 square inches. Once section of the border will require 4 large tiles, 8 medium tiles, and 10 small tiles to cover the width of the wall. Simplify the expression $4\sqrt{12}+8\sqrt{8}+10\sqrt{4}$ to determine the width of the wall.

Writing Exercises

Exercise:

Problem:

Explain the difference between like radicals and unlike radicals. Make sure your answer makes sense for radicals containing both numbers and variables.

Solution:

Answers will vary.

Exercise:

Problem:

Explain the process for determining whether two radicals are like or unlike. Make sure your answer makes sense for radicals containing both numbers and variables.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
add and subtract like square roots.			
add and subtract square roots that need simplification.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Glossary

like square roots
: Square roots with the same radicand are called like square roots.

Multiply Square Roots: ASE
By the end of this section, you will be able to:

- Multiply square roots
- Use polynomial multiplication to multiply square roots

Multiply Square Roots

We have used the Product Property of Square Roots to simplify square roots by removing the perfect square factors. The Product Property of Square Roots says
Equation:

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$$

We can use the Product Property of Square Roots 'in reverse' to multiply square roots.
Equation:

$$\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$$

Remember, we assume all variables are greater than or equal to zero.

We will rewrite the Product Property of Square Roots so we see both ways together.

Note:
Product Property of Square Roots
If *a*, *b* are nonnegative real numbers, then
Equation:

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b} \quad \text{and} \quad \sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$$

So we can multiply $\sqrt{3} \cdot \sqrt{5}$ in this way:

Equation:

$$\begin{gathered}\sqrt{3}\cdot\sqrt{5}\\ \sqrt{3\cdot 5}\\ \sqrt{15}\end{gathered}$$

Sometimes the product gives us a perfect square:
Equation:

$$\begin{gathered}\sqrt{2}\cdot\sqrt{8}\\ \sqrt{2\cdot 8}\\ \sqrt{16}\\ 4\end{gathered}$$

Even when the product is not a perfect square, we must look for perfect-square factors and simplify the radical whenever possible.

Multiplying radicals with coefficients is much like multiplying variables with coefficients. To multiply $4x\cdot 3y$ we multiply the coefficients together and then the variables. The result is $12xy$. Keep this in mind as you do these examples.

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt{2}\cdot\sqrt{6}$ ⓑ $\left(4\sqrt{3}\right)\left(2\sqrt{12}\right)$.

Solution:
Solution

ⓐ

	$\sqrt{2} \cdot \sqrt{6}$
Multiply using the Product Property.	$\sqrt{12}$
Simplify the radical.	$\sqrt{4} \cdot \sqrt{3}$
Simplify.	$2\sqrt{3}$

ⓑ

	$\left(4\sqrt{3}\right)\left(2\sqrt{12}\right)$
Multiply using the Product Property.	$8\sqrt{36}$
Simplify the radical.	$8 \cdot 6$
Simplify.	48

Notice that in (b) we multiplied the coefficients and multiplied the radicals. Also, we did not simplify $\sqrt{12}$. We waited to get the product and then simplified.

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{3} \cdot \sqrt{6}$ ⓑ $\left(2\sqrt{6}\right)\left(3\sqrt{12}\right)$.

Solution:

ⓐ $3\sqrt{2}$ ⓑ $36\sqrt{2}$

Note:

Exercise:

Problem: Simplify: ⓐ $\sqrt{5} \cdot \sqrt{10}$ ⓑ $\left(6\sqrt{3}\right)\left(5\sqrt{6}\right)$.

Solution:

ⓐ $5\sqrt{2}$ ⓑ $90\sqrt{2}$

Example:
Exercise:

Problem: Simplify: $\left(6\sqrt{2}\right)\left(3\sqrt{10}\right)$.

Solution:
Solution

	$\left(6\sqrt{2}\right)\left(3\sqrt{10}\right)$
Multiply using the Product Property.	$18\sqrt{20}$
Simplify the radical.	$18\sqrt{4} \cdot \sqrt{5}$
Simplify.	$18 \cdot 2 \cdot \sqrt{5}$
	$36\sqrt{5}$

Note:
Exercise:

Problem: Simplify: $\left(3\sqrt{2}\right)\left(2\sqrt{30}\right)$.

Solution:

$12\sqrt{15}$

Note:
Exercise:

Problem: Simplify: $\left(3\sqrt{3}\right)\left(3\sqrt{6}\right)$.

Solution:

$27\sqrt{2}$

When we have to multiply square roots, we first find the product and then remove any perfect square factors.

Example:
Exercise:

Problem: Simplify: ⓐ $\left(\sqrt{8x^3}\right)\left(\sqrt{3x}\right)$ ⓑ $\left(\sqrt{20y^2}\right)\left(\sqrt{5y^3}\right)$.

Solution:
Solution

ⓐ

$$\left(\sqrt{8x^3}\right)\left(\sqrt{3x}\right)$$

Multiply using the Product Property.	$\sqrt{24x^4}$
Simplify the radical.	$\sqrt{4x^4}\cdot\sqrt{6}$
Simplify.	$2x^2\sqrt{6}$

ⓑ

$$\left(\sqrt{20y^2}\right)\left(\sqrt{5y^3}\right)$$

Multiply using the Product Property.	$\sqrt{100y^5}$
Simplify the radical.	$10y^2\sqrt{y}$

Note:
Exercise:

Problem: Simplify: ⓐ $\left(\sqrt{6x^3}\right)\left(\sqrt{3x}\right)$ ⓑ $\left(\sqrt{2y^3}\right)\left(\sqrt{50y^2}\right)$.

Solution:

ⓐ $3x^2\sqrt{2}$ ⓑ $10y^2\sqrt{y}$

Note:
Exercise:

Problem: Simplify: ⓐ $\left(\sqrt{6x^5}\right)\left(\sqrt{2x}\right)$ ⓑ $\left(\sqrt{12y^2}\right)\left(\sqrt{3y^5}\right)$.

Solution:

ⓐ $2x^3\sqrt{3}$ ⓑ $6y^2\sqrt{y}$

Example:
Exercise:

Problem: Simplify: $\left(10\sqrt{6p^3}\right)\left(3\sqrt{18p}\right)$.

Solution:
Solution

	$\left(10\sqrt{6p^3}\right)\left(3\sqrt{18p}\right)$
Multiply.	$30\sqrt{108p^4}$
Simplify the radical.	$30\sqrt{36p^4}\cdot\sqrt{3}$
	$30\cdot 6p^2\cdot\sqrt{3}$
	$180p^2\sqrt{3}$

Note:
Exercise:

Problem: Simplify: $\left(6\sqrt{2x^2}\right)\left(8\sqrt{45x^4}\right)$.

Solution:

$144x^3\sqrt{10}$

Note:
Exercise:

Problem: Simplify: $\left(2\sqrt{6y^4}\right)\left(12\sqrt{30y}\right)$.

Solution:

$144y^2\sqrt{5y}$

Example:
Exercise:

Problem: Simplify: ⓐ $\left(\sqrt{2}\right)^2$ ⓑ $\left(-\sqrt{11}\right)^2$.

Solution:
Solution

ⓐ

	$\left(\sqrt{2}\right)^2$
Rewrite as a product.	$\left(\sqrt{2}\right)\left(\sqrt{2}\right)$
Multiply.	$\sqrt{4}$
Simplify.	2

ⓑ

	$\left(-\sqrt{11}\right)^2$
Rewrite as a product.	$\left(-\sqrt{11}\right)\left(-\sqrt{11}\right)$
Multiply.	$\sqrt{121}$
Simplify.	11

Note:
Exercise:

Problem: Simplify: ⓐ $\left(\sqrt{12}\right)^2$ ⓑ $\left(-\sqrt{15}\right)^2$.

Solution:

ⓐ 12 ⓑ 15

Note:
Exercise:

Problem: Simplify: ⓐ $\left(\sqrt{16}\right)^2$ ⓑ $\left(-\sqrt{20}\right)^2$.

Solution:

ⓐ 16 ⓑ 20

The results of the previous example lead us to this property.

Note:
Squaring a Square Root
If *a* is a nonnegative real number, then
Equation:

$$\left(\sqrt{a}\right)^2 = a$$

By realizing that squaring and taking a square root are 'opposite' operations, we can simplify $\left(\sqrt{2}\right)^2$ and get 2 right away. When we multiply the two like square roots in part (a) of the next example, it is the same as squaring.

Example:
Exercise:

Problem: Simplify: ⓐ $\left(2\sqrt{3}\right)\left(8\sqrt{3}\right)$ ⓑ $\left(3\sqrt{6}\right)^2$.

Solution:
Solution

ⓐ

	$\left(2\sqrt{3}\right)\left(8\sqrt{3}\right)$
Multiply. Remember, $\left(\sqrt{3}\right)^2 = 3$.	$16 \cdot 3$
Simplify.	48

ⓑ

	$\left(3\sqrt{6}\right)^2$
Multiply.	$9 \cdot 6$
Simplify.	54

Note:
Exercise:

Problem: Simplify: ⓐ $\left(6\sqrt{11}\right)\left(5\sqrt{11}\right)$ ⓑ $\left(5\sqrt{8}\right)^2$.

Solution:

ⓐ 330 ⓑ 200

Note:
Exercise:

Problem: Simplify: ⓐ $\left(3\sqrt{7}\right)\left(10\sqrt{7}\right)$ ⓑ $\left(-4\sqrt{6}\right)^2$.

Solution:

ⓐ 210 ⓑ 96

Use Polynomial Multiplication to Multiply Square Roots

In the next few examples, we will use the Distributive Property to multiply expressions with square roots.

We will first distribute and then simplify the square roots when possible.

Example:
Exercise:

Problem: Simplify: ⓐ $3\left(5-\sqrt{2}\right)$ ⓑ $\sqrt{2}\left(4-\sqrt{10}\right)$.

Solution:
Solution

ⓐ

	$3\left(5-\sqrt{2}\right)$
Distribute.	$15-3\sqrt{2}$

ⓑ

	$\sqrt{2}\left(4-\sqrt{10}\right)$
Distribute.	$4\sqrt{2}-\sqrt{20}$
	$4\sqrt{2}-2\sqrt{5}$

Note:
Exercise:

Problem: Simplify: ⓐ $2\left(3-\sqrt{5}\right)$ ⓑ $\sqrt{3}\left(2-\sqrt{18}\right)$.

Solution:

ⓐ $6-2\sqrt{5}$ ⓑ $2\sqrt{3}-3\sqrt{6}$

Note:
Exercise:

Problem: Simplify: ⓐ $6\left(2+\sqrt{6}\right)$ ⓑ $\sqrt{7}\left(1+\sqrt{14}\right)$.

Solution:

ⓐ $12+6\sqrt{6}$ ⓑ $\sqrt{7}+7\sqrt{2}$

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt{5}\left(7+2\sqrt{5}\right)$ ⓑ $\sqrt{6}\left(\sqrt{2}+\sqrt{18}\right)$.

Solution:
Solution

ⓐ

	$\sqrt{5}\left(7+2\sqrt{5}\right)$
Multiply.	$7\sqrt{5}+2\cdot 5$
Simplify.	$7\sqrt{5}+10$
	$10+7\sqrt{5}$

ⓑ

$$\sqrt{6}\left(\sqrt{2}+\sqrt{18}\right)$$

Multiply.	$\sqrt{12}+\sqrt{108}$
Simplify.	$\sqrt{4}\cdot\sqrt{3}+\sqrt{36}\cdot\sqrt{3}$ $2\sqrt{3}+6\sqrt{3}$
Combine like radicals.	$8\sqrt{3}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{6}\left(1+3\sqrt{6}\right)$ ⓑ $\sqrt{12}\left(\sqrt{3}+\sqrt{24}\right)$.

Solution:

ⓐ $18+\sqrt{6}$ ⓑ $6+12\sqrt{2}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{8}\left(2-5\sqrt{8}\right)$ ⓑ $\sqrt{14}\left(\sqrt{2}+\sqrt{42}\right)$.

Solution:

ⓐ $-40+4\sqrt{2}$ ⓑ $2\sqrt{7}+14\sqrt{3}$

When we worked with polynomials, we multiplied binomials by binomials. Remember, this gave us four products before we combined any like terms. To be sure to get all four products, we organized our work—usually by the FOIL method.

Example:
Exercise:

Problem: Simplify: $\left(2+\sqrt{3}\right)\left(4-\sqrt{3}\right)$.

Solution:
Solution

	$\left(2+\sqrt{3}\right)\left(4-\sqrt{3}\right)$
Multiply.	$8-2\sqrt{3}+4\sqrt{3}-3$
Combine like terms.	$5+2\sqrt{3}$

Note:
Exercise:

Problem: Simplify: $\left(1+\sqrt{6}\right)\left(3-\sqrt{6}\right)$.

Solution:

$-3+2\sqrt{6}$

Note:
Exercise:

Problem: Simplify: $\left(4-\sqrt{10}\right)\left(2+\sqrt{10}\right)$.

Solution:

$-2+2\sqrt{10}$

Example:
Exercise:

Problem: Simplify: $\left(3-2\sqrt{7}\right)\left(4-2\sqrt{7}\right)$.

Solution:
Solution

	$\left(3-2\sqrt{7}\right)\left(4-2\sqrt{7}\right)$
Multiply.	$12-6\sqrt{7}-8\sqrt{7}+4\cdot 7$
Simplify.	$12-6\sqrt{7}-8\sqrt{7}+28$
Combine like terms.	$40-14\sqrt{7}$

Note:
Exercise:

Problem: Simplify: $\left(6-3\sqrt{7}\right)\left(3+4\sqrt{7}\right)$.

Solution:

$-66+15\sqrt{7}$

Note:
Exercise:

Problem: Simplify: $\left(2-3\sqrt{11}\right)\left(4-\sqrt{11}\right)$.

Solution:

$41+14\sqrt{11}$

Example:
Exercise:

Problem: Simplify: $\left(3\sqrt{2}-\sqrt{5}\right)\left(\sqrt{2}+4\sqrt{5}\right)$.

Solution:
Solution

	$\left(3\sqrt{2}-\sqrt{5}\right)\left(\sqrt{2}+4\sqrt{5}\right)$
Multiply.	$3\cdot 2+12\sqrt{10}-\sqrt{10}-4\cdot 5$
Simplify.	$6+12\sqrt{10}-\sqrt{10}-20$
Combine like terms.	$-14+11\sqrt{10}$

Note:
Exercise:

Problem: Simplify: $\left(5\sqrt{3}-\sqrt{7}\right)\left(\sqrt{3}+2\sqrt{7}\right)$.

Solution:

$1+9\sqrt{21}$

Note:
Exercise:

Problem: Simplify: $\left(\sqrt{6}-3\sqrt{8}\right)\left(2\sqrt{6}+\sqrt{8}\right)$

Solution:

$-12-20\sqrt{3}$

Example:
Exercise:

Problem: Simplify: $\left(4-2\sqrt{x}\right)\left(1+3\sqrt{x}\right)$.

Solution:
Solution

	$\left(4-2\sqrt{x}\right)\left(1+3\sqrt{x}\right)$
Multiply.	$4+12\sqrt{x}-2\sqrt{x}-6x$
Combine like terms.	$4+10\sqrt{x}-6x$

Note:
Exercise:

Problem: Simplify: $(6-5\sqrt{m})(2+3\sqrt{m})$.

Solution:

$12-8\sqrt{m}-15m$

Note:
Exercise:

Problem: Simplify: $(10+3\sqrt{n})(1-5\sqrt{n})$.

Solution:

$10-47\sqrt{n}-15n$

Note that some special products made our work easier when we multiplied binomials earlier. This is true when we multiply square roots, too. The special product formulas we used are shown below.

Note:
Special Product Formulas
Equation:

Binomial Squares	**Product of Conjugates**
$(a+b)^2=a^2+2ab+b^2$	$(a-b)(a+b)=a^2-b^2$
$(a-b)^2=a^2-2ab+b^2$	

We will use the special product formulas in the next few examples. We will start with the Binomial Squares formula.

Example:
Exercise:

Problem: Simplify: ⓐ $\left(2+\sqrt{3}\right)^2$ ⓑ $\left(4-2\sqrt{5}\right)^2$.

Solution:
Solution

Be sure to include the $2ab$ term when squaring a binomial.

ⓐ

	$\overset{(a\ +\ b)^2}{\left(2+\sqrt{3}\right)^2}$
Multiply using the binomial square pattern.	$\overset{a^2\ +\ 2ab\ +\ b^2}{2^2+2\cdot 2\cdot\sqrt{3}+\left(\sqrt{3}\right)^2}$
Simplify.	$4+4\sqrt{3}+3$
Combine like terms.	$7+4\sqrt{3}$

ⓑ

	$(a - b)^2$ $\left(4 - 2\sqrt{5}\right)^2$
Multiply using the binomial square pattern.	$a^2 - 2ab + b^2$ $4^2 - 2 \cdot 4 \cdot 2\sqrt{5} + \left(2\sqrt{5}\right)^2$
Simplify.	$16 - 16\sqrt{5} + 4 \cdot 5$ $16 - 16\sqrt{5} + 20$
Combine like terms.	$36 - 16\sqrt{5}$

Note:
Exercise:

Problem: Simplify: ⓐ $\left(10 + \sqrt{2}\right)^2$ ⓑ $\left(1 + 3\sqrt{6}\right)^2$.

Solution:

ⓐ $102 + 20\sqrt{2}$ ⓑ $55 + 6\sqrt{6}$

Note:
Exercise:

Problem: Simplify: ⓐ $\left(6-\sqrt{5}\right)^2$ ⓑ $\left(9-2\sqrt{10}\right)^2$.

Solution:

ⓐ $31-12\sqrt{5}$ ⓑ $121-36\sqrt{10}$

Example:
Exercise:

Problem: Simplify: $\left(1+3\sqrt{x}\right)^2$.

Solution:
Solution

	$\overset{(a\ +\ \ \ b)^2}{\left(1+3\sqrt{x}\right)^2}$
Multiply using the binomial square pattern.	$\overset{a^2}{1^2}\overset{+}{+}\overset{2ab}{2\cdot 1\cdot 3\sqrt{x}}\overset{+}{+}\overset{b^2}{\left(3\sqrt{x}\right)^2}$
Simplify.	$1+6\sqrt{x}+9x$

Note:
Exercise:

Problem: Simplify: $\left(2+5\sqrt{m}\right)^2$.

Solution:

$4+20\sqrt{m}+25m$

Note:
Exercise:

Problem: Simplify: $\left(3-4\sqrt{n}\right)^2$.

Solution:

$9-24\sqrt{n}+16n$

In the next two examples, we will find the product of conjugates.

Example:
Exercise:

Problem: Simplify: $\left(4-\sqrt{2}\right)\left(4+\sqrt{2}\right)$.

Solution:
Solution

	$\begin{array}{c} \color{red}{(a - b)(a + b)} \\ (4-\sqrt{2})(4+\sqrt{2}) \end{array}$
Multiply using the binomial square pattern.	$\begin{array}{c} \color{red}{a^2 - b^2} \\ 4^2-(\sqrt{2})^2 \end{array}$
Simplify.	$16-2$ 14

Note:
Exercise:

Problem: Simplify: $\left(2-\sqrt{3}\right)\left(2+\sqrt{3}\right)$.

Solution:

1

Note:
Exercise:

Problem: Simplify: $\left(1+\sqrt{5}\right)\left(1-\sqrt{5}\right)$.

Solution:

-4

Example:
Exercise:

Problem: Simplify: $\left(5-2\sqrt{3}\right)\left(5+2\sqrt{3}\right)$.

Solution:
Solution

	$\overset{(a\ -\ \ b)}{\left(5-2\sqrt{3}\right)}\overset{(a\ +\ \ b)}{\left(5+2\sqrt{3}\right)}$
Multiply using the binomial square pattern.	$\overset{a^2\ -}{5^2}-\overset{b^2}{\left(2\sqrt{3}\right)^2}$
Simplify.	$25-4\cdot 3$ 13

Note:
Exercise:

Problem: Simplify: $\left(3-2\sqrt{5}\right)\left(3+2\sqrt{5}\right)$.

Solution:

-11

Note:
Exercise:

Problem: Simplify: $\left(4+5\sqrt{7}\right)\left(4-5\sqrt{7}\right)$.

Solution:

-159

Note:
Access these online resources for additional instruction and practice with multiplying square roots.

- Product Property
- Multiply Binomials with Square Roots

Key Concepts

- **Product Property of Square Roots** If a, b are nonnegative real numbers, then
  Equation:
  $$\sqrt{ab}=\sqrt{a}\cdot\sqrt{b} \quad \text{and} \quad \sqrt{a}\cdot\sqrt{b}=\sqrt{ab}$$
- **Special formulas** for multiplying binomials and conjugates:
  Equation:
  $$\begin{array}{ll} (a+b)^2=a^2+2ab+b^2 & (a-b)(a+b)=a^2-b^2 \\ (a-b)^2=a^2-2ab+b^2 & \end{array}$$

- The FOIL method can be used to multiply binomials containing radicals.

Practice Makes Perfect

Multiply Square Roots

In the following exercises, simplify.

Exercise:

Problem: ⓐ $\sqrt{2} \cdot \sqrt{8}$ ⓑ $\left(3\sqrt{3}\right)\left(2\sqrt{18}\right)$

Solution:

ⓐ 4 ⓑ 36

Exercise:

Problem: ⓐ $\sqrt{6} \cdot \sqrt{6}$ ⓑ $\left(3\sqrt{2}\right)\left(2\sqrt{32}\right)$

Exercise:

Problem: ⓐ $\sqrt{7} \cdot \sqrt{14}$ ⓑ $\left(4\sqrt{8}\right)\left(5\sqrt{8}\right)$

Solution:

ⓐ $7\sqrt{2}$ ⓑ 160

Exercise:

ⓐ $\sqrt{6} \cdot \sqrt{12}$

Problem: ⓑ $\left(2\sqrt{5}\right)\left(2\sqrt{10}\right)$

Exercise:

Problem: $\left(5\sqrt{2}\right)\left(3\sqrt{6}\right)$

Solution:

$30\sqrt{3}$

Exercise:

Problem: $\left(2\sqrt{3}\right)\left(4\sqrt{6}\right)$

Exercise:

Problem: $\left(-2\sqrt{3}\right)\left(3\sqrt{18}\right)$

Solution:

$-18\sqrt{6}$

Exercise:

Problem: $\left(-4\sqrt{5}\right)\left(5\sqrt{10}\right)$

Exercise:

Problem: $\left(5\sqrt{6}\right)\left(-\sqrt{12}\right)$

Solution:

$-30\sqrt{2}$

Exercise:

Problem: $\left(6\sqrt{2}\right)\left(-\sqrt{10}\right)$

Exercise:

Problem: $\left(-2\sqrt{7}\right)\left(-2\sqrt{14}\right)$

Solution:

$28\sqrt{2}$

Exercise:

Problem: $\left(-2\sqrt{11}\right)\left(-4\sqrt{22}\right)$

Exercise:

Problem: ⓐ $\left(\sqrt{15y}\right)\left(\sqrt{5y^3}\right)$ ⓑ $\left(\sqrt{2n^2}\right)\left(\sqrt{18n^3}\right)$

Solution:

ⓐ $5y^2\sqrt{3}$ ⓑ $6n^2\sqrt{n}$

Exercise:

Problem: ⓐ $\left(\sqrt{14x^3}\right)\left(\sqrt{7x^3}\right)$ ⓑ $\left(\sqrt{3q^2}\right)\left(\sqrt{48q^3}\right)$

Exercise:

Problem: ⓐ $\left(\sqrt{16y^2}\right)\left(\sqrt{8y^4}\right)$ ⓑ $\left(\sqrt{11s^6}\right)\left(\sqrt{11s}\right)$

Solution:

ⓐ $8y^3\sqrt{2}$ ⓑ $11s^3\sqrt{s}$

Exercise:

Problem: ⓐ $\left(\sqrt{8x^3}\right)\left(\sqrt{3x}\right)$ ⓑ $\left(\sqrt{7r}\right)\left(\sqrt{7r^8}\right)$

Exercise:

Problem: $\left(2\sqrt{5b^3}\right)\left(4\sqrt{15b}\right)$

Solution:

$40b^2\sqrt{3}$

Exercise:

Problem: $\left(3\sqrt{8c^5}\right)\left(2\sqrt{6c^3}\right)$

Exercise:

Problem: $\left(6\sqrt{3d^3}\right)\left(4\sqrt{12d^5}\right)$

Solution:

$144d^4$

Exercise:

Problem: $\left(2\sqrt{5b^3}\right)\left(4\sqrt{15b}\right)$

Exercise:

Problem: $\left(6\sqrt{3d^3}\right)\left(4\sqrt{12d^5}\right)$

Solution:

$54y^4\sqrt{y}$

Exercise:

Problem: $\left(-2\sqrt{7z^3}\right)\left(3\sqrt{14z^8}\right)$

Exercise:

Problem: $\left(4\sqrt{2k^5}\right)\left(-3\sqrt{32k^6}\right)$

Solution:

$-96k^5\sqrt{k}$

Exercise:

Problem:
ⓐ $\left(\sqrt{7}\right)^2$
ⓑ $\left(-\sqrt{15}\right)^2$

Exercise:

Problem:
ⓐ $\left(\sqrt{11}\right)^2$
ⓑ $\left(-\sqrt{21}\right)^2$

Solution:

ⓐ 11 ⓑ 21

Exercise:

Problem: ⓐ $\left(\sqrt{19}\right)^2$ ⓑ $\left(-\sqrt{5}\right)^2$

Exercise:

Problem: ⓐ $\left(\sqrt{23}\right)^2$ ⓑ $\left(-\sqrt{3}\right)^2$

Solution:

ⓐ 23 ⓑ 3

Exercise:

Problem: ⓐ $\left(4\sqrt{11}\right)\left(-3\sqrt{11}\right)$ ⓑ $\left(5\sqrt{3}\right)^2$

Exercise:

Problem: ⓐ $\left(2\sqrt{13}\right)\left(-9\sqrt{13}\right)$ ⓑ $\left(6\sqrt{5}\right)^2$

Solution:

ⓐ −234 ⓑ 180

Exercise:

Problem: ⓐ $\left(-3\sqrt{12}\right)\left(-2\sqrt{6}\right)$
ⓑ $\left(-4\sqrt{10}\right)^2$

Exercise:

Problem: ⓐ $\left(-7\sqrt{5}\right)\left(-3\sqrt{10}\right)$
ⓑ $\left(-2\sqrt{14}\right)^2$

Solution:

ⓐ $105\sqrt{2}$ ⓑ 56

Use Polynomial Multiplication to Multiply Square Roots

In the following exercises, simplify.

Exercise:

Problem: ⓐ $3\left(4-\sqrt{3}\right)$
ⓑ $\sqrt{2}\left(4-\sqrt{6}\right)$

Exercise:

Problem: ⓐ $4\left(6-\sqrt{11}\right)$
ⓑ $\sqrt{2}\left(5-\sqrt{12}\right)$

Solution:

ⓐ $24-4\sqrt{11}$ ⓑ $5\sqrt{2}-4\sqrt{6}$

Exercise:

Problem: ⓐ $5\left(3-\sqrt{7}\right)$ ⓑ $\sqrt{3}\left(4-\sqrt{15}\right)$

Exercise:

Problem: ⓐ $7\left(-2-\sqrt{11}\right)$ ⓑ $\sqrt{7}\left(6-\sqrt{14}\right)$

Solution:

ⓐ $-14-7\sqrt{11}$ ⓑ $6\sqrt{7}-7\sqrt{2}$

Exercise:

Problem: ⓐ $\sqrt{7}\left(5+2\sqrt{7}\right)$ ⓑ $\sqrt{5}\left(\sqrt{10}+\sqrt{18}\right)$

Exercise:

Problem: ⓐ $\sqrt{11}\left(8+4\sqrt{11}\right)$ ⓑ $\sqrt{3}\left(\sqrt{12}+\sqrt{27}\right)$

Solution:

ⓐ $44+8\sqrt{11}$ ⓑ 15

Exercise:

Problem: ⓐ $\sqrt{11}\left(-3+4\sqrt{11}\right)$ ⓑ $\sqrt{3}\left(\sqrt{15}-\sqrt{18}\right)$

Exercise:

Problem: ⓐ $\sqrt{2}\left(-5+9\sqrt{2}\right)$ ⓑ $\sqrt{7}\left(\sqrt{3}-\sqrt{21}\right)$

Solution:

ⓐ $18-5\sqrt{2}$ ⓑ $\sqrt{21}-7\sqrt{3}$

Exercise:

Problem: $\left(8+\sqrt{3}\right)\left(2-\sqrt{3}\right)$

Exercise:

Problem: $\left(7+\sqrt{3}\right)\left(9-\sqrt{3}\right)$

Solution:

$60+2\sqrt{3}$

Exercise:

Problem: $\left(8-\sqrt{2}\right)\left(3+\sqrt{2}\right)$

Exercise:

Problem: $\left(9-\sqrt{2}\right)\left(6+\sqrt{2}\right)$

Solution:

$52+3\sqrt{2}$

Exercise:

Problem: $\left(3-\sqrt{7}\right)\left(5-\sqrt{7}\right)$

Exercise:

Problem: $\left(5-\sqrt{7}\right)\left(4-\sqrt{7}\right)$

Solution:

$27-9\sqrt{7}$

Exercise:

Problem: $\left(1+3\sqrt{10}\right)\left(5-2\sqrt{10}\right)$

Exercise:

Problem: $\left(7-2\sqrt{5}\right)\left(4+9\sqrt{5}\right)$

Solution:

$-62+55\sqrt{5}$

Exercise:

Problem: $\left(\sqrt{3}+\sqrt{10}\right)\left(\sqrt{3}+2\sqrt{10}\right)$

Exercise:

Problem: $\left(\sqrt{11}+\sqrt{5}\right)\left(\sqrt{11}+6\sqrt{5}\right)$

Solution:

$161+7\sqrt{55}$

Exercise:

Problem: $\left(2\sqrt{7}-5\sqrt{11}\right)\left(4\sqrt{7}+9\sqrt{11}\right)$

Exercise:

Problem: $\left(4\sqrt{6}+7\sqrt{13}\right)\left(8\sqrt{6}-3\sqrt{13}\right)$

Solution:

$-81+44\sqrt{78}$

Exercise:

Problem: $\left(5-\sqrt{u}\right)\left(3+\sqrt{u}\right)$

Exercise:

Problem: $\left(9-\sqrt{w}\right)\left(2+\sqrt{w}\right)$

Solution:

$18+7\sqrt{w}-w$

Exercise:

Problem: $\left(7+2\sqrt{m}\right)\left(4+9\sqrt{m}\right)$

Exercise:

Problem: $\left(6+5\sqrt{n}\right)\left(11+3\sqrt{n}\right)$

Solution:

$66 + 73\sqrt{n} + 15n$

Exercise:

Problem: ⓐ $\left(3 + \sqrt{5}\right)^2$ ⓑ $\left(2 - 5\sqrt{3}\right)^2$

Exercise:

Problem: ⓐ $\left(4 + \sqrt{11}\right)^2$ ⓑ $\left(3 - 2\sqrt{5}\right)^2$

Solution:

ⓐ $27 + 8\sqrt{11}$ ⓑ $29 - 12\sqrt{5}$

Exercise:

Problem: ⓐ $\left(9 - \sqrt{6}\right)^2$ ⓑ $\left(10 + 3\sqrt{7}\right)^2$

Exercise:

Problem: ⓐ $\left(5 - \sqrt{10}\right)^2$ ⓑ $\left(8 + 3\sqrt{2}\right)^2$

Solution:

ⓐ $35 - 10\sqrt{10}$ ⓑ $82 + 48\sqrt{2}$

Exercise:

Problem: $\left(3 - \sqrt{5}\right)\left(3 + \sqrt{5}\right)$

Exercise:

Problem: $\left(10 - \sqrt{3}\right)\left(10 + \sqrt{3}\right)$

Solution:

97

Exercise:

Problem: $\left(4 + \sqrt{2}\right)\left(4 - \sqrt{2}\right)$

Exercise:

Problem: $\left(7 + \sqrt{10}\right)\left(7 - \sqrt{10}\right)$

Solution:

39

Exercise:

Problem: $\left(4 + 9\sqrt{3}\right)\left(4 - 9\sqrt{3}\right)$

Exercise:

Problem: $\left(1 + 8\sqrt{2}\right)\left(1 - 8\sqrt{2}\right)$

Solution:

-127

Exercise:

Problem: $\left(12-5\sqrt{5}\right)\left(12+5\sqrt{5}\right)$

Exercise:

Problem: $\left(9-4\sqrt{3}\right)\left(9+4\sqrt{3}\right)$

Solution:

33

Mixed Practice

In the following exercises, simplify.

Exercise:

Problem: $\sqrt{3}\cdot\sqrt{21}$

Exercise:

Problem: $\left(4\sqrt{6}\right)\left(-\sqrt{18}\right)$

Solution:

$-24\sqrt{3}$

Exercise:

Problem: $\left(-5+\sqrt{7}\right)\left(6+\sqrt{21}\right)$

Exercise:

Problem: $\left(-5\sqrt{7}\right)\left(6\sqrt{21}\right)$

Solution:

$-210\sqrt{3}$

Exercise:

Problem: $\left(-4\sqrt{2}\right)\left(2\sqrt{18}\right)$

Exercise:

Problem: $\left(\sqrt{35y^3}\right)\left(\sqrt{7y^3}\right)$

Solution:

$7y^3\sqrt{5}$

Exercise:

Problem: $\left(4\sqrt{12x^5}\right)\left(2\sqrt{6x^3}\right)$

Exercise:

Problem: $\left(\sqrt{29}\right)^2$

Solution:

29

Exercise:

Problem: $\left(-4\sqrt{17}\right)\left(-3\sqrt{17}\right)$

Exercise:

Problem: $\left(-4+\sqrt{17}\right)\left(-3+\sqrt{17}\right)$

Solution:

$29-7\sqrt{17}$

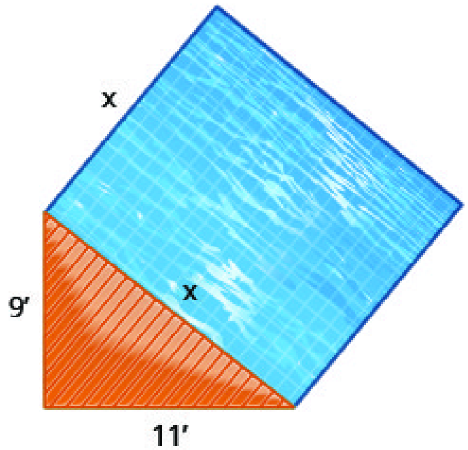
Everyday Math

Exercise:

Problem:

A landscaper wants to put a square reflecting pool next to a triangular deck, as shown below. The triangular deck is a right triangle, with legs of length 9 feet and 11 feet, and the pool will be adjacent to the hypotenuse.

ⓐ Use the Pythagorean Theorem to find the length of a side of the pool. Round your answer to the nearest tenth of a foot.
ⓑ Find the exact area of the pool.

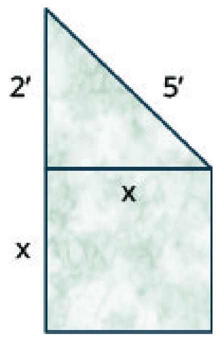


Exercise:

Problem:

An artist wants to make a small monument in the shape of a square base topped by a right triangle, as shown below. The square base will be adjacent to one leg of the triangle. The other leg of the triangle will measure 2 feet and the hypotenuse will be 5 feet.

ⓐ Use the Pythagorean Theorem to find the length of a side of the square base. Round your answer to the nearest tenth of a foot.



ⓑ Find the exact area of the face of the square base.

Solution:

ⓐ 4.6 feet ⓑ 21 sq. feet

Exercise:

Problem:

A square garden will be made with a stone border on one edge. If only $3+\sqrt{10}$ feet of stone are available, simplify $\left(3+\sqrt{10}\right)^2$ to determine the area of the largest such garden.

Exercise:

Problem:

A garden will be made so as to contain two square sections, one section with side length $\sqrt{5}+\sqrt{6}$ yards and one section with side length $\sqrt{2}+\sqrt{3}$ yards. Simplify $\left(\sqrt{5}+\sqrt{6}\right)\left(\sqrt{2}+\sqrt{3}\right)$ to determine the total area of the garden.

Exercise:

Problem:

Suppose a third section will be added to the garden in the previous exercise. The third section is to have a width of $\sqrt{432}$ feet. Write an expression that gives the total area of the garden.

Writing Exercises

Exercise:

Problem:

ⓐ Explain why $\left(-\sqrt{n}\right)^2$ is always positive, for $n \geq 0$.
ⓑ Explain why $\left(-\sqrt{n}\right)^2$ is always negative, for $n \geq 0$.

Solution:

ⓐ when squaring a negative, it becomes a positive ⓑ since the negative is not included in the parenthesis, it is not squared, and remains negative

Exercise:

Problem:

Use the binomial square pattern to simplify $\left(3+\sqrt{2}\right)^2$. Explain all your steps.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
multiply square roots.			
use polynomial multiplication to multiply square roots.			

ⓑ On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Divide Square Roots: ASE
By the end of this section, you will be able to:

- Divide square roots
- Rationalize a one-term denominator
- Rationalize a two-term denominator

Divide Square Roots

We know that we simplify fractions by removing factors common to the numerator and the denominator. When we have a fraction with a square root in the numerator, we first simplify the square root. Then we can look for common factors.

Common Factors	No common factors
$\frac{3\sqrt{2}}{3\cdot 5}$	$\frac{2\sqrt{3}}{3\cdot 5}$

Example:
Exercise:

Problem: Simplify: $\frac{\sqrt{54}}{6}$.

Solution:
Solution

	$\frac{\sqrt{54}}{6}$
Simplify the radical.	$\frac{\sqrt{9}\cdot\sqrt{6}}{6}$
Simplify.	$\frac{3\sqrt{6}}{6}$
Remove the common factors.	$\frac{\cancel{3}\sqrt{6}}{\cancel{3}\cdot 2}$
Simplify.	$\frac{\sqrt{6}}{2}$

Note:
Exercise:

Problem: Simplify: $\frac{\sqrt{32}}{8}$.

Solution:

$\frac{\sqrt{2}}{2}$

Note:
Exercise:

Problem: Simplify: $\frac{\sqrt{75}}{15}$.

Solution:

$\frac{\sqrt{3}}{3}$

Example:
Exercise:

Problem: Simplify: $\frac{6-\sqrt{24}}{12}$.

Solution:
Solution

	$\frac{6-\sqrt{24}}{12}$
Simplify the radical.	$\frac{6-\sqrt{4}\cdot\sqrt{6}}{12}$
Simplify.	$\frac{6-2\sqrt{6}}{12}$
Factor the common factor from the numerator.	$\frac{2\left(3-\sqrt{6}\right)}{2\cdot 6}$
Remove the common factors.	$\frac{\cancel{2}\left(3-\sqrt{6}\right)}{\cancel{2}\cdot 6}$
Simplify.	$\frac{3-\sqrt{6}}{6}$

Note:
Exercise:

Problem: Simplify: $\frac{8-\sqrt{40}}{10}$.

Solution:

$\frac{4-\sqrt{10}}{5}$

Note:
Exercise:

Problem: Simplify: $\frac{10-\sqrt{75}}{20}$.

Solution:

$\frac{5-\sqrt{3}}{4}$

We have used the Quotient Property of Square Roots to simplify square roots of fractions. The Quotient Property of Square Roots says

Equation:

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}, b \neq 0$$

Sometimes we will need to use the Quotient Property of Square Roots ‘in reverse’ to simplify a fraction with square roots.

Equation:

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}, b \neq 0$$

We will rewrite the Quotient Property of Square Roots so we see both ways together. Remember: we assume all variables are greater than or equal to zero so that their square roots are real numbers.

Note:
Quotient Property of Square Roots
If a, b are non-negative real numbers and $b \neq 0$, then
Equation:

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} \quad \text{and} \quad \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

We will use the Quotient Property of Square Roots ‘in reverse’ when the fraction we start with is the quotient of two square roots, and neither radicand is a perfect square. When we write the fraction in a single square root, we may find common factors in the numerator and denominator.

Example:
Exercise:

Problem: Simplify: $\frac{\sqrt{27}}{\sqrt{75}}$.

Solution:
Solution

$$\frac{\sqrt{27}}{\sqrt{75}}$$

Neither radicand is a perfect square, so rewrite using the quotient property of square roots.

$$\sqrt{\frac{27}{75}}$$

Remove common factors in the numerator and denominator.

$$\sqrt{\frac{\cancel{3}\cdot 9}{\cancel{3}\cdot 25}}$$

Simplify.

$$\sqrt{\frac{9}{25}}$$

$$\frac{3}{5}$$

Note:
Exercise:

Problem: Simplify: $\frac{\sqrt{48}}{\sqrt{108}}$.

Solution:

$\frac{2}{3}$

Note:
Exercise:

Problem: Simplify: $\frac{\sqrt{96}}{\sqrt{54}}$.

Solution:

$\frac{4}{3}$

We will use the Quotient Property for Exponents, $\frac{a^m}{a^n} = a^{m-n}$, when we have variables with exponents in the radicands.

Example:
Exercise:

Problem: Simplify: $\frac{\sqrt{6y^5}}{\sqrt{2y}}$.

Solution:
Solution

	$\frac{\sqrt{6y^5}}{\sqrt{2y}}$
Neither radicand is a perfect square, so rewrite using the quotient property of square roots.	$\sqrt{\frac{6y^5}{2y}}$
Remove common factors in the numerator and denominator.	$\sqrt{\frac{\cancel{2}\cdot 3\cdot y^4\cdot \cancel{y}}{\cancel{2}\cdot \cancel{y}}}$
Simplify.	$\sqrt{3y^4}$
Simplify the radical.	$y^2\sqrt{3}$

Note:
Exercise:

Problem: Simplify: $\frac{\sqrt{12r^3}}{\sqrt{6r}}$.

Solution:

$n\sqrt{2}$

Note:
Exercise:

Problem: Simplify: $\frac{\sqrt{14p^9}}{\sqrt{2p^5}}$.

Solution:

$p^2\sqrt{7}$

Example:
Exercise:

Problem: Simplify: $\frac{\sqrt{72x^3}}{\sqrt{162x}}$.

Solution:
Solution

	$\frac{\sqrt{72x^3}}{\sqrt{162x}}$
Rewrite using the quotient property of square roots.	$\sqrt{\frac{72x^3}{162x}}$
Remove common factors.	$\sqrt{\frac{\cancel{18}\cdot 4\cdot x^2\cdot \cancel{x}}{\cancel{18}\cdot 9\cdot \cancel{x}}}$
Simplify.	$\sqrt{\frac{4x^2}{9}}$
Simplify the radical.	$\frac{2x}{3}$

Note:
Exercise:

Problem: Simplify: $\frac{\sqrt{50s^3}}{\sqrt{128s}}$.

Solution:

$\frac{5s}{8}$

Note:
Exercise:

Problem: Simplify: $\frac{\sqrt{75q^5}}{\sqrt{108q}}$.

Solution:

$\frac{5q^2}{6}$

Example:
Exercise:

Problem: Simplify: $\frac{\sqrt{147ab^8}}{\sqrt{3a^3b^4}}$.

Solution:
Solution

	$\frac{\sqrt{147ab^8}}{\sqrt{3a^3b^4}}$
Rewrite using the quotient property of square roots.	$\sqrt{\frac{147ab^8}{3a^3b^4}}$
Remove common factors.	$\sqrt{\frac{49b^4}{a^2}}$
Simplify the radical.	$\frac{7b^2}{a}$

Note:
Exercise:

Problem: Simplify: $\frac{\sqrt{162x^{10}y^2}}{\sqrt{2x^6y^6}}$.

Solution:

$\frac{9x^2}{y^2}$

Note:
Exercise:

Problem: Simplify: $\frac{\sqrt{300m^3n^7}}{\sqrt{3m^5n}}$.

Solution:

$$\frac{10n^3}{m}$$

Rationalize a One Term Denominator

Before the calculator became a tool of everyday life, tables of square roots were used to find approximate values of square roots. [link] shows a portion of a table of squares and square roots. Square roots are approximated to five decimal places in this table.

n	n^2	$\sqrt{n}$
200	40,000	14.14214
201	40,401	14.17745
202	40,804	14.21267
203	41,209	14.24781
204	41,616	14.28286
205	42,025	14.31782
206	42,436	14.35270
207	42,849	14.38749
208	43,264	14.42221
209	43,681	14.45683
210	44,100	14.49138

A table of square roots was used to find approximate values of square roots before there were calculators.

If someone needed to approximate a fraction with a square root in the denominator, it meant doing long division with a five decimal-place divisor. This was a very cumbersome process.

For this reason, a process called rationalizing the denominator was developed. A fraction with a radical in the denominator is converted to an equivalent fraction whose denominator is an integer. This process is still used today and is useful in other areas of mathematics, too.

Note:
Rationalizing the Denominator
The process of converting a fraction with a radical in the denominator to an equivalent fraction whose denominator is an integer is called **rationalizing the denominator**.

Square roots of numbers that are not perfect squares are irrational numbers. When we **rationalize the denominator**, we write an equivalent fraction with a rational number in the denominator.

Let's look at a numerical example.

Suppose we need an approximate value for the fraction.	$\frac{1}{\sqrt{2}}$
A five decimal place approximation to $\sqrt{2}$ is 1.41421.	$\frac{1}{1.41421}$
Without a calculator, would you want to do this division?	$1.41421\overline{)1.0}$

But we can find a fraction equivalent to $\frac{1}{\sqrt{2}}$ by multiplying the numerator and denominator by $\sqrt{2}$.

$$\frac{1}{\sqrt{2}}$$

$$\frac{1 \cdot \sqrt{2}}{\sqrt{2} \cdot \sqrt{2}}$$

$$\frac{\sqrt{2}}{2}$$

Now if we need an approximate value, we divide $2\overline{)1.41421}$. This is much easier.

Even though we have calculators available nearly everywhere, a fraction with a radical in the denominator still must be rationalized. It is not considered simplified if the denominator contains a square root.

Similarly, a square root is not considered simplified if the radicand contains a fraction.

Note:

Simplified Square Roots

A square root is considered simplified if there are

- no perfect-square factors in the radicand
- no fractions in the radicand
- no square roots in the denominator of a fraction

To rationalize a denominator, we use the property that $\left(\sqrt{a}\right)^2 = a$. If we square an irrational square root, we get a rational number.

We will use this property to rationalize the denominator in the next example.

Example:
Exercise:

Problem: Simplify: $\frac{4}{\sqrt{3}}$.

Solution:
Solution

To rationalize a denominator, we can multiply a square root by itself. To keep the fraction equivalent, we multiply both the numerator and denominator by the same factor.

	$\frac{4}{\sqrt{3}}$
Multiply both the numerator and denominator by $\sqrt{3}$.	$\frac{4 \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}}$
Simplify.	$\frac{4\sqrt{3}}{3}$

Note:
Exercise:

Problem: Simplify: $\frac{5}{\sqrt{3}}$.

Solution:

$\frac{5\sqrt{3}}{3}$

Note:
Exercise:

Problem: Simplify: $\frac{6}{\sqrt{5}}$.

Solution:

$\frac{6\sqrt{5}}{5}$

Example:
Exercise:

Problem: Simplify: $-\frac{8}{3\sqrt{6}}$.

Solution:
Solution

To remove the square root from the denominator, we multiply it by itself. To keep the fractions equivalent, we multiply both the numerator and denominator by $\sqrt{6}$.

	$-\frac{8}{3\sqrt{6}}$
Multiply both the numerator and the denominator by $\sqrt{6}$.	$-\frac{8 \cdot \sqrt{6}}{3\sqrt{6} \cdot \sqrt{6}}$
Simplify.	$-\frac{8\sqrt{6}}{3 \cdot 6}$
Remove common factors.	$-\frac{4 \cdot \cancel{2}\sqrt{6}}{3 \cdot \cancel{2} \cdot 3}$
Simplify.	$-\frac{4\sqrt{6}}{9}$

Note:
Exercise:

Problem: Simplify: $\frac{5}{2\sqrt{5}}$.

Solution:

$\frac{\sqrt{5}}{2}$

Note:
Exercise:

Problem: Simplify: $-\frac{9}{4\sqrt{3}}$.

Solution:

$-\frac{3\sqrt{3}}{4}$

Always simplify the radical in the denominator first, before you rationalize it. This way the numbers stay smaller and easier to work with.

Example:
Exercise:

Problem: Simplify: $\sqrt{\frac{5}{12}}$.

Solution:
Solution

	$\sqrt{\frac{5}{12}}$
The fraction is not a perfect square, so rewrite using the Quotient Property.	$\frac{\sqrt{5}}{\sqrt{12}}$
Simplify the denominator	$\frac{\sqrt{5}}{2\sqrt{3}}$

Rationalize the denominator.	$\frac{\sqrt{5}\cdot\sqrt{3}}{2\sqrt{3}\cdot\sqrt{3}}$
Simplify.	$\frac{\sqrt{15}}{2\cdot 3}$
Simplify.	$\frac{\sqrt{15}}{6}$

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{7}{18}}$.

Solution:

$\frac{\sqrt{14}}{6}$

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{3}{32}}$.

Solution:

$\frac{\sqrt{6}}{8}$

Example:
Exercise:

Problem: Simplify: $\sqrt{\frac{11}{28}}$.

Solution:
Solution

	$\sqrt{\frac{11}{28}}$
Rewrite using the Quotient Property.	$\frac{\sqrt{11}}{\sqrt{28}}$
Simplify the denominator.	$\frac{\sqrt{11}}{2\sqrt{7}}$
Rationalize the denominator.	$\frac{\sqrt{11}\cdot\sqrt{7}}{2\sqrt{7}\cdot\sqrt{7}}$
Simplify.	$\frac{\sqrt{77}}{2\cdot 7}$
Simplify.	$\frac{\sqrt{77}}{14}$

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{3}{27}}$.

Solution:

$\frac{1}{3}$

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{10}{50}}$.

Solution:

$\frac{\sqrt{5}}{5}$

Rationalize a Two-Term Denominator

When the denominator of a fraction is a sum or difference with square roots, we use the Product of Conjugates pattern to rationalize the denominator.
Equation:

$$\begin{array}{cc} (a-b)(a+b) & (2-\sqrt{5})(2+\sqrt{5}) \\ a^2-b^2 & 2^2-(\sqrt{5})^2 \\ & 4-5 \\ & -1 \end{array}$$

When we multiply a binomial that includes a square root by its conjugate, the product has no square roots.

Example:
Exercise:

Problem: Simplify: $\frac{4}{4+\sqrt{2}}$.

Solution:
Solution

	$\frac{4}{4+\sqrt{2}}$
Multiply the numerator and denominator by the conjugate of the denominator.	$\frac{4(4-\sqrt{2})}{(4+\sqrt{2})(4-\sqrt{2})}$
Multiply the conjugates in the denominator.	$\frac{4(4-\sqrt{2})}{4^2-(\sqrt{2})^2}$
Simplify the denominator.	$\frac{4(4-\sqrt{2})}{16-2}$
Simplify the denominator.	$\frac{4(4-\sqrt{2})}{14}$
Remove common factors from the numerator and denominator.	$\frac{2(4-\sqrt{2})}{7}$
We leave the numerator in factored form to make it easier to look for common factors after we have simplified the	

denominator.

Note:
Exercise:

Problem: Simplify: $\frac{2}{2+\sqrt{3}}$.

Solution:

$$\frac{2\left(2-\sqrt{3}\right)}{1}$$

Note:
Exercise:

Problem: Simplify: $\frac{5}{5+\sqrt{3}}$.

Solution:

$$\frac{5\left(5-\sqrt{3}\right)}{22}$$

Example:
Exercise:

Problem: Simplify: $\frac{5}{2-\sqrt{3}}$.

Solution:
Solution

	$\frac{5}{2-\sqrt{3}}$
Multiply the numerator and denominator by the conjugate of the denominator.	$\frac{5(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})}$
Multiply the conjugates in the denominator.	$\frac{5(2+\sqrt{3})}{2^2-(\sqrt{3})^2}$
Simplify the denominator.	$\frac{5(2+\sqrt{3})}{4-3}$
Simplify the denominator.	$\frac{5(2+\sqrt{3})}{1}$
Simplify.	$5(2+\sqrt{3})$

Note:
Exercise:

Problem: Simplify: $\frac{3}{1-\sqrt{5}}$.

Solution:

$-\frac{3\left(1+\sqrt{5}\right)}{4}$

Note:
Exercise:

Problem: Simplify: $\frac{2}{4-\sqrt{6}}$.

Solution:

$\frac{4+\sqrt{6}}{5}$

Example:
Exercise:

Problem: Simplify: $\frac{\sqrt{3}}{\sqrt{u}-\sqrt{6}}$.

Solution:
Solution

	$\frac{\sqrt{3}}{\sqrt{u}-\sqrt{6}}$
Multiply the numerator and denominator by the conjugate of the denominator.	$\frac{\sqrt{3}(\sqrt{u}+\sqrt{6})}{(\sqrt{u}-\sqrt{6})(\sqrt{u}+\sqrt{6})}$
Multiply the conjugates in the denominator.	$\frac{\sqrt{3}(\sqrt{u}+\sqrt{6})}{(\sqrt{u})^2-(\sqrt{6})^2}$
Simplify the denominator.	$\frac{\sqrt{3}(\sqrt{u}+\sqrt{6})}{u-6}$

Note:
Exercise:

Problem: Simplify: $\frac{\sqrt{5}}{\sqrt{x}+\sqrt{2}}$.

Solution:

$\frac{\sqrt{5}\left(\sqrt{x}-\sqrt{2}\right)}{x-2}$

Note:
Exercise:

Problem: Simplify: $\frac{\sqrt{10}}{\sqrt{y}-\sqrt{3}}$.

Solution:

$\frac{\sqrt{10}\left(\sqrt{y}+\sqrt{3}\right)}{y-3}$

Example:
Exercise:

Problem: Simplify: $\frac{\sqrt{x}+\sqrt{7}}{\sqrt{x}-\sqrt{7}}$.

Solution:
Solution

	$\frac{\sqrt{x}+\sqrt{7}}{\sqrt{x}-\sqrt{7}}$
Multiply the numerator and denominator by the conjugate of the denominator.	$\frac{(\sqrt{x}+\sqrt{7})(\sqrt{x}+\sqrt{7})}{(\sqrt{x}-\sqrt{7})(\sqrt{x}+\sqrt{7})}$
Multiply the conjugates in the denominator.	$\frac{(\sqrt{x}+\sqrt{7})(\sqrt{x}+\sqrt{7})}{(\sqrt{x})^2-(\sqrt{7})^2}$
Simplify the denominator.	$\frac{(\sqrt{x}+\sqrt{7})^2}{x-7}$
We do not square the numerator. In factored form, we can see there are no common factors to remove from the numerator and denominator.	

Note:
Exercise:

Problem: Simplify: $\frac{\sqrt{p}+\sqrt{2}}{\sqrt{p}-\sqrt{2}}$.

Solution:

$$\frac{\left(\sqrt{p}+\sqrt{2}\right)^2}{p-2}$$

Note:
Exercise:

Problem: Simplify: $\frac{\sqrt{q}-\sqrt{10}}{\sqrt{q}+\sqrt{10}}$.

Solution:

$$\frac{\left(\sqrt{q}-\sqrt{10}\right)^2}{q-10}$$

Note:
Access this online resource for additional instruction and practice with dividing and rationalizing.

- Dividing and Rationalizing

Key Concepts

- **Quotient Property of Square Roots**
  - If a, b are non-negative real numbers and $b \neq 0$, then
    Equation:

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} \quad \text{and} \quad \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

- **Simplified Square Roots**
  A square root is considered simplified if there are
  - no perfect square factors in the radicand
  - no fractions in the radicand
  - no square roots in the denominator of a fraction

Practice Makes Perfect

Divide Square Roots

In the following exercises, simplify.
Exercise:

Problem: $\frac{\sqrt{27}}{6}$

Solution:

$\frac{\sqrt{3}}{2}$

Exercise:

Problem: $\frac{\sqrt{50}}{10}$

Exercise:

Problem: $\frac{\sqrt{72}}{9}$

Solution:

$\frac{2\sqrt{2}}{3}$

Exercise:

Problem: $\frac{\sqrt{243}}{6}$

Exercise:

Problem: $\frac{2-\sqrt{32}}{8}$

Solution:

$\frac{1-2\sqrt{2}}{4}$

Exercise:

Problem: $\frac{3+\sqrt{27}}{9}$

Exercise:

Problem: $\frac{6+\sqrt{45}}{6}$

Solution:

$\frac{2+\sqrt{5}}{2}$

Exercise:

Problem: $\frac{10-\sqrt{200}}{20}$

Exercise:

Problem: $\frac{\sqrt{80}}{\sqrt{125}}$

Solution:

$\frac{4}{5}$

Exercise:

Problem: $\frac{\sqrt{72}}{\sqrt{200}}$

Exercise:

Problem: $\frac{\sqrt{128}}{\sqrt{72}}$

Solution:

$\frac{4}{3}$

Exercise:

Problem: $\frac{\sqrt{48}}{\sqrt{75}}$

Exercise:

Problem: ⓐ $\frac{\sqrt{8x^6}}{\sqrt{2x^2}}$ ⓑ $\frac{\sqrt{200m^5}}{\sqrt{98m}}$

Solution:

ⓐ $2x^2$ ⓑ $\frac{10m^2}{7}$

Exercise:

Problem: ⓐ $\frac{\sqrt{10y^3}}{\sqrt{5y}}$ ⓑ $\frac{\sqrt{108n^7}}{\sqrt{243n^3}}$

Exercise:

Problem: $\frac{\sqrt{75r^3}}{\sqrt{108r}}$

Solution:

$\frac{5r}{6}$

Exercise:

Problem: $\frac{\sqrt{196q^5}}{\sqrt{484q}}$

Exercise:

Problem: $\frac{\sqrt{108p^5q^2}}{\sqrt{34p^3q^6}}$

Solution:

$\frac{6p}{q^2}$

Exercise:

Problem: $\frac{\sqrt{98rs^{10}}}{\sqrt{2r^3s^4}}$

Exercise:

Problem: $\frac{\sqrt{320mn^5}}{\sqrt{45m^7n^3}}$

Solution:

$\frac{8n}{3m^3}$

Exercise:

Problem: $\frac{\sqrt{810c^3d^7}}{\sqrt{1000c^5d}}$

Exercise:

Problem: $\frac{\sqrt{98}}{14}$

Solution:

$\frac{\sqrt{2}}{2}$

Exercise:

Problem: $\frac{\sqrt{72}}{18}$

Exercise:

Problem: $\frac{5+\sqrt{125}}{15}$

Solution:

$\frac{1+\sqrt{3}}{3}$

Exercise:

Problem: $\frac{6-\sqrt{45}}{12}$

Exercise:

Problem: $\frac{\sqrt{96}}{\sqrt{150}}$

Solution:

$\frac{4}{5}$

Exercise:

Problem: $\frac{\sqrt{28}}{\sqrt{63}}$

Exercise:

Problem: $\frac{\sqrt{26y^7}}{\sqrt{2y}}$

Solution:

$y^3\sqrt{13}$

Exercise:

Problem: $\frac{\sqrt{15x^3}}{\sqrt{3x}}$

Rationalize a One-Term Denominator

In the following exercises, simplify and rationalize the denominator.

Exercise:

Problem: $\frac{10}{\sqrt{6}}$

Solution:

$\frac{5\sqrt{6}}{3}$

Exercise:

Problem: $\frac{8}{\sqrt{3}}$

Exercise:

Problem: $\frac{6}{\sqrt{7}}$

Solution:

$\frac{6\sqrt{7}}{7}$

Exercise:

Problem: $\frac{4}{\sqrt{5}}$

Exercise:

Problem: $\frac{3}{\sqrt{13}}$

Solution:

$\frac{3\sqrt{13}}{13}$

Exercise:

Problem: $\frac{10}{\sqrt{11}}$

Exercise:

Problem: $\frac{10}{3\sqrt{10}}$

Solution:

$\frac{\sqrt{10}}{3}$

Exercise:

Problem: $\frac{2}{5\sqrt{2}}$

Exercise:

Problem: $\frac{4}{9\sqrt{5}}$

Solution:

$\frac{4\sqrt{5}}{45}$

Exercise:

Problem: $\frac{9}{2\sqrt{7}}$

Exercise:

Problem: $-\frac{9}{2\sqrt{3}}$

Solution:

$-\frac{3\sqrt{3}}{2}$

Exercise:

Problem: $-\frac{8}{3\sqrt{6}}$

Exercise:

Problem: $\sqrt{\frac{3}{20}}$

Solution:

$\frac{\sqrt{15}}{10}$

Exercise:

Problem: $\sqrt{\frac{4}{27}}$

Exercise:

Problem: $\sqrt{\frac{7}{40}}$

Solution:

$\frac{\sqrt{70}}{20}$

Exercise:

Problem: $\sqrt{\frac{8}{45}}$

Exercise:

Problem: $\sqrt{\frac{19}{175}}$

Solution:

$\frac{\sqrt{133}}{35}$

Exercise:

Problem: $\sqrt{\frac{17}{192}}$

Rationalize a Two-Term Denominator

In the following exercises, simplify by rationalizing the denominator.

Exercise:

Problem: ⓐ $\frac{3}{3+\sqrt{11}}$ ⓑ $\frac{8}{1-\sqrt{5}}$

Solution:

ⓐ $\frac{3\left(3-\sqrt{11}\right)}{-2}$ ⓑ $-2\left(1+\sqrt{5}\right)$

Exercise:

Problem: ⓐ $\frac{4}{4+\sqrt{7}}$ ⓑ $\frac{7}{2-\sqrt{6}}$

Exercise:

Problem: ⓐ $\frac{5}{5+\sqrt{6}}$ ⓑ $\frac{6}{3-\sqrt{7}}$

Solution:

ⓐ $\frac{5\left(5-\sqrt{6}\right)}{19}$ ⓑ $3\left(3+\sqrt{7}\right)$

Exercise:

Problem: ⓐ $\frac{6}{6+\sqrt{5}}$ ⓑ $\frac{5}{4-\sqrt{11}}$

Exercise:

Problem: $\frac{\sqrt{3}}{\sqrt{m}-\sqrt{5}}$

Solution:

$$\frac{\sqrt{3}\left(\sqrt{m}+\sqrt{5}\right)}{m-5}$$

Exercise:

Problem: $\frac{\sqrt{5}}{\sqrt{n}-\sqrt{7}}$

Exercise:

Problem: $\frac{\sqrt{2}}{\sqrt{x}-\sqrt{6}}$

Solution:

$$\frac{\sqrt{2}\left(\sqrt{x}+\sqrt{3}\right)}{x-6}$$

Exercise:

Problem: $\frac{\sqrt{7}}{\sqrt{y}+\sqrt{3}}$

Exercise:

Problem: $\frac{\sqrt{r}+\sqrt{5}}{\sqrt{r}-\sqrt{5}}$

Solution:

$$\frac{\left(\sqrt{r}+\sqrt{5}\right)^2}{r-5}$$

Exercise:

Problem: $\frac{\sqrt{s}-\sqrt{6}}{\sqrt{s}+\sqrt{6}}$

Exercise:

Problem: $\frac{\sqrt{150x^2y^6}}{\sqrt{6x^4y^2}}$

Solution:

$\frac{5y^2}{x}$

Exercise:

Problem: $\frac{\sqrt{80p^3q}}{\sqrt{5pq^5}}$

Exercise:

Problem: $\frac{15}{\sqrt{5}}$

Solution:

$3\sqrt{5}$

Exercise:

Problem: $\frac{3}{5\sqrt{8}}$

Exercise:

Problem: $\sqrt{\frac{8}{54}}$

Solution:

$\frac{2\sqrt{3}}{9}$

Exercise:

Problem: $\sqrt{\frac{12}{20}}$

Exercise:

Problem: $\frac{3}{5+\sqrt{5}}$

Solution:

$\frac{3\left(5-\sqrt{5}\right)}{20}$

Exercise:

Problem: $\frac{20}{4-\sqrt{3}}$

Exercise:

Problem: $\frac{\sqrt{2}}{\sqrt{x}-\sqrt{3}}$

Solution:

$\frac{\sqrt{2}\left(\sqrt{x}+\sqrt{3}\right)}{x-3}$

Exercise:

Problem: $\frac{\sqrt{5}}{\sqrt{y}-\sqrt{7}}$

Exercise:

Problem: $\frac{\sqrt{x}+\sqrt{8}}{\sqrt{x}-\sqrt{8}}$

Solution:

$\frac{\left(\sqrt{x}+2\sqrt{2}\right)^2}{x-8}$

Exercise:

Problem: $\frac{\sqrt{m}-\sqrt{3}}{\sqrt{m}+\sqrt{3}}$

Everyday Math

Exercise:

Problem:

A supply kit is dropped from an airplane flying at an altitude of 250 feet. Simplify $\sqrt{\frac{250}{16}}$ to determine how many seconds it takes for the supply kit to reach the ground.

Solution:

$\frac{5\sqrt{10}}{4}$ seconds

Exercise:

Problem:

A flare is dropped into the ocean from an airplane flying at an altitude of 1,200 feet. Simplify $\sqrt{\frac{1200}{16}}$ to determine how many seconds it takes for the flare to reach the ocean.

Writing Exercises

Exercise:

Problem:

ⓐ Simplify $\sqrt{\frac{27}{3}}$ and explain all your steps.

ⓑ Simplify $\sqrt{\frac{27}{5}}$ and explain all your steps.

ⓒ Why are the two methods of simplifying square roots different?

Solution:

Answers will vary.

Exercise:

Problem:

ⓐ Approximate $\frac{1}{\sqrt{2}}$ by dividing $\frac{1}{1.414}$ using long division without a calculator.

ⓑ Rationalizing the denominator of $\frac{1}{\sqrt{2}}$ gives $\frac{\sqrt{2}}{2}$. Approximate $\frac{\sqrt{2}}{2}$ by dividing $\frac{1.414}{2}$ using long division without a calculator.
ⓒ Do you agree that rationalizing the denominator makes calculations easier? Why or why not?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
divide square roots.			
rationalize a one-term denominator.			
rationalize a two-term denominator.			

ⓑ After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

Glossary

rationalizing the denominator
: The process of converting a fraction with a radical in the denominator to an equivalent fraction whose denominator is an integer is called rationalizing the denominator.

Solve Equations with Square Roots: ASE
By the end of this section, you will be able to:

- Solve radical equations
- Use square roots in applications

Solve Radical Equations

In this section we will solve equations that have the variable in the radicand of a square root. Equations of this type are called radical equations.

Note:
Radical Equation
An equation in which the variable is in the radicand of a square root is called a **radical equation**.

As usual, in solving these equations, what we do to one side of an equation we must do to the other side as well. Since squaring a quantity and taking a square root are 'opposite' operations, we will square both sides in order to remove the radical sign and solve for the variable inside.

But remember that when we write $\sqrt{a}$ we mean the principal square root. So $\sqrt{a} \geq 0$ always. When we solve radical equations by squaring both sides we may get an algebraic solution that would make $\sqrt{a}$ negative. This algebraic solution would not be a solution to the original radical equation; it is an *extraneous solution.* We saw extraneous solutions when we solved rational equations, too.

Example:
Exercise:

Problem: For the equation $\sqrt{x+2} = x$:

ⓐ Is $x = 2$ a solution? ⓑ Is $x = -1$ a solution?

Solution:
Solution

ⓐ Is $x = 2$ a solution?

	$\sqrt{x+2} = x$
Let x = 2.	$\sqrt{2+2} \stackrel{?}{=} 2$

Simplify.	$\sqrt{4} \stackrel{?}{=} 2$
	$2 = 2$ ✓
	2 is a solution.

ⓑ Is $x = -1$ a solution?

	$\sqrt{x+2} = x$
Let $x = -1$.	$\sqrt{-1+2} \stackrel{?}{=} -1$
Simplify.	$\sqrt{1} \stackrel{?}{=} -1$
	$1 \neq -1$
	−1 is not a solution.
	−1 is an extraneous solution to the equation.

Note:
Exercise:

Problem: For the equation $\sqrt{x+6} = x$:

ⓐ Is $x = -2$ a solution? ⓑ Is $x = 3$ a solution?

Solution:

ⓐ no ⓑ yes

Note:
Exercise:

Problem: For the equation $\sqrt{-x+2}=x$:

ⓐ Is $x=-2$ a solution? ⓑ Is $x=1$ a solution?

Solution:

ⓐ no ⓑ yes

Now we will see how to solve a radical equation. Our strategy is based on the relation between taking a square root and squaring.
Equation:

$$\text{For } a \geq 0, \quad \left(\sqrt{a}\right)^2 = a$$

Example:
How to Solve Radical Equations
Exercise:

Problem: Solve: $\sqrt{2x-1}=7$.

Solution:
Solution

Step 1. Isolate the radical on one side of the equation.	$\sqrt{2x-1}$ is already isolated on the left side.	$\sqrt{2x-1}=7$
Step 2. Square both sides of the equation.	Remember, $\left(\sqrt{a}\right)^2=a$	$\left(\sqrt{2x-1}\right)^2=(7)^2$
Step 3. Solve the new equation.		$2x-1=49$ $2x=50$ $x=25$

Step 4. Check the answer.		Check: $\sqrt{2x-1}=7$ $\sqrt{2(25)-1}\stackrel{?}{=}7$ $\sqrt{50-1}\stackrel{?}{=}7$ $\sqrt{49}\stackrel{?}{=}7$ $7=7$ ✓ The solution is $x=25$.

Note:
Exercise:

Problem: Solve: $\sqrt{3x-5}=5$.

Solution:

10

Note:
Exercise:

Problem: Solve: $\sqrt{4x+8}=6$.

Solution:

7

Note:
Solve a radical equation.

Isolate the radical on one side of the equation.
Square both sides of the equation.
Solve the new equation.
Check the answer.

Example:
Exercise:

Problem: Solve: $\sqrt{5n-4}-9=0$.

Solution:
Solution

	$\sqrt{5n-4}-9=0$
To isolate the radical, add 9 to both sides.	$\sqrt{5n-4}-9+9=0+9$
Simplify.	$\sqrt{5n-4}=9$
Square both sides of the equation.	$\left(\sqrt{5n-4}\right)^2=(9)^2$
Solve the new equation.	$5n-4=81$
	$5n=85$
	$n=17$
Check the answer.	
$\sqrt{5n-4}-9=0$ $\sqrt{5(17)-4}-9\stackrel{?}{=}0$ $\sqrt{85-4}-9\stackrel{?}{=}0$ $\sqrt{81}-9\stackrel{?}{=}0$ $9-9\stackrel{?}{=}0$ $0=0\checkmark$	
	The solution is n = 17.

Note:
Exercise:

Problem: Solve: $\sqrt{3m+2}-5=0$.

Solution:

$\frac{23}{3}$

Note:
Exercise:

Problem: Solve: $\sqrt{10z+1}-2=0$.

Solution:

$\frac{3}{10}$

Example:
Exercise:

Problem: Solve: $\sqrt{3y+5}+2=5$.

Solution:
Solution

	$\sqrt{3y+5}+2=5$
To isolate the radical, subtract 2 from both sides.	$\sqrt{3y+5}+2-2=5-2$
Simplify.	$\sqrt{3y+5}=3$
Square both sides of the equation.	

	$\left(\sqrt{3y+5}\right)^2 = (3)^2$
Solve the new equation.	$3y+5=9$
	$3y=4$
	$y=\frac{4}{3}$
Check the answer.	
$\sqrt{3y+5}+2=5$ $\sqrt{3\left(\frac{4}{3}\right)+5}+2\stackrel{?}{=}5$ $\sqrt{4+5}+2\stackrel{?}{=}5$ $\sqrt{9}+2\stackrel{?}{=}5$ $3+2\stackrel{?}{=}5$ $5=5\checkmark$	
	The solution is $y=\frac{4}{3}$.

Note:
Exercise:

Problem: Solve: $\sqrt{3p+3}+3=5$.

Solution:

$\frac{1}{2}$

Note:
Exercise:

Problem: Solve: $\sqrt{5q+1}+4=6$.

Solution:

$\frac{3}{5}$

When we use a radical sign, we mean the principal or positive root. If an equation has a square root equal to a negative number, that equation will have no solution.

Example:
Exercise:

Problem: Solve: $\sqrt{9k-2}+1=0$.

Solution:
Solution

	$\sqrt{9k-2}+1=0$
To isolate the radical, subtract 1 from both sides.	$\sqrt{9k-2}+1-1=0-1$
Simplify.	$\sqrt{9k-2}=-1$
Since the square root is equal to a negative number, the equation has no solution.	

Note:
Exercise:

Problem: Solve: $\sqrt{2r-3}+5=0$.

Solution:

no solution

Note:

Exercise:

Problem: Solve: $\sqrt{7s-3}+2=0$.

Solution:

no solution

If one side of the equation is a binomial, we use the binomial squares formula when we square it.

Note:
Binomial Squares
Equation:

$$(a+b)^2 = a^2+2ab+b^2 \qquad (a-b)^2 = a^2-2ab+b^2$$

Don’t forget the middle term!

Example:
Exercise:

Problem: Solve: $\sqrt{p-1}+1=p$.

Solution:
Solution

	$\sqrt{p-1}+1=p$
To isolate the radical, subtract 1 from both sides.	$\sqrt{p-1}+1-1=p-1$
Simplify.	$\sqrt{p-1}=p-1$
Square both sides of the equation.	

	$\left(\sqrt{p-1}\right)^2 = (p-1)^2$
Simplify, then solve the new equation.	$p - 1 = p^2 - 2p + 1$
It is a quadratic equation, so get zero on one side.	$0 = p^2 - 3p + 2$
Factor the right side.	$0 = (p-1)(p-2)$
Use the zero product property.	$0 = p - 1 \quad 0 = p - 2$
Solve each equation.	$p = 1 \quad p = 2$
Check the answers.	
$p = 1$: $\sqrt{p-1} + 1 = p$; $\sqrt{1-1} + 1 \stackrel{?}{=} 1$; $\sqrt{0} + 1 \stackrel{?}{=} 1$; $1 = 1$ ✓ $p = 2$: $\sqrt{p-1} + 1 = p$; $\sqrt{2-1} + 1 \stackrel{?}{=} 2$; $\sqrt{1} + 1 \stackrel{?}{=} 2$; $2 = 2$ ✓	
	The solutions are $p = 1$, $p = 2$.

Note:
Exercise:

Problem: Solve: $\sqrt{x-2} + 2 = x$.

Solution:

$2, 3$

Note:
Exercise:

Problem: Solve: $\sqrt{y-5}+5=y$.

Solution:

$5, 6$

Example:
Exercise:

Problem: Solve: $\sqrt{r+4}-r+2=0$.

Solution:
Solution

	$\sqrt{r+4}-r+2=0$
Isolate the radical.	$\sqrt{r+4}=r-2$
Square both sides of the equation.	$\left(\sqrt{r+4}\right)^2=(r-2)^2$
Solve the new equation.	$r+4=r^2-4r+4$
It is a quadratic equation, so get zero on one side.	$0=r^2-5r$
Factor the right side.	$0=r(r-5)$
Use the zero product property.	$0=r \quad 0=r-5$
Solve the equation.	$r=0 \quad r=5$
Check the answer.	
$r=0$: $\sqrt{r+4}-r+2=0$, $\sqrt{0+4}-0+2\stackrel{?}{=}0$, $\sqrt{4}+2\stackrel{?}{=}0$, $4\neq 0$; $r=5$: $\sqrt{r+4}-r+2=0$, $\sqrt{5+4}-5+2\stackrel{?}{=}0$, $\sqrt{9}-3\stackrel{?}{=}0$, $0=0$ ✓	The solution is $r=5$.

	$r = 0$ is an extraneous solution.

Note:
Exercise:

Problem: Solve: $\sqrt{m+9} - m + 3 = 0$.

Solution:

7

Note:
Exercise:

Problem: Solve: $\sqrt{n+1} - n + 1 = 0$.

Solution:

3

When there is a coefficient in front of the radical, we must square it, too.

Example:
Exercise:

Problem: Solve: $3\sqrt{3x-5} - 8 = 4$.

Solution:
Solution

	$3\sqrt{3x-5} - 8 = 4$
Isolate the radical.	$3\sqrt{3x-5} = 12$
Square both sides of the equation.	

	$\left(3\sqrt{3x-5}\right)^2 = (12)^2$
Simplify, then solve the new equation.	$9(3x-5) = 144$
Distribute.	$27x - 45 = 144$
Solve the equation.	$27x = 189$
	$x = 7$
Check the answer.	
$x = 7$ $\quad 3\sqrt{3x-5} - 8 = 4$ $3\sqrt{3(7)-5} - 8 \stackrel{?}{=} 4$ $3\sqrt{21-5} - 8 \stackrel{?}{=} 4$ $3\sqrt{16} - 8 \stackrel{?}{=} 4$ $3(4) - 8 \stackrel{?}{=} 4$ $4 = 4 \checkmark$	The solution is $x = 7$.

Note:
Exercise:

Problem: Solve: $2\sqrt{4a+2} - 16 = 16$.

Solution:

$\frac{127}{2}$

Note:
Exercise:

Problem: Solve: $3\sqrt{6b+3} - 25 = 50$.

Solution:

$\frac{311}{3}$

Example:
Exercise:

Problem: Solve: $\sqrt{4z-3}=\sqrt{3z+2}$.

Solution:
Solution

	$\sqrt{4z-3}=\sqrt{3z+2}$
The radical terms are isolated.	$\sqrt{4z-3}=\sqrt{3z+2}$
Square both sides of the equation.	$\left(\sqrt{4z-3}\right)^2=\left(\sqrt{3z+2}\right)^2$
Simplify, then solve the new equation.	$4z-3=3z+2$ $z-3=2$ $z=5$
Check the answer. We leave it to you to show that 5 checks!	The solution is $z=5$.

Note:
Exercise:

Problem: Solve: $\sqrt{2x-5}=\sqrt{5x+3}$.

Solution:

no solution

Note:
Exercise:

Problem: Solve: $\sqrt{7y+1}=\sqrt{2y-5}$.

Solution:

no solution

Sometimes after squaring both sides of an equation, we still have a variable inside a radical. When that happens, we repeat Step 1 and Step 2 of our procedure. We isolate the radical and square both sides of the equation again.

Example:
Exercise:

Problem: Solve: $\sqrt{m}+1=\sqrt{m+9}$.

Solution:
Solution

	$\sqrt{m}+1=\sqrt{m+9}$
The radical on the right side is isolated. Square both sides.	$\left(\sqrt{m}+1\right)^2=\left(\sqrt{m+9}\right)^2$
Simplify—be very careful as you multiply! There is still a radical in the equation.	$m+2\sqrt{m}+1=m+9$
So we must repeat the previous steps. Isolate the radical.	$2\sqrt{m}=8$
Square both sides.	$\left(2\sqrt{m}\right)^2=(8)^2$
Simplify, then solve the new equation.	$4m=64$ $m=16$
Check the answer.	
We leave it to you to show that $m=16$ checks!	The solution is $m=16$.

Note:
Exercise:

Problem: Solve: $\sqrt{x}+3=\sqrt{x+5}$.

Solution:

no solution

Note:
Exercise:

Problem: Solve: $\sqrt{m}+5=\sqrt{m+16}$.

Solution:

no solution

Example:
Exercise:

Problem: Solve: $\sqrt{q-2}+3=\sqrt{4q+1}$.

Solution:
Solution

	$\sqrt{q-2}+3=\sqrt{4q+1}$
The radical on the right side is isolated. Square both sides.	$\left(\sqrt{q-2}+3\right)^2=\left(\sqrt{4q+1}\right)^2$
Simplify.	$q-2+6\sqrt{q-2}+9=4q+1$
There is still a radical in the equation. So we must repeat the previous steps. Isolate the radical.	$6\sqrt{q-2}=3q-6$
Square both sides.	$\left(6\sqrt{q-2}\right)^2=(3q-6)^2$
Simplify, then solve the new equation.	$36\,(q-2)=9q^2-36q+36$
Distribute.	$36q-72=9q^2-36q+36$
It is a quadratic equation, so get zero on one side.	$0=9q^2-72q+108$
Factor the right side.	$0=9\left(q^2-8q+12\right)$ $0=9\,(q-6)\,(q-2)$
Use the zero product property.	$q-6=0 \qquad q-2=0$ $q=6 \qquad q=2$
The checks are left to you. (Both solutions should work.)	The solutions are $q=6$ and $q=2$.

Note:
Exercise:

Problem: Solve: $\sqrt{y-3}+2=\sqrt{4y+2}$.

Solution:

no solution

Note:
Exercise:

Problem: Solve: $\sqrt{n-4}+5=\sqrt{3n+3}$.

Solution:

no solution

Use Square Roots in Applications

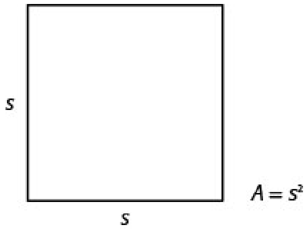
As you progress through your college courses, you'll encounter formulas that include square roots in many disciplines. We have already used formulas to solve geometry applications.

We will use our Problem Solving Strategy for Geometry Applications, with slight modifications, to give us a plan for solving applications with formulas from any discipline.

Note:
Solve applications with formulas.

Read the problem and make sure all the words and ideas are understood. When appropriate, draw a figure and label it with the given information.
Identify what we are looking for.
Name what we are looking for by choosing a variable to represent it.
Translate into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
Solve the equation using good algebra techniques.
Check the answer in the problem and make sure it makes sense.
Answer the question with a complete sentence.

We used the formula $A=L\cdot W$ to find the area of a rectangle with length L and width W. A square is a rectangle in which the length and width are equal. If we let s be the length of a side of a square, the area of the square is s^2.



The formula $A=s^2$ gives us the area of a square if we know the length of a side. What if we want to find the length of a side for a given area? Then we need to solve the equation for s.

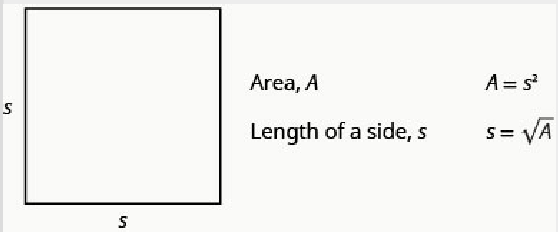
$$A = s^2$$

Take the square root of both sides. $\sqrt{A} = \sqrt{s^2}$

Simplify. $\sqrt{A} = s$

We can use the formula $s = \sqrt{A}$ to find the length of a side of a square for a given area.

Note:
Area of a Square



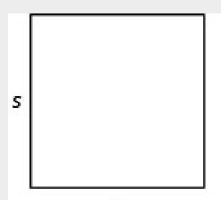
We will show an example of this in the next example.

Example:
Exercise:

Problem:

Mike and Lychelle want to make a square patio. They have enough concrete to pave an area of 200 square feet. Use the formula $s = \sqrt{A}$ to find the length of each side of the patio. Round your answer to the nearest tenth of a foot.

Solution:
Solution

Step 1. Read the problem. Draw a figure and label it with the given information.	
	A = 200 square feet

Step 2. Identify what you are looking for.	The length of a side of the square patio.
Step 3. Name what you are looking for by choosing a variable to represent it.	Let s = the length of a side.
Step 4. Translate into an equation by writing the appropriate formula or model for the situation. Substitute the given information.	$s=\sqrt{A}$, and $A=200$ $s=\sqrt{200}$
Step 5. Solve the equation using good algebra techniques. Round to one decimal place.	$s=14.14213...$ $s\approx 14.1$
Step 6. Check the answer in the problem and make sure it makes sense.	
$14.1^2 \overset{?}{\approx} 200$ $14.1^2 \approx 198.81$ ✓	
This is close enough because we rounded the square root. Is a patio with side 14.1 feet reasonable? Yes.	
Step 7. Answer the question with a complete sentence.	Each side of the patio should be 14.1 feet.

Note:
Exercise:

Problem:

Katie wants to plant a square lawn in her front yard. She has enough sod to cover an area of 370 square feet. Use the formula $s=\sqrt{A}$ to find the length of each side of her lawn. Round your answer to the nearest tenth of a foot.

Solution:

19.2 yards

Note:
Exercise:

Problem:

Sergio wants to make a square mosaic as an inlay for a table he is building. He has enough tile to cover an area of 2704 square centimeters. Use the formula $s = \sqrt{A}$ to find the length of each side of his mosaic. Round your answer to the nearest tenth of a foot.

Solution:

52.0 cm

Another application of square roots has to do with gravity.

Note:
Falling Objects
On Earth, if an object is dropped from a height of h feet, the time in seconds it will take to reach the ground is found by using the formula,
Equation:

$$t = \frac{\sqrt{h}}{4}$$

For example, if an object is dropped from a height of 64 feet, we can find the time it takes to reach the ground by substituting $h = 64$ into the formula.

	$t = \frac{\sqrt{h}}{4}$
	$t = \frac{\sqrt{64}}{4}$
Take the square root of 64.	$t = \frac{8}{4}$
Simplify the fraction.	$t = 2$

It would take 2 seconds for an object dropped from a height of 64 feet to reach the ground.

Example:
Exercise:

Problem:

Christy dropped her sunglasses from a bridge 400 feet above a river. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the sunglasses to reach the river.

Solution:
Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	The time it takes for the sunglasses to reach the river.
Step 3. Name what you are looking for by choosing a variable to represent it.	Let t = time.
Step 4. Translate into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.	$t=\frac{\sqrt{h}}{4}$, and $h=400$ $t=\frac{\sqrt{400}}{4}$
Step 5. Solve the equation using good algebra techniques.	$t=\frac{20}{4}$ $t=5$
Step 6. Check the answer in the problem and make sure it makes sense.	
$5\stackrel{?}{=}\frac{\sqrt{400}}{4}$	

$5 \stackrel{?}{=} \frac{20}{4}$

$5 = 5\checkmark$

Does 5 seconds seem reasonable? Yes.	
Step 7. Answer the question with a complete sentence.	It will take 5 seconds for the sunglasses to hit the water.

Note:
Exercise:

Problem:

A helicopter dropped a rescue package from a height of 1,296 feet. Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the package to reach the ground.

Solution:

9 seconds

Note:
Exercise:

Problem:

A window washer dropped a squeegee from a platform 196 feet above the sidewalk Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the squeegee to reach the sidewalk.

Solution:

3.5 seconds

Police officers investigating car accidents measure the length of the skid marks on the pavement. Then they use square roots to determine the speed, in miles per hour, a car was going before applying the brakes.

Note:
Skid Marks and Speed of a Car
If the length of the skid marks is *d* feet, then the speed, *s*, of the car before the brakes were applied can be found by using the formula,

Equation:

$$s = \sqrt{24d}$$

Example:
Exercise:

Problem:

After a car accident, the skid marks for one car measured 190 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Solution:
Solution

Step 1. **Read** the problem.	
Step 2. **Identify** what we are looking for.	The speed of a car.
Step 3. **Name** what we are looking for.	Let s = the speed.
Step 4. **Translate** into an equation by writing the appropriate formula.	$s = \sqrt{24d}$, and $d = 190$
Substitute the given information.	$s = \sqrt{24(190)}$
Step 5. **Solve the equation**.	$s = \sqrt{4560}$
	$s = 67.52777...$
Round to 1 decimal place.	$s \approx 67.5$
Step 6. **Check** the answer in the problem. $67.5 \overset{?}{\approx} \sqrt{24\,(190)}$ $67.5 \overset{?}{\approx} \sqrt{4560}$ $67.5 \overset{?}{\approx} 67.5277...$	

Is 67.5 mph a reasonable speed?	Yes.
Step 7. **Answer** the question with a complete sentence.	The speed of the car was approximately 67.5 miles per hour.

Note:
Exercise:

Problem:

An accident investigator measured the skid marks of the car. The length of the skid marks was 76 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Solution:

42.7 feet

Note:
Exercise:

Problem:

The skid marks of a vehicle involved in an accident were 122 feet long. Use the formula $s = \sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

Solution:

54.1 feet

Key Concepts

- **To Solve a Radical Equation:**

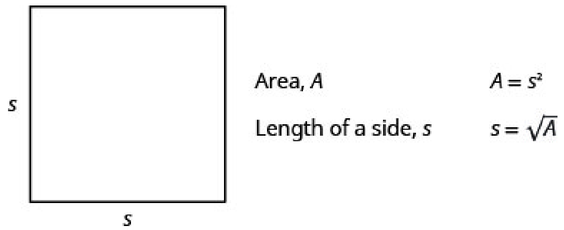
  Isolate the radical on one side of the equation.
  Square both sides of the equation.
  Solve the new equation.
  Check the answer. Some solutions obtained may not work in the original equation.

- **Solving Applications with Formulas**

  Read the problem and make sure all the words and ideas are understood. When appropriate, draw a figure and label it with the given information.
  Identify what we are looking for.
  Name what we are looking for by choosing a variable to represent it.

Translate into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
Solve the equation using good algebra techniques.
Check the answer in the problem and make sure it makes sense.
Answer the question with a complete sentence.

- **Area of a Square**



- **Falling Objects**
  - On Earth, if an object is dropped from a height of h feet, the time in seconds it will take to reach the ground is found by using the formula $t = \frac{\sqrt{h}}{4}$.
- **Skid Marks and Speed of a Car**
  - If the length of the skid marks is d feet, then the speed, s, of the car before the brakes were applied can be found by using the formula $s = \sqrt{24d}$.

Practice Makes Perfect

Solve Radical Equations

In the following exercises, check whether the given values are solutions.

Exercise:

Problem: For the equation $\sqrt{x+12} = x$:
ⓐ Is $x = 4$ a solution?
ⓑ Is $x = -3$ a solution?

Solution:

ⓐ yes ⓑ no

Exercise:

Problem: For the equation $\sqrt{-y+20} = y$:
ⓐ Is $y = 4$ a solution?
ⓑ Is $y = -5$ a solution?

Exercise:

For the equation $\sqrt{t+6} = t$:
ⓐ Is $t = -2$ a solution?
Problem: ⓑ Is $t = 3$ a solution?

Solution:

ⓐ no ⓑ yes

Exercise:

For the equation $\sqrt{u+42} = u$:
ⓐ Is $u = -6$ a solution?
Problem: ⓑ Is $u = 7$ a solution?

In the following exercises, solve.

Exercise:

Problem: $\sqrt{5y+1} = 4$

Solution:

3

Exercise:

Problem: $\sqrt{7z+15} = 6$

Exercise:

Problem: $\sqrt{5x-6} = 8$

Solution:

14

Exercise:

Problem: $\sqrt{4x-3} = 7$

Exercise:

Problem: $\sqrt{2m-3} - 5 = 0$

Solution:

14

Exercise:

Problem: $\sqrt{2n-1} - 3 = 0$

Exercise:

Problem: $\sqrt{6v-2}-10=0$

Solution:

17

Exercise:

Problem: $\sqrt{4u+2}-6=0$

Exercise:

Problem: $\sqrt{5q+3}-4=0$

Solution:

$\frac{13}{5}$

Exercise:

Problem: $\sqrt{4m+2}+2=6$

Exercise:

Problem: $\sqrt{6n+1}+4=8$

Solution:

$\frac{5}{2}$

Exercise:

Problem: $\sqrt{2u-3}+2=0$

Exercise:

Problem: $\sqrt{5v-2}+5=0$

Solution:

no solution

Exercise:

Problem: $\sqrt{3z-5}+2=0$

Exercise:

Problem: $\sqrt{2m+1}+4=0$

Solution:

no solution

Exercise:

Problem: ⓐ $\sqrt{u-3}+3=u$
ⓑ $\sqrt{x+1}-x+1=0$

Exercise:

Problem: ⓐ $\sqrt{v-10}+10=v$
ⓑ $\sqrt{y+4}-y+2=0$

Solution:

ⓐ $10, 11$ ⓑ 5

Exercise:

Problem: ⓐ $\sqrt{r-1}-r=-1$
ⓑ $\sqrt{z+100}-z+10=0$

Exercise:

Problem: ⓐ $\sqrt{s-8}-s=-8$
ⓑ $\sqrt{w+25}-w+5=0$

Solution:

ⓐ $8, 9$ ⓑ 11

Exercise:

Problem: $3\sqrt{2x-3}-20=7$

Exercise:

Problem: $2\sqrt{5x+1}-8=0$

Solution:

3

Exercise:

Problem: $2\sqrt{8r+1}-8=2$

Exercise:

Problem: $3\sqrt{7y+1}-10=8$

Solution:

5

Exercise:

Problem: $\sqrt{3u-2}=\sqrt{5u+1}$

Exercise:

Problem: $\sqrt{4v+3}=\sqrt{v-6}$

Solution:

not a real number

Exercise:

Problem: $\sqrt{8+2r}=\sqrt{3r+10}$

Exercise:

Problem: $\sqrt{12c+6}=\sqrt{10-4c}$

Solution:

$\frac{1}{4}$

Exercise:

Problem: ⓐ $\sqrt{a}+2=\sqrt{a+4}$
ⓑ $\sqrt{b-2}+1=\sqrt{3b+2}$

Exercise:

Problem: ⓐ $\sqrt{r}+6=\sqrt{r+8}$
ⓑ $\sqrt{s-3}+2=\sqrt{s+4}$

Solution:

ⓐ no solution ⓑ $\frac{57}{16}$

Exercise:

Problem: ⓐ $\sqrt{u}+1=\sqrt{u+4}$
ⓑ $\sqrt{n-5}+4=\sqrt{3n+7}$

Exercise:

Problem: ⓐ $\sqrt{x}+10=\sqrt{x+2}$ ⓑ $\sqrt{y-2}+2=\sqrt{2y+4}$

Solution:

ⓐ no solution ⓑ 6

Exercise:

Problem: $\sqrt{2y+4}+6=0$

Exercise:

Problem: $\sqrt{8u+1}+9=0$

Solution:

no solution

Exercise:

Problem: $\sqrt{a}+1=\sqrt{a+5}$

Exercise:

Problem: $\sqrt{d}-2=\sqrt{d-20}$

Solution:

36

Exercise:

Problem: $\sqrt{6s+4}=\sqrt{8s-28}$

Exercise:

Problem: $\sqrt{9p+9}=\sqrt{10p-6}$

Solution:

15

Use Square Roots in Applications

In the following exercises, solve. Round approximations to one decimal place.

Exercise:

Problem:

Landscaping Reed wants to have a square garden plot in his backyard. He has enough compost to cover an area of 75 square feet. Use the formula $s = \sqrt{A}$ to find the length of each side of his garden. Round your answer to the nearest tenth of a foot.

Exercise:

Problem:

Landscaping Vince wants to make a square patio in his yard. He has enough concrete to pave an area of 130 square feet. Use the formula $s = \sqrt{A}$ to find the length of each side of his patio. Round your answer to the nearest tenth of a foot.

Solution:

11.4 feet

Exercise:

Problem:

Gravity While putting up holiday decorations, Renee dropped a light bulb from the top of a 64 foot tall tree. Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the light bulb to reach the ground.

Exercise:

Problem:

Gravity An airplane dropped a flare from a height of 1024 feet above a lake. Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the flare to reach the water.

Solution:

8 seconds

Exercise:

Problem:

Gravity A hang glider dropped his cell phone from a height of 350 feet. Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the cell phone to reach the ground.

Exercise:

Problem:

Gravity A construction worker dropped a hammer while building the Grand Canyon skywalk, 4000 feet above the Colorado River. Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the hammer to reach the river.

Solution:

15.8 seconds

Exercise:

Problem:

Accident investigation The skid marks for a car involved in an accident measured 54 feet. Use the formula $s=\sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Exercise:

Problem:

Accident investigation The skid marks for a car involved in an accident measured 216 feet. Use the formula $s=\sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Solution:

72 feet

Exercise:

Problem:

Accident investigation An accident investigator measured the skid marks of one of the vehicles involved in an accident. The length of the skid marks was 175 feet. Use the formula $s=\sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

Exercise:

Problem:

Accident investigation An accident investigator measured the skid marks of one of the vehicles involved in an accident. The length of the skid marks was 117 feet. Use the formula $s=\sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

Solution:

53.0 feet

Writing Exercises

Exercise:

Problem: Explain why an equation of the form $\sqrt{x}+1=0$ has no solution.

Exercise:

Problem:

ⓐ Solve the equation $\sqrt{r+4}-r+2=0$.
ⓑ Explain why one of the “solutions” that was found was not actually a solution to the equation.

Solution:

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
use square roots in applications.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

Glossary

radical equation
: An equation in which the variable is in the radicand of a square root is called a radical equation

Higher Roots: ASE
By the end of this section, you will be able to:

- Simplify expressions with higher roots
- Use the Product Property to simplify expressions with higher roots
- Use the Quotient Property to simplify expressions with higher roots
- Add and subtract higher roots

Simplify Expressions with Higher Roots

Up to now, in this chapter we have worked with squares and square roots. We will now extend our work to include higher powers and higher roots.

Let's review some vocabulary first.
Equation:

$$\begin{array}{ll} \textbf{We write:} & \textbf{We say:} \\ n^2 & n \text{ squared} \\ n^3 & n \text{ cubed} \\ n^4 & n \text{ to the fourth} \\ n^5 & n \text{ to the fifth} \end{array}$$

The terms 'squared' and 'cubed' come from the formulas for area of a square and volume of a cube.

It will be helpful to have a table of the powers of the integers from -5 to 5. See [link].

Number	Square	Cube	Fourth power	Fifth power
n	n^2	n^3	n^4	n^5
1	1	1	1	1
2	4	8	16	32
3	9	27	81	243
4	16	64	256	1024
5	25	125	625	3125
x	x^2	x^3	x^4	x^5
x^2	x^4	x^6	x^8	x^{10}

Number	Square	Cube	Fourth power	Fifth power
n	n^2	n^3	n^4	n^5
–1	1	–1	1	–1
–2	4	–8	16	–32
–3	9	–27	81	–243
–4	16	–64	256	–1024
–5	25	–125	625	–3125

First through fifth powers of integers from -5 to 5.

Notice the signs in [link]. All powers of positive numbers are positive, of course. But when we have a negative number, the even powers are positive and the odd powers are negative. We'll copy the row with the powers of -2 below to help you see this.

n	n^2	n^3	n^4	n^5
–2	4	–8	16	–32

Earlier in this chapter we defined the square root of a number.
Equation:

$$\text{If } n^2 = m, \text{ then } n \text{ is a square root of } m.$$

And we have used the notation $\sqrt{m}$ to denote the **principal square root**. So $\sqrt{m} \geq 0$ always.

We will now extend the definition to higher roots.

Note:
*n*th Root of a Number
If $b^n = a$, then b is an ***n*th root of a number** a.
The principal *n*th root of a is written $\sqrt[n]{a}$.
n is called the **index** of the radical.

We do not write the index for a square root. Just like we use the word 'cubed' for b^3, we use the term 'cube root' for $\sqrt[3]{a}$.

We refer to [link] to help us find higher roots.
Equation:

$$\begin{array}{rclcrcl} 4^3 & = & 64 & \qquad & \sqrt[3]{64} & = & 4 \\ 3^4 & = & 81 & & \sqrt[4]{81} & = & 3 \\ (-2)^5 & = & -32 & & \sqrt[5]{-32} & = & -2 \end{array}$$

Could we have an even root of a negative number? No. We know that the square root of a negative number is not a real number. The same is true for any even root. Even roots of negative numbers are not real numbers. Odd roots of negative numbers are real numbers.

Note:
Properties of $\sqrt[n]{a}$
When n is an even number and

- $a \geq 0$, then $\sqrt[n]{a}$ is a real number
- $a < 0$, then $\sqrt[n]{a}$ is not a real number

When n is an odd number, $\sqrt[n]{a}$ is a real number for all values of a.

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{8}$ ⓑ $\sqrt[4]{81}$ ⓒ $\sqrt[5]{32}$.

Solution:
Solution

ⓐ

	$\sqrt[3]{8}$
Since $(2)^3 = 8$.	2

ⓑ

	$\sqrt[4]{81}$
Since $(3)^4 = 81$.	3

ⓒ

	$\sqrt[5]{32}$
Since $(2)^5 = 32$.	2

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{27}$ ⓑ $\sqrt[4]{256}$ ⓒ $\sqrt[5]{243}$.

Solution:

ⓐ 3 ⓑ 4 ⓒ 3

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{1000}$ ⓑ $\sqrt[4]{16}$ ⓒ $\sqrt[5]{32}$.

Solution:

ⓐ 10 ⓑ 2 ⓒ 2

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{-64}$ ⓑ $\sqrt[4]{-16}$ ⓒ $\sqrt[5]{-243}$.

Solution:
Solution

ⓐ

	$\sqrt[3]{-64}$
Since $(-4)^3 = -64$.	-4

ⓑ

$$\sqrt[4]{-16}$$

Think, $(?)^4 = -16.$ No real number raised to the fourth power is positive. | Not a real number.

ⓒ

$$\sqrt[5]{-243}$$

Since $(-3)^5 = -243.$ | -3

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{-125}$ ⓑ $\sqrt[4]{-16}$ ⓒ $\sqrt[5]{-32}$.

Solution:

ⓐ -5 ⓑ not real ⓒ -2

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{-216}$ ⓑ $\sqrt[4]{-81}$ ⓒ $\sqrt[5]{-1024}$.

Solution:

ⓐ -6 ⓑ not real ⓒ -4

When we worked with square roots that had variables in the radicand, we restricted the variables to non-negative values. Now we will remove this restriction.

The odd root of a number can be either positive or negative. We have seen that $\sqrt[3]{-64} = -4$.

But the even root of a non-negative number is always non-negative, because we take the principal *n*th root.

Suppose we start with $a = -5$.
Equation:

$$(-5)^4 = 625 \qquad \sqrt[4]{625} = 5$$

How can we make sure the fourth root of −5 raised to the fourth power, $(-5)^4$ is 5? We will see in the following property.

Note:
Simplifying Odd and Even Roots
For any integer $n \geq 1$,
Equation:

$$\begin{array}{ll} \text{when } n \text{ is odd} & \sqrt[n]{a^n} = a \\ \text{when } n \text{ is even} & \sqrt[n]{a^n} = |a| \end{array}$$

We must use the absolute value signs when we take an even root of an expression with a variable in the radical.

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt{x^2}$ ⓑ $\sqrt[3]{n^3}$ ⓒ $\sqrt[4]{p^4}$ ⓓ $\sqrt[5]{y^5}$.

Solution:
Solution

We use the absolute value to be sure to get the positive root.

ⓐ

	$\sqrt{x^2}$
Since $(x)^2 = x^2$ and we want the positive root.	$\|x\|$

ⓑ

	$\sqrt[3]{n^3}$
Since $(n)^3 = n^3$. It is an odd root so there is no absolute value sign.	n

ⓒ

	$\sqrt[4]{p^4}$
Since $(p)^4 = p^4$ and we want the positive root.	$\|p\|$

ⓓ

	$\sqrt[5]{y^5}$
Since $(y)^5 = y^5$. It is an odd root so there is no absolute value sign.	y

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{b^2}$ ⓑ $\sqrt[3]{w^3}$ ⓒ $\sqrt[4]{m^4}$ ⓓ $\sqrt[5]{q^5}$.

Solution:

ⓐ $|b|$ ⓑ w ⓒ $|m|$ ⓓ q

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{y^2}$ ⓑ $\sqrt[3]{p^3}$ ⓒ $\sqrt[4]{z^4}$ ⓓ $\sqrt[5]{q^5}$.

Solution:

ⓐ $|y|$ ⓑ p ⓒ $|z|$ ⓓ q

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{y^{18}}$ ⓑ $\sqrt[4]{z^8}$.

Solution:
Solution

ⓐ

	$\sqrt[3]{y^{18}}$
Since $\left(y^6\right)^3 = y^{18}$.	$\sqrt[3]{\left(y^6\right)^3}$
	y^6

ⓑ

	$\sqrt[4]{z^8}$
Since $\left(z^2\right)^4 = z^8$.	$\sqrt[4]{\left(z^2\right)^4}$
Since z^2 is positive, we do not need an absolute value sign.	z^2

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[4]{u^{12}}$ ⓑ $\sqrt[3]{v^{15}}$.

Solution:

ⓐ u^3 ⓑ v^5

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[5]{c^{20}}$ ⓑ $\sqrt[6]{d^{24}}$.

Solution:

ⓐ c^4 ⓑ d^4

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{64p^6}$ ⓑ $\sqrt[4]{16q^{12}}$.

Solution:
Solution

ⓐ

	$\sqrt[3]{64p^6}$
Rewrite $64p^6$ as $\left(4p^2\right)^3$.	$\sqrt[3]{\left(4p^2\right)^3}$
Take the cube root.	$4p^2$

ⓑ

	$\sqrt[4]{16q^{12}}$
Rewrite the radicand as a fourth power.	$\sqrt[4]{\left(2q^3\right)^4}$
Take the fourth root.	$2\left\|q^3\right\|$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{27x^{27}}$ ⓑ $\sqrt[4]{81q^{28}}$.

Solution:

ⓐ $3x^9$ ⓑ $3\left|q^7\right|$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{125p^9}$ ⓑ $\sqrt[5]{243q^{25}}$.

Solution:

ⓐ $5p^3$ ⓑ $3q^5$

Use the Product Property to Simplify Expressions with Higher Roots

We will simplify expressions with higher roots in much the same way as we simplified expressions with square roots. An *n*th root is considered simplified if it has no factors of m^n.

Note:
Simplified *n*th Root
$\sqrt[n]{a}$ is considered simplified if a has no factors of m^n.

We will generalize the Product Property of Square Roots to include any integer root $n \geq 2$.

Note:
Product Property of *n*th Roots
Equation:

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} \quad \text{and} \quad \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

when $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers and for any integer $n \geq 2$

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{x^4}$ ⓑ $\sqrt[4]{x^7}$.

Solution:
Solution

ⓐ

	$\sqrt[3]{x^4}$
Rewrite the radicand as a product using the largest perfect cube factor.	$\sqrt[3]{x^3 \cdot x}$
Rewrite the radical as the product of two radicals.	$\sqrt[3]{x^3} \cdot \sqrt[3]{x}$
Simplify.	$x\sqrt[3]{x}$

ⓑ

	$\sqrt[4]{x^7}$
Rewrite the radicand as a product using the greatest perfect fourth power factor.	$\sqrt[4]{x^4 \cdot x^3}$
Rewrite the radical as the product of two radicals.	$\sqrt[4]{x^4} \cdot \sqrt[4]{x^3}$
Simplify.	$\lvert x\rvert\sqrt[4]{x^3}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[4]{y^6}$ ⓑ $\sqrt[3]{z^5}$.

Solution:

ⓐ $\lvert y\rvert\sqrt[4]{y^2}$ ⓑ $z\sqrt[3]{z^2}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[5]{p^8}$ ⓑ $\sqrt[6]{q^{13}}$.

Solution:

ⓐ $p\sqrt[5]{p^3}$ ⓑ $q^2\sqrt[6]{q}$

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{16}$ ⓑ $\sqrt[4]{243}$.

Solution:
Solution

ⓐ

	$\sqrt[3]{16}$
	$\sqrt[3]{2^4}$
Rewrite the radicand as a product using the greatest perfect cube factor.	$\sqrt[3]{2^3 \cdot 2}$
Rewrite the radical as the product of two radicals.	$\sqrt[3]{2^3} \cdot \sqrt[3]{2}$
Simplify.	$2\sqrt[3]{2}$

ⓑ

	$\sqrt[4]{243}$
	$\sqrt[4]{3^5}$
Rewrite the radicand as a product using the greatest perfect fourth power factor.	$\sqrt[4]{3^4 \cdot 3}$
Rewrite the radical as the product of two radicals.	$\sqrt[4]{3^4} \cdot \sqrt[4]{3}$
Simplify.	$3\sqrt[4]{3}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{81}$ ⓑ $\sqrt[4]{64}$.

Solution:

ⓐ $3\sqrt[3]{4}$ ⓑ $2\sqrt[4]{4}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{625}$ ⓑ $\sqrt[4]{729}$.

Solution:

ⓐ $5\sqrt[3]{5}$ ⓑ $3\sqrt[4]{9}$

Don’t forget to use the absolute value signs when taking an even root of an expression with a variable in the radical.

Example:

Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{24x^7}$ ⓑ $\sqrt[4]{80y^{14}}$.

Solution:
Solution

ⓐ

	$\sqrt[3]{24x^7}$
Rewrite the radicand as a product using perfect cube factors.	$\sqrt[3]{2^3x^6 \cdot 3x}$
Rewrite the radical as the product of two radicals.	$\sqrt[3]{2^3x^6} \cdot \sqrt[3]{3x}$
Rewrite the first radicand as $\left(2x^2\right)^3$.	$\sqrt[3]{\left(2x^2\right)^3} \cdot \sqrt[3]{3x}$
Simplify.	$2x^2\sqrt[3]{3x}$

ⓑ

	$\sqrt[4]{80y^{14}}$
Rewrite the radicand as a product using perfect fourth power factors.	$\sqrt[4]{2^4y^{12} \cdot 5y^2}$
Rewrite the radical as the product of two radicals.	$\sqrt[4]{2^4y^{12}} \cdot \sqrt[4]{5y^2}$
Rewrite the first radicand as $\left(2y^3\right)^4$.	$\sqrt[4]{\left(2y^3\right)^4} \cdot \sqrt[4]{5y^2}$
Simplify.	$2\left\|y^3\right\|\sqrt[4]{5y^2}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{54p^{10}}$ ⓑ $\sqrt[4]{64q^{10}}$.

Solution:

ⓐ $3p^3\sqrt[3]{2p}$ ⓑ $2q^2\sqrt[4]{4q^2}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{128m^{11}}$ ⓑ $\sqrt[4]{162n^7}$.

Solution:

ⓐ $4m^3\sqrt[3]{2m^2}$ ⓑ $3\left|n\right|\sqrt[4]{2n^3}$

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{-27}$ ⓑ $\sqrt[4]{-16}$.

Solution:
Solution

ⓐ

	$\sqrt[3]{-27}$
Rewrite the radicand as a product using perfect cube factors.	$\sqrt[3]{(-3)^3}$
Take the cube root.	-3

ⓑ

	$\sqrt[4]{-16}$
There is no real number n where $n^4 = -16$.	Not a real number.

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{-108}$ ⓑ $\sqrt[4]{-48}$.

Solution:

ⓐ $-3\sqrt[3]{4}$ ⓑ not real

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{-625}$ ⓑ $\sqrt[4]{-324}$.

Solution:

ⓐ $-5\sqrt[3]{5}$ ⓑ not real

Use the Quotient Property to Simplify Expressions with Higher Roots

We can simplify higher roots with quotients in the same way we simplified square roots. First we simplify any fractions inside the radical.

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{\frac{a^8}{a^5}}$ ⓑ $\sqrt[4]{\frac{a^{10}}{a^2}}$.

Solution:
Solution

ⓐ

	$\sqrt[3]{\frac{a^8}{a^5}}$
Simplify the fraction under the radical first.	$\sqrt[3]{a^3}$
Simplify.	a

ⓑ

	$\sqrt[4]{\frac{a^{10}}{a^2}}$
Simplify the fraction under the radical first.	$\sqrt[4]{a^8}$
Rewrite the radicand using perfect fourth power factors.	$\sqrt[4]{(a^2)^4}$
Simplify.	a^2

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[4]{\frac{x^7}{x^3}}$ ⓑ $\sqrt[4]{\frac{y^{17}}{y^5}}$.

Solution:

ⓐ $|x|$ ⓑ y^3

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{\frac{m^{13}}{m^7}}$ ⓑ $\sqrt[5]{\frac{n^{12}}{n^2}}$.

Solution:

ⓐ m^2 ⓑ n^2

Previously, we used the Quotient Property ‘in reverse’ to simplify square roots. Now we will generalize the formula to include higher roots.

Note:
Quotient Property of *n*th Roots
Equation:

$$\sqrt[n]{\frac{a}{b}}=\frac{\sqrt[n]{a}}{\sqrt[n]{b}}\quad\text{and}\quad\frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\frac{a}{b}}$$

when $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b\ne 0,$ and for any integer $n\ge 2$

Example:
Exercise:

Problem: Simplify: ⓐ $\frac{\sqrt[3]{-108}}{\sqrt[3]{2}}$ ⓑ $\frac{\sqrt[4]{96x^7}}{\sqrt[4]{3x^2}}$.

Solution:
Solution

ⓐ

	$\frac{\sqrt[3]{-108}}{\sqrt[3]{2}}$
Neither radicand is a perfect cube, so use the Quotient Property to write as one radical.	$\sqrt[3]{\frac{-108}{2}}$
Simplify the fraction under the radical.	$\sqrt[3]{-54}$
Rewrite the radicand as a product using perfect cube factors.	$\sqrt[3]{(-3)^3\cdot 2}$
Rewrite the radical as the product of two radicals.	$\sqrt[3]{(-3)^3}\cdot\sqrt[3]{2}$
Simplify.	$-3\sqrt[3]{2}$

ⓑ

$$\frac{\sqrt[4]{96x^7}}{\sqrt[4]{3x^2}}$$

Neither radicand is a perfect fourth power, so use the Quotient Property to write as one radical. $\sqrt[4]{\frac{96x^7}{3x^2}}$

Simplify the fraction under the radical. $\sqrt[4]{32x^5}$

Rewrite the radicand as a product using perfect fourth power factors. $\sqrt[4]{2^4x^4 \cdot 2x}$

Rewrite the radical as the product of two radicals. $\sqrt[4]{(2x)^4} \cdot \sqrt[4]{2x}$

Simplify. $2\left|x\right|\sqrt[4]{2x}$

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{\sqrt[3]{-532}}{\sqrt[3]{2}}$ ⓑ $\frac{\sqrt[4]{486m^{11}}}{\sqrt[4]{3m^5}}$.

Solution:

ⓐ not real ⓑ $3\left|m\right|\sqrt[4]{2m^2}$

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{\sqrt[3]{-192}}{\sqrt[3]{3}}$ ⓑ $\frac{\sqrt[4]{324n^7}}{\sqrt[4]{2n^3}}$.

Solution:

ⓐ -4 ⓑ $3\left|n\right|\sqrt[4]{2}$

If the fraction inside the radical cannot be simplified, we use the first form of the Quotient Property to rewrite the expression as the quotient of two radicals.

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{\frac{24x^7}{y^3}}$ ⓑ $\sqrt[4]{\frac{48x^{10}}{y^8}}$.

Solution:
Solution

ⓐ

	$\sqrt[3]{\frac{24x^7}{y^3}}$
The fraction in the radicand cannot be simplified. Use the Quotient Property to write as two radicals.	$\frac{\sqrt[3]{24x^7}}{\sqrt[3]{y^3}}$
Rewrite each radicand as a product using perfect cube factors.	$\frac{\sqrt[3]{8x^6 \cdot 3x}}{\sqrt[3]{y^3}}$
Rewrite the numerator as the product of two radicals.	$\frac{\sqrt[3]{(2x^2)^3}\,\sqrt[3]{3x}}{\sqrt[3]{y^3}}$
Simplify.	$\frac{2x^2\sqrt[3]{3x}}{y}$

ⓑ

	$\sqrt[4]{\frac{48x^{10}}{y^8}}$
The fraction in the radicand cannot be simplified. Use the Quotient Property to write as two radicals.	$\frac{\sqrt[4]{48x^{10}}}{\sqrt[4]{y^8}}$
Rewrite each radicand as a product using perfect fourth power factors.	$\frac{\sqrt[4]{16x^8 \cdot 3x^2}}{\sqrt[4]{y^8}}$
Rewrite the numerator as the product of two radicals.	$\frac{\sqrt[4]{(2x^2)^4}\,\sqrt[4]{3x^2}}{\sqrt[4]{(y^2)^4}}$
Simplify.	$\frac{2x^2\sqrt[4]{3x^2}}{y^2}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{\frac{108c^{10}}{d^6}}$ ⓑ $\sqrt[4]{\frac{80x^{10}}{y^5}}$.

Solution:

ⓐ $\frac{3c^3\sqrt[3]{4c}}{d^2}$ ⓑ $\frac{x^2}{|y|}\sqrt[4]{\frac{80x^2}{y}}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{\frac{40r^3}{s}}$ ⓑ $\sqrt[4]{\frac{162m^{14}}{n^{12}}}$.

Solution:

ⓐ $r\sqrt[3]{\frac{40}{s}}$ ⓑ $\frac{3m^3\sqrt[4]{2m^2}}{|n^3|}$

Add and Subtract Higher Roots

We can add and subtract higher roots like we added and subtracted square roots. First we provide a formal definition of like radicals.

Note:
Like Radicals
Radicals with the same index and same radicand are called **like radicals**.

Like radicals have the same index and the same radicand.

- $9\sqrt[4]{42x}$ and $-2\sqrt[4]{42x}$ are like radicals.
- $5\sqrt[3]{125x}$ and $6\sqrt[3]{125y}$ are not like radicals. The radicands are different.
- $2\sqrt[5]{1000q}$ and $-4\sqrt[4]{1000q}$ are not like radicals. The indices are different.

We add and subtract like radicals in the same way we add and subtract like terms. We can add $9\sqrt[4]{42x}+\left(-2\sqrt[4]{42x}\right)$ and the result is $7\sqrt[4]{42x}$.

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{4x}+\sqrt[3]{4x}$ ⓑ $4\sqrt[4]{8}-2\sqrt[4]{8}$.

Solution:
Solution

ⓐ

$$\sqrt[3]{4x}+\sqrt[3]{4x}$$

The radicals are like, so we add the coefficients. $2\sqrt[3]{4x}$

ⓑ

$$4\sqrt[4]{8}-2\sqrt[4]{8}$$

The radicals are like, so we subtract the coefficients. $2\sqrt[4]{8}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[5]{3x}+\sqrt[5]{3x}$ ⓑ $3\sqrt[3]{9}-\sqrt[3]{9}$.

Solution:

ⓐ $2\sqrt[5]{3x}$ ⓑ $2\sqrt[3]{9}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[4]{10y}+\sqrt[4]{10y}$ ⓑ $5\sqrt[6]{32}-3\sqrt[6]{32}$.

Solution:

ⓐ $2\sqrt[4]{10y}$ ⓑ $2\sqrt[6]{32}$

When an expression does not appear to have like radicals, we will simplify each radical first. Sometimes this leads to an expression with like radicals.

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{54}-\sqrt[3]{16}$ ⓑ $\sqrt[4]{48}+\sqrt[4]{243}$.

Solution:
Solution

ⓐ

$$\sqrt[3]{54} - \sqrt[3]{16}$$

Rewrite each radicand using perfect cube factors.	$\sqrt[3]{27} \cdot \sqrt[3]{2} - \sqrt[3]{8} \cdot \sqrt[3]{2}$
Rewrite the perfect cubes.	$\sqrt[3]{(3)^3}\,\sqrt[3]{2} - \sqrt[3]{(2)^3}\,\sqrt[3]{2}$
Simplify the radicals where possible.	$3\sqrt[3]{2} - 2\sqrt[3]{2}$
Combine like radicals.	$\sqrt[3]{2}$

ⓑ

$$\sqrt[4]{48} + \sqrt[4]{243}$$

Rewrite using perfect fourth power factors.	$\sqrt[4]{16} \cdot \sqrt[4]{3} + \sqrt[4]{81} \cdot \sqrt[4]{3}$
Rewrite the perfect fourth powers.	$\sqrt[4]{(2)^4}\,\sqrt[4]{3} + \sqrt[4]{(3)^4}\,\sqrt[4]{3}$
Simplify the radicals where possible.	$2\sqrt[4]{3} + 3\sqrt[4]{3}$
Combine like radicals.	$5\sqrt[4]{3}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{192} - \sqrt[3]{81}$ ⓑ $\sqrt[4]{32} + \sqrt[4]{512}$.

Solution:

ⓐ $\sqrt[3]{3}$ ⓑ $6\sqrt[4]{2}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{108} - \sqrt[3]{250}$ ⓑ $\sqrt[5]{64} + \sqrt[5]{486}$.

Solution:

ⓐ $-\sqrt[3]{2}$ ⓑ $5\sqrt[5]{2}$

Example:

Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{24x^4} - \sqrt[3]{-81x^7}$ ⓑ $\sqrt[4]{162y^9} + \sqrt[4]{516y^5}$.

Solution:
Solution

ⓐ

	$\sqrt[3]{24x^4} - \sqrt[3]{-81x^7}$
Rewrite each radicand using perfect cube factors.	$\sqrt[3]{8x^3} \cdot \sqrt[3]{3x} - \sqrt[3]{-27x^6} \cdot \sqrt[3]{3x}$
Rewrite the perfect cubes.	$\sqrt[3]{(2x)^3}\,\sqrt[3]{3x} - \sqrt[3]{(-3x^2)^3}\,\sqrt[3]{3x}$
Simplify the radicals where possible.	$2x\sqrt[3]{3x} - \left(-3x^2\sqrt[3]{3x}\right)$

ⓑ

	$\sqrt[4]{162y^9} + \sqrt[4]{516y^5}$
Rewrite each radicand using perfect fourth power factors.	$\sqrt[4]{81y^8} \cdot \sqrt[4]{2y} + \sqrt[4]{256y^4} \cdot \sqrt[4]{2y}$
Rewrite the perfect fourth powers.	$\sqrt[4]{(3y^2)^4} \cdot \sqrt[4]{2y} + \sqrt[4]{(4y)^4} \cdot \sqrt[4]{2y}$
Simplify the radicals where possible.	$3y^2\sqrt[4]{2y} + 4\left\|y\right\|\sqrt[4]{2y}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{32y^5} - \sqrt[3]{-108y^8}$ ⓑ $\sqrt[4]{243r^{11}} + \sqrt[4]{768r^{10}}$.

Solution:

ⓐ $2y\sqrt[3]{4y^2} + 3y^2\sqrt[3]{4y^2}$ ⓑ $3r^2\sqrt[4]{3r^3} + 4r^2\sqrt[4]{3r^2}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{40z^7} - \sqrt[3]{-135z^4}$ ⓑ $\sqrt[4]{80s^{13}} + \sqrt[4]{1280s^6}$.

Solution:

ⓐ $2z^2\sqrt[3]{5z} + 3z\sqrt[3]{5z}$ ⓑ $2\left|s^3\right|\sqrt[4]{5s} + 4\left|s\right|\sqrt[4]{5s}$

Note:
Access these online resources for additional instruction and practice with simplifying higher roots.

- Simplifying Higher Roots
- Add/Subtract Roots with Higher Indices

Key Concepts

- **Properties of**
- $\sqrt[n]{a}$ when n is an even number and
  - $a \geq 0$, then $\sqrt[n]{a}$ is a real number
  - $a < 0$, then $\sqrt[n]{a}$ is not a real number
  - When n is an odd number, $\sqrt[n]{a}$ is a real number for all values of a.
  - For any integer $n \geq 2$, when n is odd $\sqrt[n]{a^n} = a$
  - For any integer $n \geq 2$, when n is even $\sqrt[n]{a^n} = |a|$
- $\sqrt[n]{a}$ is considered simplified if a has no factors of m^n.
- **Product Property of nth Roots**
  Equation:

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} \quad \text{and} \quad \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

- **Quotient Property of nth Roots**
  Equation:

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \quad \text{and} \quad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

- To combine like radicals, simply add or subtract the coefficients while keeping the radical the same.

Practice Makes Perfect

Simplify Expressions with Higher Roots

In the following exercises, simplify.
Exercise:

Problem:
ⓐ $\sqrt[3]{216}$
ⓑ $\sqrt[4]{256}$
ⓒ $\sqrt[5]{32}$

Exercise:

ⓐ $\sqrt[3]{27}$
ⓑ $\sqrt[4]{16}$
Problem: ⓒ $\sqrt[5]{243}$

Solution:

ⓐ 3 ⓑ 2 ⓒ 3

Exercise:

ⓐ $\sqrt[3]{512}$
ⓑ $\sqrt[4]{81}$
Problem: ⓒ $\sqrt[5]{1}$

Exercise:

ⓐ $\sqrt[3]{125}$
ⓑ $\sqrt[4]{1296}$
Problem: ⓒ $\sqrt[5]{1024}$

Solution:

ⓐ 5 ⓑ 6 ⓒ 4

Exercise:

ⓐ $\sqrt[3]{-8}$
ⓑ $\sqrt[4]{-81}$
Problem: ⓒ $\sqrt[5]{-32}$

Exercise:

ⓐ $\sqrt[3]{-64}$
ⓑ $\sqrt[4]{-16}$
Problem: ⓒ $\sqrt[5]{-243}$

Solution:

ⓐ -4 ⓑ not real ⓒ -3

Exercise:

ⓐ $\sqrt[3]{-125}$
ⓑ $\sqrt[4]{-1296}$
Problem: ⓒ $\sqrt[5]{-1024}$

Exercise:

Problem: ⓐ $\sqrt[3]{-512}$
ⓑ $\sqrt[4]{-81}$
ⓒ $\sqrt[5]{-1}$

Solution:

ⓐ -8 ⓑ not a real number ⓒ -1

Exercise:

Problem: ⓐ $\sqrt[5]{u^5}$
ⓑ $\sqrt[8]{v^8}$

Exercise:

Problem: ⓐ $\sqrt[3]{a^3}$
ⓑ

$\sqrt[12]{b^{12}}$

Solution:

ⓐ a ⓑ $|b|$

Exercise:

Problem: ⓐ $\sqrt[4]{y^4}$
ⓑ $\sqrt[7]{m^7}$

Exercise:

Problem: ⓐ $\sqrt[8]{k^8}$
ⓑ $\sqrt[6]{p^6}$

Solution:

ⓐ $|k|$ ⓑ $|p|$

Exercise:

Problem: ⓐ $\sqrt[3]{x^9}$
ⓑ $\sqrt[4]{y^{12}}$

Exercise:

Problem: ⓐ $\sqrt[5]{a^{10}}$
ⓑ $\sqrt[3]{b^{27}}$

Solution:

ⓐ a^2 ⓑ b^9

Exercise:

Problem: ⓐ $\sqrt[4]{m^8}$
ⓑ $\sqrt[5]{n^{20}}$

Exercise:

Problem: ⓐ $\sqrt[6]{r^{12}}$
ⓑ $\sqrt[3]{s^{30}}$

Solution:

ⓐ r^2 ⓑ s^{10}

Exercise:

Problem: ⓐ $\sqrt[4]{16x^8}$
ⓑ $\sqrt[6]{64y^{12}}$

Exercise:

Problem: ⓐ $\sqrt[3]{-8c^9}$
ⓑ $\sqrt[3]{125d^{15}}$

Solution:

ⓐ $-2c^3$ ⓑ $5d^5$

Exercise:

Problem: ⓐ $\sqrt[3]{216a^6}$
ⓑ $\sqrt[5]{32b^{20}}$

Exercise:

ⓐ $\sqrt[7]{128r^{14}}$
Problem: ⓑ $\sqrt[4]{81s^{24}}$

Solution:

ⓐ $2r^2$ ⓑ $3s^6$

Use the Product Property to Simplify Expressions with Higher Roots

In the following exercises, simplify.
Exercise:

Problem: ⓐ $\sqrt[3]{r^5}$ ⓑ $\sqrt[4]{s^{10}}$

Exercise:

Problem: ⓐ $\sqrt[5]{u^7}$ ⓑ $\sqrt[6]{v^{11}}$

Solution:

ⓐ $u\sqrt[5]{u^2}$ ⓑ $v\sqrt[6]{v^5}$

Exercise:

Problem: ⓐ $\sqrt[4]{m^5}$ ⓑ $\sqrt[8]{n^{10}}$

Exercise:

Problem: ⓐ $\sqrt[5]{p^8}$ ⓑ $\sqrt[3]{q^8}$

Solution:

ⓐ $p\sqrt[5]{p^3}$ ⓑ $q^2\sqrt[3]{q^2}$

Exercise:

Problem: ⓐ $\sqrt[4]{32}$ ⓑ $\sqrt[5]{64}$

Exercise:

Problem: ⓐ $\sqrt[3]{625}$ ⓑ $\sqrt[6]{128}$

Solution:

ⓐ $5\sqrt[3]{5}$ ⓑ $2\sqrt[6]{2}$

Exercise:

Problem: ⓐ $\sqrt[5]{64}$ ⓑ $\sqrt[3]{256}$

Exercise:

Problem: ⓐ $\sqrt[4]{3125}$ ⓑ $\sqrt[3]{81}$

Solution:

ⓐ $5\sqrt[4]{5}$ ⓑ $3\sqrt[3]{3}$

Exercise:

Problem: ⓐ $\sqrt[3]{108x^5}$ ⓑ $\sqrt[4]{48y^6}$

Exercise:

Problem: ⓐ $\sqrt[5]{96a^7}$ ⓑ $\sqrt[3]{375b^4}$

Solution:

ⓐ $2a\sqrt[5]{3a^2}$ ⓑ $5b\sqrt[3]{3b}$

Exercise:

Problem: ⓐ $\sqrt[4]{405m^{10}}$ ⓑ $\sqrt[5]{160n^8}$

Exercise:

Problem: ⓐ $\sqrt[3]{512p^5}$ ⓑ $\sqrt[4]{324q^7}$

Solution:

ⓐ $8p\sqrt[3]{p^2}$ ⓑ $3q\sqrt[4]{4q^3}$

Exercise:

Problem: ⓐ $\sqrt[3]{-864}$ ⓑ $\sqrt[4]{-256}$

Exercise:

Problem: ⓐ $\sqrt[5]{-486}$ ⓑ $\sqrt[6]{-64}$

Solution:

ⓐ $-3\sqrt[5]{2}$ ⓑ not real

Exercise:

Problem: ⓐ $\sqrt[5]{-32}$ ⓑ $\sqrt[8]{-1}$

Exercise:

Problem: ⓐ $\sqrt[3]{-8}$ ⓑ $\sqrt[4]{-16}$

Solution:

ⓐ -2 ⓑ not real

Use the Quotient Property to Simplify Expressions with Higher Roots

In the following exercises, simplify.

Exercise:

Problem: ⓐ $\sqrt[3]{\frac{p^{11}}{p^2}}$ ⓑ $\sqrt[4]{\frac{q^{17}}{q^{13}}}$

Exercise:

Problem: ⓐ $\sqrt[5]{\frac{d^{12}}{d^7}}$ ⓑ $\sqrt[8]{\frac{m^{12}}{m^4}}$

Solution:

ⓐ d ⓑ $|m|$

Exercise:

Problem: ⓐ $\sqrt[5]{\frac{u^{21}}{u^{11}}}$ ⓑ $\sqrt[6]{\frac{v^{30}}{v^{12}}}$

Exercise:

Problem: ⓐ $\sqrt[3]{\frac{r^{14}}{r^5}}$ ⓑ $\sqrt[4]{\frac{c^{21}}{c^9}}$

Solution:

ⓐ r^2 ⓑ $\left|c^3\right|$

Exercise:

Problem: ⓐ $\frac{\sqrt[4]{64}}{\sqrt[4]{2}}$ ⓑ $\frac{\sqrt[5]{128x^8}}{\sqrt[5]{2x^2}}$

Exercise:

Problem: ⓐ $\frac{\sqrt[3]{-625}}{\sqrt[3]{5}}$ ⓑ $\frac{\sqrt[4]{80m^7}}{\sqrt[4]{5m}}$

Solution:

ⓐ -5 ⓑ $4m\sqrt[4]{m^2}$

Exercise:

Problem: ⓐ $\sqrt[3]{\frac{1050}{2}}$ ⓑ $\sqrt[4]{\frac{486y^9}{2y^3}}$

Exercise:

Problem: ⓐ $\sqrt[3]{\frac{162}{6}}$ ⓑ $\sqrt[4]{\frac{160r^{10}}{5r^3}}$

Solution:

ⓐ $3\sqrt[3]{6}$ ⓑ $2\left|r\right|\sqrt[4]{2r^3}$

Exercise:

Problem: ⓐ $\sqrt[3]{\frac{54a^8}{b^3}}$ ⓑ $\sqrt[4]{\frac{64c^5}{d^2}}$

Exercise:

Problem: ⓐ $\sqrt[5]{\frac{96r^{11}}{s^3}}$ ⓑ $\sqrt[6]{\frac{128u^7}{v^3}}$

Solution:

ⓐ $\frac{2r^2\sqrt[5]{3r}}{s^3}$ ⓑ $\frac{2u^3\sqrt[6]{2uv3}}{v}$

Exercise:

Problem: ⓐ $\sqrt[3]{\frac{81s^8}{t^3}}$ ⓑ $\sqrt[4]{\frac{64p^{15}}{q^{12}}}$

Exercise:

Problem: ⓐ $\sqrt[3]{\frac{625u^{10}}{v^3}}$ ⓑ $\sqrt[4]{\frac{729c^{21}}{d^8}}$

Solution:

ⓐ $\frac{5u^3\sqrt[3]{5u}}{v}$ ⓑ $\frac{3c^5\sqrt[4]{9c}}{d^2}$

Add and Subtract Higher Roots

In the following exercises, simplify.

Exercise:

Problem: ⓐ $\sqrt[7]{8p}+\sqrt[7]{8p}$
ⓑ $3\sqrt[3]{25}-\sqrt[3]{25}$

Exercise:

Problem: ⓐ $\sqrt[3]{15q}+\sqrt[3]{15q}$
ⓑ $2\sqrt[4]{27}-6\sqrt[4]{27}$

Solution:

ⓐ $2\sqrt[3]{15q}$ ⓑ $-4\sqrt[4]{27}$

Exercise:

Problem: ⓐ $3\sqrt[5]{9x}+7\sqrt[5]{9x}$
ⓑ $8\sqrt[7]{3q}-2\sqrt[7]{3q}$

Exercise:

Problem: ⓐ

$23\sqrt[12]{4y}+19\sqrt[12]{4y}$

ⓑ

$31\sqrt[19]{5z}-17\sqrt[19]{5z}$

Solution:

ⓐ

$42\sqrt[12]{4y}$

ⓑ

$14\sqrt[19]{5z}$

Exercise:

Problem: ⓐ $\sqrt[3]{81}-\sqrt[3]{192}$
ⓑ $\sqrt[4]{512}-\sqrt[4]{32}$

Exercise:

Problem: ⓐ $\sqrt[3]{250}-\sqrt[3]{54}$
ⓑ $\sqrt[4]{243}-\sqrt[4]{1875}$

Solution:

ⓐ $5\sqrt[3]{5}-3\sqrt[3]{2}$ ⓑ $-2\sqrt[4]{3}$

Exercise:

Problem: ⓐ $\sqrt[3]{128}+\sqrt[3]{250}$
ⓑ $\sqrt[5]{729}+\sqrt[5]{96}$

Exercise:

Problem: ⓐ $\sqrt[4]{243}+\sqrt[4]{1250}$ ⓑ $\sqrt[3]{2000}+\sqrt[3]{54}$

Solution:

ⓐ $3\sqrt[4]{3}+5\sqrt[4]{2}$ ⓑ $13\sqrt[3]{2}$

Exercise:

Problem: ⓐ $\sqrt[3]{64a^{10}}-\sqrt[3]{-216a^{12}}$ ⓑ $\sqrt[4]{486u^7}+\sqrt[4]{768u^3}$

Exercise:

Problem: ⓐ $\sqrt[3]{80b^5}-\sqrt[3]{-270b^3}$ ⓑ $\sqrt[4]{160v^{10}}-\sqrt[4]{1280v^3}$

Solution:

ⓐ $2b\sqrt[3]{10b^2}+3b\sqrt[3]{10}$ ⓑ $2v^2\sqrt[4]{10v^2}-4\sqrt[4]{5v^3}$

Mixed Practice

In the following exercises, simplify.

Exercise:

Problem: $\sqrt[4]{16}$

Exercise:

Problem: $\sqrt[6]{64}$

Solution:

2

Exercise:

Problem: $\sqrt[3]{a^3}$

Exercise:

Problem:

$\sqrt[12]{b^{12}}$

Solution:

$|b|$

Exercise:

Problem: $\sqrt[3]{-8c^9}$

Exercise:

Problem: $\sqrt[3]{125d^{15}}$

Solution:

$5d^5$

Exercise:

Problem: $\sqrt[3]{r^5}$

Exercise:

Problem: $\sqrt[4]{s^{10}}$

Solution:

$s^2\sqrt[4]{s^2}$

Exercise:

Problem: $\sqrt[3]{108x^5}$

Exercise:

Problem: $\sqrt[4]{48y^6}$

Solution:

$2y\sqrt[4]{3y^2}$

Exercise:

Problem: $\sqrt[5]{-486}$

Exercise:

Problem: $\sqrt[6]{-64}$

Solution:

not real

Exercise:

Problem: $\frac{\sqrt[4]{64}}{\sqrt[4]{2}}$

Exercise:

Problem: $\frac{\sqrt[5]{128x^8}}{\sqrt[5]{2x^2}}$

Solution:

$2x\sqrt[5]{2x}$

Exercise:

Problem: $\sqrt[5]{\frac{96r^{11}}{s^3}}$

Exercise:

Problem: $\sqrt[6]{\frac{128u^7}{v^3}}$

Solution:

$\frac{2u^3\sqrt[6]{2uv3}}{v}$

Exercise:

Problem: $\sqrt[3]{81}-\sqrt[3]{192}$

Exercise:

Problem: $\sqrt[4]{512}-\sqrt[4]{32}$

Solution:

$4\sqrt[4]{2}$

Exercise:

Problem: $\sqrt[3]{64a^{10}}-\sqrt[3]{-216a^{12}}$

Exercise:

Problem: $\sqrt[4]{486u^7}+\sqrt[4]{768u^3}$

Solution:

$3u\sqrt[4]{6u^3}+4\sqrt[4]{3u^3}$

Everyday Math

Exercise:

Problem:

Population growth The expression $10 \cdot x^n$ models the growth of a mold population after n generations. There were 10 spores at the start, and each had x offspring. So $10 \cdot x^n$ is the number of offspring at the fifth generation. At the fifth generation there were 10,240 offspring. Simplify the expression $\sqrt[5]{\frac{10{,}240}{10}}$ to determine the number of offspring of each spore.

Exercise:

Problem:

Spread of a virus The expression $3 \cdot x^n$ models the spread of a virus after n cycles. There were three people originally infected with the virus, and each of them infected x people. So $3 \cdot x^4$ is the number of people infected on the fourth cycle. At the fourth cycle 1875 people were infected. Simplify the expression $\sqrt[4]{\frac{1875}{3}}$ to determine the number of people each person infected.

Solution:

5

Writing Exercises

Exercise:

Problem: Explain how you know that $\sqrt[5]{x^{10}} = x^2$.

Exercise:

Problem: Explain why $\sqrt[4]{-64}$ is not a real number but $\sqrt[3]{-64}$ is.

Solution:

Answers may vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
simplify expressions with higher roots.			
use the Product Property to simplify expressions with higher roots.			
use the Quotient Property to simplify expressions with higher roots.			
add and subtract higher roots.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Glossary

*n*th root of a number
: If $b^n = a$, then b is an *n*th root of a.

principal *n*th root
: The principal *n*th root of a is written $\sqrt[n]{a}$.

index
: $\sqrt[n]{a}$ n is called the *index* of the radical.

like radicals
: Radicals with the same index and same radicand are called like radicals.

Rational Exponents: ASE
By the end of this section, you will be able to:

- Simplify expressions with $a^{\frac{1}{n}}$
- Simplify expressions with $a^{\frac{m}{n}}$
- Use the Laws of Exponents to simply expressions with rational exponents

Simplify Expressions with $a^{\frac{1}{n}}$

Rational exponents are another way of writing expressions with radicals. When we use **rational exponents**, we can apply the properties of exponents to simplify expressions.

The Power Property for Exponents says that $\left(a^m\right)^n = a^{m \cdot n}$ when *m* and *n* are whole numbers. Let's assume we are now not limited to whole numbers.

Suppose we want to find a number *p* such that $\left(8^p\right)^3 = 8$. We will use the Power Property of Exponents to find the value of *p*.

$$\begin{array}{lrcl} & \left(8^p\right)^3 & = & 8 \\ \text{Multiply the exponents on the left.} & 8^{3p} & = & 8 \\ \text{Write the exponent 1 on the right.} & 8^{3p} & = & 8^1 \\ \text{The exponents must be equal.} & 3p & = & 1 \\ \text{Solve for } p. & p & = & \frac{1}{3} \end{array}$$

$$\text{So } \left(8^{\frac{1}{3}}\right)^3 \;=\; 8.$$

But we know also $\left(\sqrt[3]{8}\right)^3 = 8$. Then it must be that $8^{\frac{1}{3}} = \sqrt[3]{8}$.

This same logic can be used for any positive integer exponent *n* to show that $a^{\frac{1}{n}} = \sqrt[n]{a}$.

Note:
Rational Exponent $a^{\frac{1}{n}}$
If $\sqrt[n]{a}$ is a real number and $n \geq 2$, $a^{\frac{1}{n}} = \sqrt[n]{a}$.

There will be times when working with expressions will be easier if you use rational exponents and times when it will be easier if you use radicals. In the first few examples, you'll practice converting expressions between these two notations.

Example:
Exercise:

Problem: Write as a radical expression: ⓐ $x^{\frac{1}{2}}$ ⓑ $y^{\frac{1}{3}}$ ⓒ $z^{\frac{1}{4}}$.

Solution:
Solution

We want to write each expression in the form $\sqrt[n]{a}$.

ⓐ

	$x^{\frac{1}{2}}$
The denominator of the exponent is 2, so the index of the radical is 2. We do not show the index when it is 2.	$\sqrt{x}$

ⓑ

	$y^{\frac{1}{3}}$
The denominator of the exponent is 3, so the index is 3.	$\sqrt[3]{y}$

ⓒ

The denominator of the exponent is 4, so the index is 4.

$$z^{\frac{1}{4}}$$

$$\sqrt[4]{z}$$

Note:
Exercise:

Problem: Write as a radical expression: ⓐ $t^{\frac{1}{2}}$ ⓑ $m^{\frac{1}{3}}$ ⓒ $r^{\frac{1}{4}}$.

Solution:

ⓐ $\sqrt{t}$ ⓑ $\sqrt[3]{m}$ ⓒ $\sqrt[4]{r}$

Note:
Exercise:

Problem: Write as a radial expression: ⓐ $b^{\frac{1}{2}}$ ⓑ $z^{\frac{1}{3}}$ ⓒ $p^{\frac{1}{4}}$.

Solution:

ⓐ $\sqrt{b}$ ⓑ $\sqrt[3]{z}$ ⓒ $\sqrt[4]{p}$

Example:
Exercise:

Problem: Write with a rational exponent: ⓐ $\sqrt{x}$ ⓑ $\sqrt[3]{y}$ ⓒ $\sqrt[4]{z}$.

Solution:
Solution

We want to write each radical in the form $a^{\frac{1}{n}}$.

ⓐ

	$\sqrt{x}$
No index is shown, so it is 2. The denominator of the exponent will be 2.	$x^{\frac{1}{2}}$

ⓑ

	$\sqrt[3]{y}$
The index is 3, so the denominator of the exponent is 3.	$y^{\frac{1}{3}}$

ⓒ

	$\sqrt[4]{z}$
The index is 4, so the denominator of the exponent is 4.	$z^{\frac{1}{4}}$

Note:
Exercise:

Problem: Write with a rational exponent: ⓐ $\sqrt{s}$ ⓑ $\sqrt[3]{x}$ ⓒ $\sqrt[4]{b}$.

Solution:

ⓐ $s^{\frac{1}{2}}$ ⓑ $x^{\frac{1}{3}}$ ⓒ $b^{\frac{1}{4}}$

Note:

Exercise:

Problem: Write with a rational exponent: ⓐ $\sqrt{v}$ ⓑ $\sqrt[3]{p}$ ⓒ $\sqrt[4]{p}$.

Solution:

ⓐ $v^{\frac{1}{2}}$ ⓑ $p^{\frac{1}{3}}$ ⓒ $p^{\frac{1}{4}}$

Example:
Exercise:

Problem: Write with a rational exponent: ⓐ $\sqrt{5y}$ ⓑ $\sqrt[3]{4x}$ ⓒ $3\sqrt[4]{5z}$.

Solution:
Solution

We want to write each radical in the form $a^{\frac{1}{n}}$.

ⓐ

	$\sqrt{5y}$
No index is shown, so it is 2. The denominator of the exponent will be 2.	$(5y)^{\frac{1}{2}}$

ⓑ

	$\sqrt[3]{4x}$
The index is 3, so the denominator of the exponent is 3.	$(4x)^{\frac{1}{3}}$

ⓒ

	$3\sqrt[4]{5z}$
The index is 4, so the denominator of the exponent is 4.	$3(5z)^{\frac{1}{4}}$

Note:
Exercise:

Problem:

Write with a rational exponent: ⓐ $\sqrt{10m}$ ⓑ $\sqrt[5]{3n}$ ⓒ $3\sqrt[4]{6y}$.

Solution:

ⓐ $(10m)^{\frac{1}{2}}$ ⓑ $(3n)^{\frac{1}{5}}$ ⓒ $(486y)^{\frac{1}{4}}$

Note:
Exercise:

Problem: Write with a rational exponent: ⓐ $\sqrt[7]{3k}$ ⓑ $\sqrt[4]{5j}$ ⓒ $8\sqrt[3]{2a}$.

Solution:

ⓐ $(3k)^{\frac{1}{7}}$ ⓑ $(5j)^{\frac{1}{4}}$ ⓒ $(1024a)^{\frac{1}{3}}$

In the next example, you may find it easier to simplify the expressions if you rewrite them as radicals first.

Example:
Exercise:

Problem: Simplify: ⓐ $25^{\frac{1}{2}}$ ⓑ $64^{\frac{1}{3}}$ ⓒ $256^{\frac{1}{4}}$.

Solution:
Solution

ⓐ

	$25^{\frac{1}{2}}$
Rewrite as a square root.	$\sqrt{25}$
Simplify.	5

ⓑ

	$64^{\frac{1}{3}}$
Rewrite as a cube root.	$\sqrt[3]{64}$
Recognize 64 is a perfect cube.	$\sqrt[3]{4^3}$
Simplify.	4

ⓒ

	$256^{\frac{1}{4}}$
Rewrite as a fourth root.	$\sqrt[4]{256}$
Recognize 256 is a perfect fourth power.	$\sqrt[4]{4^4}$
Simplify.	4

Note:
Exercise:

Problem: Simplify: ⓐ $36^{\frac{1}{2}}$ ⓑ $8^{\frac{1}{3}}$ ⓒ $16^{\frac{1}{4}}$.

Solution:

ⓐ 6 ⓑ 2 ⓒ 2

Note:
Exercise:

Problem: Simplify: ⓐ $100^{\frac{1}{2}}$ ⓑ $27^{\frac{1}{3}}$ ⓒ $81^{\frac{1}{4}}$.

Solution:

ⓐ 10 ⓑ 3 ⓒ 3

Be careful of the placement of the negative signs in the next example. We will need to use the property $a^{-n} = \frac{1}{a^n}$ in one case.

Example:
Exercise:

Problem: Simplify: ⓐ $(-64)^{\frac{1}{3}}$ ⓑ $-64^{\frac{1}{3}}$ ⓒ $(64)^{-\frac{1}{3}}$.

Solution:
Solution

ⓐ

	$(-64)^{\frac{1}{3}}$
Rewrite as a cube root.	$\sqrt[3]{-64}$
Rewrite -64 as a perfect cube.	$\sqrt[3]{(-4)^3}$
Simplify.	-4

ⓑ

$$-64^{\frac{1}{3}}$$

The exponent applies only to the 64. $-\left(64^{\frac{1}{3}}\right)$

Rewrite as a cube root. $-\sqrt[3]{64}$

Rewrite 64 as 4^3. $-\sqrt[3]{4^3}$

Simplify. -4

ⓒ

$$(64)^{-\frac{1}{3}}$$

Rewrite as a fraction with a positive exponent, using the property, $a^{-n} = \frac{1}{a^n}$. $\frac{1}{\sqrt[3]{64}}$

Write as a cube root.

Rewrite 64 as 4^3. $\frac{1}{\sqrt[3]{4^3}}$

Simplify. $\frac{1}{4}$

Note:
Exercise:

Problem: Simplify: ⓐ $(-125)^{\frac{1}{3}}$ ⓑ $-125^{\frac{1}{3}}$ ⓒ $(125)^{-\frac{1}{3}}$.

Solution:

ⓐ -5 ⓑ -5 ⓒ $\frac{1}{5}$

Note:
Exercise:

Problem: Simplify: ⓐ $(-32)^{\frac{1}{5}}$ ⓑ $-32^{\frac{1}{5}}$ ⓒ $(32)^{-\frac{1}{5}}$.

Solution:

ⓐ -2 ⓑ -2 ⓒ $\frac{1}{2}$

Example:
Exercise:

Problem: Simplify: ⓐ $(-16)^{\frac{1}{4}}$ ⓑ $-16^{\frac{1}{4}}$ ⓒ $(16)^{-\frac{1}{4}}$.

Solution:
Solution

ⓐ

	$(-16)^{\frac{1}{4}}$
Rewrite as a fourth root.	$\sqrt[4]{-16}$
There is no real number whose fourth power is -16.	

ⓑ

	$-16^{\frac{1}{4}}$
The exponent only applies to the 16. Rewrite as a fourth root.	$-\sqrt[4]{16}$
Rewrite 16 as 2^4.	$-\sqrt[4]{2^4}$
Simplify.	-2

ⓒ

	$(16)^{-\frac{1}{4}}$
Rewrite using the property $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{(16)^{\frac{1}{4}}}$
Rewrite as a fourth root.	$\frac{1}{\sqrt[4]{16}}$
Rewrite 16 as 2^4.	$\frac{1}{\sqrt[4]{2^4}}$
Simplify.	$\frac{1}{2}$

Note:
Exercise:

Problem: Simplify: ⓐ $(-64)^{\frac{1}{2}}$ ⓑ $-64^{\frac{1}{2}}$ ⓒ $(64)^{-\frac{1}{2}}$.

Solution:

ⓐ -8 ⓑ -8 ⓒ $\frac{1}{8}$

Note:
Exercise:

Problem: Simplify: ⓐ $(-256)^{\frac{1}{4}}$ ⓑ $-256^{\frac{1}{4}}$ ⓒ $(256)^{-\frac{1}{4}}$.

Solution:

ⓐ -4 ⓑ -4 ⓒ $\frac{1}{4}$

Simplify Expressions with $a^{\frac{m}{n}}$

Let’s work with the Power Property for Exponents some more.

Suppose we raise $a^{\frac{1}{n}}$ to the power *m*.

$$\left(a^{\frac{1}{n}}\right)^{m}$$

Multiply the exponents. $\quad a^{\frac{1}{n}\cdot m}$

Simplify. $\quad a^{\frac{m}{n}}$

So $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^{m}$.

Now suppose we take a^{m} to the $\frac{1}{n}$ power.

$$\left(a^{m}\right)^{\frac{1}{n}}$$

Multiply the exponents. $\quad a^{m\cdot\frac{1}{n}}$

Simplify. $\quad a^{\frac{m}{n}}$

So $a^{\frac{m}{n}} = \sqrt[n]{a^{m}}$ also.

Which form do we use to simplify an expression? We usually take the root first—that way we keep the numbers in the radicand smaller.

Note:
Rational Exponent $a^{\frac{m}{n}}$
For any positive integers *m* and *n*,
Equation:

$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^{m}$$
$$a^{\frac{m}{n}} = \sqrt[n]{a^{m}}$$

Example:
Exercise:

Problem: Write with a rational exponent: ⓐ $\sqrt{y^3}$ ⓑ $\sqrt[3]{x^2}$ ⓒ $\sqrt[4]{z^3}$.

Solution:
Solution

We want to use $a^{\frac{m}{n}} = \sqrt[n]{a^m}$ to write each radical in the form $a^{\frac{m}{n}}$.

ⓐ

The numerator of the exponent is the exponent of y, 3.	$\sqrt{y^3}$
The denominator of the exponent is the index of the radical, 2.	$y^{\frac{3}{2}}$

ⓑ

The numerator of the exponent is the exponent of x, 2.	$\sqrt[3]{x^2}$
The denominator of the exponent is the index of the radical, 3.	$x^{\frac{2}{3}}$

ⓒ

The numerator of the exponent is the exponent of z, 3.	$\sqrt[4]{z^3}$
The denominator of the exponent is the index of the radical, 4.	$z^{\frac{3}{4}}$

Note:
Exercise:

Problem: Write with a rational exponent: ⓐ $\sqrt{x^5}$ ⓑ $\sqrt[4]{z^3}$ ⓒ $\sqrt[5]{y^2}$.

Solution:

ⓐ $x^{\frac{5}{2}}$ ⓑ $z^{\frac{3}{4}}$ ⓒ $y^{\frac{2}{5}}$

Note:
Exercise:

Problem: Write with a rational exponent: ⓐ $\sqrt[5]{a^2}$ ⓑ $\sqrt[3]{b^7}$ ⓒ $\sqrt[4]{m^5}$.

Solution:

ⓐ $a^{\frac{2}{5}}$ ⓑ $b^{\frac{7}{3}}$ ⓒ $m^{\frac{5}{4}}$

Example:
Exercise:

Problem: Simplify: ⓐ $9^{\frac{3}{2}}$ ⓑ $125^{\frac{2}{3}}$ ⓒ $81^{\frac{3}{4}}$.

Solution:
Solution

We will rewrite each expression as a radical first using the property, $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$. This form lets us take the root first and so we keep the numbers in the radicand smaller than if we used the other form.

ⓐ

$$9^{\frac{3}{2}}$$

The power of the radical is the numerator of the exponent, 3. Since the denominator of the exponent is 2, this is a square root.

$$\left(\sqrt{9}\right)^3$$

Simplify.

$$(3)^3$$
$$27$$

ⓑ

$$125^{\frac{2}{3}}$$

The power of the radical is the numerator of the exponent, 2. The index of the radical is the denominator of the exponent, 3.

$$\left(\sqrt[3]{125}\right)^2$$

Simplify.

$$(5)^2$$
$$25$$

ⓒ

$$81^{\frac{3}{4}}$$

The power of the radical is the numerator of the exponent, 3. The index of the radical is the denominator of the exponent, 4.

$$\left(\sqrt[4]{81}\right)^3$$

Simplify.

$$(3)^3$$
$$27$$

Note:
Exercise:

Problem: Simplify: ⓐ $4^{\frac{3}{2}}$ ⓑ $27^{\frac{2}{3}}$ ⓒ $625^{\frac{3}{4}}$.

Solution:

ⓐ 8 ⓑ 9 ⓒ 125

Note:
Exercise:

Problem: Simplify: ⓐ $8^{\frac{5}{3}}$ ⓑ $81^{\frac{3}{2}}$ ⓒ $16^{\frac{3}{4}}$.

Solution:

ⓐ 32 ⓑ 729 ⓒ 8

Remember that $b^{-p} = \frac{1}{b^p}$. The negative sign in the exponent does not change the sign of the expression.

Example:
Exercise:

Problem: Simplify: ⓐ $16^{-\frac{3}{2}}$ ⓑ $32^{-\frac{2}{5}}$ ⓒ $4^{-\frac{5}{2}}$.

Solution:
Solution

We will rewrite each expression first using $b^{-p} = \frac{1}{b^p}$ and then change to radical form.

ⓐ

$$16^{-\frac{3}{2}}$$

Rewrite using $b^{-p} = \frac{1}{b^p}$.

$$\frac{1}{16^{\frac{3}{2}}}$$

Change to radical form. The power of the radical is the numerator of the exponent, 3. The index is the denominator of the exponent, 2.

$$\frac{1}{\left(\sqrt{16}\right)^3}$$

Simplify.

$$\frac{1}{4^3}$$

$$\frac{1}{64}$$

ⓑ

$$32^{-\frac{2}{5}}$$

Rewrite using $b^{-p} = \frac{1}{b^p}$.

$$\frac{1}{32^{\frac{2}{5}}}$$

Change to radical form.

$$\frac{1}{\left(\sqrt[5]{32}\right)^2}$$

Rewrite the radicand as a power.

$$\frac{1}{\left(\sqrt[5]{2^5}\right)^2}$$

Simplify.

$$\frac{1}{2^2}$$

$$\frac{1}{4}$$

ⓒ

$$4^{-\frac{5}{2}}$$

Rewrite using $b^{-p} = \frac{1}{b^p}$. $\frac{1}{4^{\frac{5}{2}}}$

Change to radical form. $\frac{1}{\left(\sqrt{4}\right)^5}$

Simplify. $\frac{1}{2^5}$

$\frac{1}{32}$

Note:
Exercise:

Problem: Simplify: ⓐ $8^{-\frac{5}{3}}$ ⓑ $81^{-\frac{3}{2}}$ ⓒ $16^{-\frac{3}{4}}$.

Solution:

ⓐ $\frac{1}{32}$ ⓑ $\frac{1}{729}$ ⓒ $\frac{1}{8}$

Note:
Exercise:

Problem: Simplify: ⓐ $4^{-\frac{3}{2}}$ ⓑ $27^{-\frac{2}{3}}$ ⓒ $625^{-\frac{3}{4}}$.

Solution:

ⓐ $\frac{1}{8}$ ⓑ $\frac{1}{9}$ ⓒ $\frac{1}{125}$

Example:
Exercise:

Problem: Simplify: ⓐ $-25^{\frac{3}{2}}$ ⓑ $-25^{-\frac{3}{2}}$ ⓒ $(-25)^{\frac{3}{2}}$.

Solution:
Solution

ⓐ

	$-25^{\frac{3}{2}}$
Rewrite in radical form.	$-\left(\sqrt{25}\right)^{3}$
Simplify the radical.	$-(5)^{3}$
Simplify.	-125

ⓑ

	$-25^{-\frac{3}{2}}$
Rewrite using $b^{-p} = \frac{1}{b^{p}}$.	$-\left(\frac{1}{25^{\frac{3}{2}}}\right)$
Rewrite in radical form.	$-\left(\frac{1}{\left(\sqrt{25}\right)^{3}}\right)$
Simplify the radical.	$-\left(\frac{1}{(5)^{3}}\right)$
Simplify.	$-\frac{1}{125}$

ⓒ

	$(-25)^{\frac{3}{2}}$
Rewrite in radical form.	$\left(\sqrt{-25}\right)^3$
There is no real number whose square root is -25.	Not a real number.

Note:
Exercise:

Problem: Simplify: ⓐ $-16^{\frac{3}{2}}$ ⓑ $-16^{-\frac{3}{2}}$ ⓒ $(-16)^{-\frac{3}{2}}$.

Solution:

ⓐ -64 ⓑ $-\frac{1}{64}$ ⓒ not a real number

Note:
Exercise:

Problem: Simplify: ⓐ $-81^{\frac{3}{2}}$ ⓑ $-81^{-\frac{3}{2}}$ ⓒ $(-81)^{-\frac{3}{2}}$.

Solution:

ⓐ -729 ⓑ $-\frac{1}{729}$ ⓒ not a real number

Use the Laws of Exponents to Simplify Expressions with Rational Exponents

The same laws of exponents that we already used apply to rational exponents, too. We will list the Exponent Properties here to have them for reference as we simplify expressions.

Note:
Summary of Exponent Properties
If a, b are real numbers and m, n are rational numbers, then

Product Property	$a^m \cdot a^n = a^{m+n}$
Power Property	$(a^m)^n = a^{m \cdot n}$
Product to a Power	$(ab)^m = a^m b^m$
Quotient Property	$\frac{a^m}{a^n} = a^{m-n},\ a \neq 0,\ m > n$
	$\frac{a^m}{a^n} = \frac{1}{a^{n-m}},\ a \neq 0,\ n > m$
Zero Exponent Definition	$a^0 = 1,\ a \neq 0$
Quotient to a Power Property	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m},\ b \neq 0$

When we multiply the same base, we add the exponents.

Example:
Exercise:

Problem: Simplify: ⓐ $2^{\frac{1}{2}} \cdot 2^{\frac{5}{2}}$ ⓑ $x^{\frac{2}{3}} \cdot x^{\frac{4}{3}}$ ⓒ $z^{\frac{3}{4}} \cdot z^{\frac{5}{4}}$.

Solution:
Solution

ⓐ

$$2^{\frac{1}{2}} \cdot 2^{\frac{5}{2}}$$

The bases are the same, so we add the exponents.	$2^{\frac{1}{2}+\frac{5}{2}}$
Add the fractions.	$2^{\frac{6}{2}}$
Simplify the exponent.	2^3
Simplify.	8

ⓑ

$$x^{\frac{2}{3}} \cdot x^{\frac{4}{3}}$$

The bases are the same, so we add the exponents.	$x^{\frac{2}{3}+\frac{4}{3}}$
Add the fractions.	$x^{\frac{6}{3}}$
Simplify.	x^2

ⓒ

$$z^{\frac{3}{4}} \cdot z^{\frac{5}{4}}$$

The bases are the same, so we add the exponents.	$z^{\frac{3}{4}+\frac{5}{4}}$
Add the fractions.	$z^{\frac{8}{4}}$
Simplify.	z^2

Note:
Exercise:

Problem: Simplify: ⓐ $3^{\frac{2}{3}} \cdot 3^{\frac{4}{3}}$ ⓑ $y^{\frac{1}{3}} \cdot y^{\frac{8}{3}}$ ⓒ $m^{\frac{1}{4}} \cdot m^{\frac{3}{4}}$.

Solution:

ⓐ 9 ⓑ y^3 ⓒ m

Note:
Exercise:

Problem: Simplify: ⓐ $5^{\frac{3}{5}} \cdot 5^{\frac{7}{5}}$ ⓑ $z^{\frac{1}{8}} \cdot z^{\frac{7}{8}}$ ⓒ $n^{\frac{2}{7}} \cdot n^{\frac{5}{7}}$.

Solution:

ⓐ 25 ⓑ z ⓒ n

We will use the Power Property in the next example.

Example:
Exercise:

Problem: Simplify: ⓐ $\left(x^4\right)^{\frac{1}{2}}$ ⓑ $\left(y^6\right)^{\frac{1}{3}}$ ⓒ $\left(z^9\right)^{\frac{2}{3}}$.

Solution:
Solution

ⓐ

	$\left(x^4\right)^{\frac{1}{2}}$
To raise a power to a power, we multiply the exponents.	$x^{4 \cdot \frac{1}{2}}$
Simplify.	x^2

ⓑ

	$\left(y^6\right)^{\frac{1}{3}}$
To raise a power to a power, we multiply the exponents.	$y^{6 \cdot \frac{1}{3}}$
Simplify.	y^2

ⓒ

	$\left(z^9\right)^{\frac{2}{3}}$
To raise a power to a power, we multiply the exponents.	$z^{9 \cdot \frac{2}{3}}$
Simplify.	z^6

Note:
Exercise:

Problem: Simplify: ⓐ $\left(p^{10}\right)^{\frac{1}{5}}$ ⓑ $\left(q^8\right)^{\frac{3}{4}}$ ⓒ $\left(x^6\right)^{\frac{4}{3}}$.

Solution:

ⓐ p^2 ⓑ q^6 ⓒ x^8

Note:
Exercise:

Problem: Simplify: ⓐ $\left(r^6\right)^{\frac{5}{3}}$ ⓑ $\left(s^{12}\right)^{\frac{3}{4}}$ ⓒ $\left(m^9\right)^{\frac{2}{9}}$.

Solution:

ⓐ r^{10} ⓑ s^9 ⓒ m^2

The Quotient Property tells us that when we divide with the same base, we subtract the exponents.

Example:
Exercise:

Problem: Simplify: ⓐ $\frac{x^{\frac{4}{3}}}{x^{\frac{1}{3}}}$ ⓑ $\frac{y^{\frac{3}{4}}}{y^{\frac{1}{4}}}$ ⓒ $\frac{z^{\frac{2}{3}}}{z^{\frac{5}{3}}}$.

Solution:
Solution

ⓐ

	$\frac{x^{\frac{4}{3}}}{x^{\frac{1}{3}}}$
To divide with the same base, we subtract the exponents.	$x^{\frac{4}{3}-\frac{1}{3}}$
Simplify.	x

ⓑ

	$\frac{y^{\frac{3}{4}}}{y^{\frac{1}{4}}}$
To divide with the same base, we subtract the exponents.	$y^{\frac{3}{4}-\frac{1}{4}}$
Simplify.	$y^{\frac{1}{2}}$

ⓒ

	$\frac{z^{\frac{2}{3}}}{z^{\frac{5}{3}}}$
To divide with the same base, we subtract the exponents.	$z^{\frac{2}{3}-\frac{5}{3}}$
Rewrite without a negative exponent.	$\frac{1}{z}$

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{u^{\frac{5}{4}}}{u^{\frac{1}{4}}}$ ⓑ $\frac{v^{\frac{3}{5}}}{v^{\frac{2}{5}}}$ ⓒ $\frac{x^{\frac{2}{3}}}{x^{\frac{5}{3}}}$.

Solution:

ⓐ u ⓑ $v^{\frac{1}{5}}$ ⓒ $\frac{1}{x}$

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{c^{\frac{12}{5}}}{c^{\frac{2}{5}}}$ ⓑ $\frac{m^{\frac{5}{4}}}{m^{\frac{9}{4}}}$ ⓒ $\frac{d^{\frac{1}{5}}}{d^{\frac{6}{5}}}$.

Solution:

ⓐ c^6 ⓑ $\frac{1}{m}$ ⓒ $\frac{1}{d}$

Sometimes we need to use more than one property. In the next two examples, we will use both the Product to a Power Property and then the Power Property.

Example:
Exercise:

Problem: Simplify: ⓐ $\left(27u^{\frac{1}{2}}\right)^{\frac{2}{3}}$ ⓑ $\left(8v^{\frac{1}{4}}\right)^{\frac{2}{3}}$.

Solution:
Solution

ⓐ

	$\left(27u^{\frac{1}{2}}\right)^{\frac{2}{3}}$
First we use the Product to a Power Property.	$(27)^{\frac{2}{3}}\left(u^{\frac{1}{2}}\right)^{\frac{2}{3}}$
Rewrite 27 as a power of 3.	$(3^3)^{\frac{2}{3}}\left(u^{\frac{1}{2}}\right)^{\frac{2}{3}}$
To raise a power to a power, we multiply the exponents.	$(3^2)\left(u^{\frac{1}{3}}\right)$
Simplify.	$9u^{\frac{1}{3}}$

ⓑ

	$\left(8v^{\frac{1}{4}}\right)^{\frac{2}{3}}$
First we use the Product to a Power Property.	$(8)^{\frac{2}{3}}\left(v^{\frac{1}{4}}\right)^{\frac{2}{3}}$
Rewrite 8 as a power of 2.	$(2^3)^{\frac{2}{3}}\left(v^{\frac{1}{4}}\right)^{\frac{2}{3}}$
To raise a power to a power, we multiply the exponents.	$(2^2)\left(v^{\frac{1}{6}}\right)$
Simplify.	$4v^{\frac{1}{6}}$

Note:
Exercise:

Problem: Simplify: ⓐ $\left(32x^{\frac{1}{3}}\right)^{\frac{3}{5}}$ ⓑ $\left(64y^{\frac{2}{3}}\right)^{\frac{1}{3}}$.

Solution:

ⓐ $8x^{\frac{1}{5}}$ ⓑ $4y^{\frac{2}{9}}$

Note:
Exercise:

Problem: Simplify: ⓐ $\left(16m^{\frac{1}{3}}\right)^{\frac{3}{2}}$ ⓑ $\left(81n^{\frac{2}{5}}\right)^{\frac{3}{2}}$.

Solution:

ⓐ $64m^{\frac{1}{2}}$ ⓑ $729n^{\frac{3}{5}}$

Example:
Exercise:

Problem: Simplify: ⓐ $\left(m^3n^9\right)^{\frac{1}{3}}$ ⓑ $\left(p^4q^8\right)^{\frac{1}{4}}$.

Solution:
Solution

ⓐ

$$\left(m^3 n^9\right)^{\frac{1}{3}}$$

First we use the Product to a Power Property. $\left(m^3\right)^{\frac{1}{3}}\left(n^9\right)^{\frac{1}{3}}$

To raise a power to a power, we multiply the exponents. mn^3

ⓑ

$$\left(p^4 q^8\right)^{\frac{1}{4}}$$

First we use the Product to a Power Property. $\left(p^4\right)^{\frac{1}{4}}\left(q^8\right)^{\frac{1}{4}}$

To raise a power to a power, we multiply the exponents. pq^2

We will use both the Product and Quotient Properties in the next example.

Example:
Exercise:

Problem: Simplify: ⓐ $\frac{x^{\frac{3}{4}} \cdot x^{-\frac{1}{4}}}{x^{-\frac{6}{4}}}$ ⓑ $\frac{y^{\frac{4}{3}} \cdot y}{y^{-\frac{2}{3}}}$.

Solution:
Solution

ⓐ

$$\frac{x^{\frac{3}{4}} \cdot x^{-\frac{1}{4}}}{x^{-\frac{6}{4}}}$$

Use the Product Property in the numerator, add the exponents.

$$\frac{x^{\frac{2}{4}}}{x^{-\frac{6}{4}}}$$

Use the Quotient Property, subtract the exponents.

$$x^{\frac{8}{4}}$$

Simplify.

$$x^2$$

ⓑ

$$\frac{y^{\frac{4}{3}} \cdot y}{y^{-\frac{2}{3}}}$$

Use the Product Property in the numerator, add the exponents.

$$\frac{y^{\frac{7}{3}}}{y^{-\frac{2}{3}}}$$

Use the Quotient Property, subtract the exponents.

$$y^{\frac{9}{3}}$$

Simplify.

$$y^3$$

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{m^{\frac{2}{3}} \cdot m^{-\frac{1}{3}}}{m^{-\frac{5}{3}}}$ ⓑ $\frac{n^{\frac{1}{6}} \cdot n}{n^{-\frac{11}{6}}}$.

Solution:

ⓐ m^2 ⓑ n^3

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{u^{\frac{4}{5}} \cdot u^{-\frac{2}{5}}}{u^{-\frac{13}{5}}}$ ⓑ $\frac{v^{\frac{1}{2}} \cdot v}{v^{-\frac{7}{2}}}$.

Solution:

ⓐ u^3 ⓑ v^5

Key Concepts

- **Summary of Exponent Properties**
- If a, b are real numbers and m, n are rational numbers, then
  - **Product Property** $a^m \cdot a^n = a^{m+n}$
  - **Power Property** $(a^m)^n = a^{m \cdot n}$
  - **Product to a Power** $(ab)^m = a^m b^m$
  - **Quotient Property**:
    Equation:

    $$\frac{a^m}{a^n} = a^{m-n}, \;\; a \neq 0, \;\; m > n$$

    Equation:

    $$\frac{a^m}{a^n} = \frac{1}{a^{n-m}}, \;\; a \neq 0, \;\; n > m$$

  - **Zero Exponent Definition** $a^0 = 1$, $a \neq 0$
  - **Quotient to a Power Property** $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, \;\; b \neq 0$

Section Exercises

Practice Makes Perfect

Simplify Expressions with $a^{\frac{1}{n}}$

In the following exercises, write as a radical expression.

Exercise:

Problem: ⓐ $x^{\frac{1}{2}}$
ⓑ $y^{\frac{1}{3}}$
ⓒ $z^{\frac{1}{4}}$

Exercise:

Problem: ⓐ $r^{\frac{1}{2}}$
ⓑ $s^{\frac{1}{3}}$
ⓒ $t^{\frac{1}{4}}$

Solution:

ⓐ $\sqrt{r}$ ⓑ $\sqrt[3]{s}$ ⓒ $\sqrt[4]{t}$

Exercise:

Problem: ⓐ $u^{\frac{1}{5}}$
ⓑ $v^{\frac{1}{9}}$
ⓒ $w^{\frac{1}{20}}$

Exercise:

Problem: ⓐ $g^{\frac{1}{7}}$
ⓑ $h^{\frac{1}{5}}$
ⓒ $j^{\frac{1}{25}}$

Solution:

ⓐ $\sqrt[7]{g}$ ⓑ $\sqrt[5]{h}$ ⓒ $\sqrt[25]{j}$

In the following exercises, write with a rational exponent.

Exercise:

Problem: ⓐ $-\sqrt[7]{x}$
ⓑ $\sqrt[9]{y}$
ⓒ $\sqrt[5]{f}$

Exercise:

Problem: ⓐ $\sqrt[8]{r}$
ⓑ

$\sqrt[10]{s}$

ⓒ $\sqrt[4]{t}$

Solution:

ⓐ $r^{\frac{1}{8}}$ ⓑ $s^{\frac{1}{10}}$ ⓒ $t^{\frac{1}{4}}$

Exercise:

Problem: ⓐ $\sqrt[3]{a}$
ⓑ

$\sqrt[12]{b}$

ⓒ $\sqrt{c}$

Exercise:

Problem: ⓐ $\sqrt[5]{u}$
ⓑ $\sqrt{v}$
ⓒ

$\sqrt[16]{w}$

Solution:

ⓐ $u^{\frac{1}{5}}$ ⓑ $v^{\frac{1}{2}}$ ⓒ $w^{\frac{1}{16}}$

Exercise:

Problem: ⓐ $\sqrt[3]{7c}$
ⓑ $\sqrt[7]{12d}$
ⓒ $3\sqrt[4]{5f}$

Exercise:

Problem: ⓐ $\sqrt[4]{5x}$
ⓑ $\sqrt[8]{9y}$
ⓒ $7\sqrt[5]{3z}$

Solution:

ⓐ $(5x)^{\frac{1}{4}}$ ⓑ $(9y)^{\frac{1}{8}}$ ⓒ $7(3z)^{\frac{1}{5}}$

Exercise:

ⓐ $\sqrt{21p}$
ⓑ $\sqrt[4]{8q}$
Problem: ⓒ $4\sqrt[6]{36r}$

Exercise:

ⓐ $\sqrt[3]{25a}$
ⓑ $\sqrt{3b}$
Problem: ⓒ

$\sqrt[10]{40c}$

Solution:

ⓐ $(25a)^{\frac{1}{3}}$ ⓑ $(3b)^{\frac{1}{2}}$ ⓒ $(40c)^{\frac{1}{10}}$

In the following exercises, simplify.

Exercise:

ⓐ $81^{\frac{1}{2}}$
ⓑ $125^{\frac{1}{3}}$
Problem: ⓒ $64^{\frac{1}{2}}$

Exercise:

ⓐ $625^{\frac{1}{4}}$
ⓑ $243^{\frac{1}{5}}$
Problem: ⓒ $32^{\frac{1}{5}}$

Solution:

ⓐ 5 ⓑ 3 ⓒ 2

Exercise:

Problem: ⓐ $16^{\frac{1}{4}}$
ⓑ $16^{\frac{1}{2}}$
ⓒ $3125^{\frac{1}{5}}$

Exercise:

Problem: ⓐ $216^{\frac{1}{3}}$
ⓑ $32^{\frac{1}{5}}$
ⓒ $81^{\frac{1}{4}}$

Solution:

ⓐ 6 ⓑ 2 ⓒ 3

Exercise:

Problem: ⓐ $(-216)^{\frac{1}{3}}$
ⓑ $-216^{\frac{1}{3}}$
ⓒ $(216)^{-\frac{1}{3}}$

Exercise:

Problem: ⓐ $(-243)^{\frac{1}{5}}$
ⓑ $-243^{\frac{1}{5}}$
ⓒ $(243)^{-\frac{1}{5}}$

Solution:

ⓐ -3 ⓑ -3 ⓒ $\frac{1}{3}$

Exercise:

Problem:
ⓐ $(-1)^{\frac{1}{3}}$
ⓑ $-1^{\frac{1}{3}}$
ⓒ $(1)^{-\frac{1}{3}}$

Exercise:

Problem:
ⓐ $(-1000)^{\frac{1}{3}}$
ⓑ $-1000^{\frac{1}{3}}$
ⓒ $(1000)^{-\frac{1}{3}}$

Solution:

ⓐ -10 ⓑ -10 ⓒ $\frac{1}{10}$

Exercise:

Problem:
ⓐ $(-81)^{\frac{1}{4}}$
ⓑ $-81^{\frac{1}{4}}$
ⓒ $(81)^{-\frac{1}{4}}$

Exercise:

ⓐ $(-49)^{\frac{1}{2}}$
ⓑ $-49^{\frac{1}{2}}$
Problem: ⓒ $(49)^{-\frac{1}{2}}$

Solution:

ⓐ not a real number ⓑ -7 ⓒ $\frac{1}{7}$

Exercise:

ⓐ $(-36)^{\frac{1}{2}}$
ⓑ $-36^{\frac{1}{2}}$
Problem: ⓒ $(36)^{-\frac{1}{2}}$

Exercise:

ⓐ $(-1)^{\frac{1}{4}}$
ⓑ $(1)^{-\frac{1}{4}}$
Problem: ⓒ $-1^{\frac{1}{4}}$

Solution:

ⓐ not a real number ⓑ 1 ⓒ -1

Exercise:

ⓐ $(-100)^{\frac{1}{2}}$
ⓑ $-100^{\frac{1}{2}}$
Problem: ⓒ $(100)^{-\frac{1}{2}}$

Exercise:

Problem: ⓐ $(-32)^{\frac{1}{5}}$
ⓑ $(243)^{-\frac{1}{5}}$
ⓒ $-125^{\frac{1}{3}}$

Solution:

ⓐ -2 ⓑ $\frac{1}{3}$
ⓒ -5

Simplify Expressions with $a^{\frac{m}{n}}$

In the following exercises, write with a rational exponent.

Exercise:

Problem: ⓐ $\sqrt{m^5}$
ⓑ $\sqrt[3]{n^2}$
ⓒ $\sqrt[4]{p^3}$

Exercise:

Problem: ⓐ $\sqrt[4]{r^7}$
ⓑ $\sqrt[5]{s^3}$
ⓒ $\sqrt[3]{t^7}$

Solution:

ⓐ $r^{\frac{7}{4}}$ ⓑ $s^{\frac{3}{5}}$ ⓒ $t^{\frac{7}{3}}$

Exercise:

Problem: ⓐ $\sqrt[5]{u^2}$
ⓑ $\sqrt[5]{v^8}$
ⓒ $\sqrt[9]{w^4}$

Exercise:

Problem: ⓐ $\sqrt[3]{a}$
ⓑ $\sqrt{b^5}$
ⓒ $\sqrt[3]{c^5}$

Solution:

ⓐ $a^{\frac{1}{3}}$ ⓑ $b^{\frac{1}{5}}$ ⓒ $c^{\frac{5}{3}}$

In the following exercises, simplify.

Exercise:

Problem: ⓐ $16^{\frac{3}{2}}$
ⓑ $8^{\frac{2}{3}}$
ⓒ $10{,}000^{\frac{3}{4}}$

Exercise:

Problem: ⓐ $1000^{\frac{2}{3}}$
ⓑ $25^{\frac{3}{2}}$
ⓒ $32^{\frac{3}{5}}$

Solution:

ⓐ 100 ⓑ 125 ⓒ 8

Exercise:

Problem: ⓐ $27^{\frac{5}{3}}$
ⓑ $16^{\frac{5}{4}}$
ⓒ $32^{\frac{2}{5}}$

Exercise:

Problem: ⓐ $16^{\frac{3}{2}}$
ⓑ $125^{\frac{5}{3}}$
ⓒ $64^{\frac{4}{3}}$

Solution:

ⓐ 64 ⓑ 3125 ⓒ 256

Exercise:

Problem: ⓐ $32^{\frac{2}{5}}$
ⓑ $27^{-\frac{2}{3}}$
ⓒ $25^{-\frac{3}{2}}$

Exercise:

Problem: ⓐ $64^{\frac{5}{2}}$
ⓑ $81^{-\frac{3}{2}}$
ⓒ $27^{-\frac{4}{3}}$

Solution:

ⓐ 32,768 ⓑ $\frac{1}{729}$ ⓒ $\frac{1}{81}$

Exercise:

Problem: ⓐ $25^{\frac{3}{2}}$
ⓑ $9^{-\frac{3}{2}}$
ⓒ $(-64)^{\frac{2}{3}}$

Exercise:

Problem: ⓐ $100^{\frac{3}{2}}$
ⓑ $49^{-\frac{5}{2}}$
ⓒ $(-100)^{\frac{3}{2}}$

Solution:

ⓐ 1000 ⓑ $\frac{1}{16{,}807}$ ⓒ not a real number

Exercise:

Problem: ⓐ $-9^{\frac{3}{2}}$
ⓑ $-9^{-\frac{3}{2}}$
ⓒ $(-9)^{\frac{3}{2}}$

Exercise:

Problem: ⓐ $-64^{\frac{3}{2}}$
ⓑ $-64^{-\frac{3}{2}}$
ⓒ $(-64)^{\frac{3}{2}}$

Solution:

ⓐ -512
ⓑ $-\frac{1}{512}$ ⓒ not a real number

Exercise:

ⓐ $-100^{\frac{3}{2}}$
ⓑ $-100^{-\frac{3}{2}}$
Problem: ⓒ $(-100)^{\frac{3}{2}}$

Exercise:

ⓐ $-49^{\frac{3}{2}}$
ⓑ $-49^{-\frac{3}{2}}$
Problem: ⓒ $(-49)^{\frac{3}{2}}$

Solution:

ⓐ -343 ⓑ $-\frac{1}{343}$ ⓒ not a real number

Use the Laws of Exponents to Simplify Expressions with Rational Exponents

In the following exercises, simplify.
Exercise:

ⓐ $4^{\frac{5}{8}} \cdot 4^{\frac{11}{8}}$
ⓑ $m^{\frac{7}{12}} \cdot m^{\frac{17}{12}}$
Problem: ⓒ $p^{\frac{3}{7}} \cdot p^{\frac{18}{7}}$

Exercise:

Problem: ⓐ $6^{\frac{5}{2}} \cdot 6^{\frac{1}{2}}$
ⓑ $n^{\frac{2}{10}} \cdot n^{\frac{8}{10}}$
ⓒ $q^{\frac{2}{5}} \cdot q^{\frac{13}{5}}$

Solution:

ⓐ 216 ⓑ n ⓒ q^3

Exercise:

Problem: ⓐ $5^{\frac{1}{2}} \cdot 5^{\frac{7}{2}}$
ⓑ $c^{\frac{3}{4}} \cdot c^{\frac{9}{4}}$
ⓒ $d^{\frac{3}{5}} \cdot d^{\frac{2}{5}}$

Exercise:

Problem: ⓐ $10^{\frac{1}{3}} \cdot 10^{\frac{5}{3}}$
ⓑ $x^{\frac{5}{6}} \cdot x^{\frac{7}{6}}$
ⓒ $y^{\frac{11}{8}} \cdot y^{\frac{21}{8}}$

Solution:

ⓐ 100 ⓑ x^2 ⓒ y^4

Exercise:

Problem: ⓐ $\left(m^6\right)^{\frac{5}{2}}$
ⓑ $\left(n^9\right)^{\frac{4}{3}}$
ⓒ $\left(p^{12}\right)^{\frac{3}{4}}$

Exercise:

Problem: ⓐ $\left(a^{12}\right)^{\frac{1}{6}}$
ⓑ $\left(b^{15}\right)^{\frac{3}{5}}$
ⓒ $\left(c^{11}\right)^{\frac{1}{11}}$

Solution:

ⓐ a^2 ⓑ b^9
ⓒ c

Exercise:

Problem: ⓐ $\left(x^{12}\right)^{\frac{2}{3}}$
ⓑ $\left(y^{20}\right)^{\frac{2}{5}}$
ⓒ $\left(z^{16}\right)^{\frac{1}{16}}$

Exercise:

Problem: ⓐ $\left(h^{6}\right)^{\frac{4}{3}}$
ⓑ $\left(k^{12}\right)^{\frac{3}{4}}$
ⓒ $\left(j^{10}\right)^{\frac{7}{5}}$

Solution:

ⓐ h^8 ⓑ k^9 ⓒ j^{14}

Exercise:

ⓐ $\dfrac{x^{\frac{7}{2}}}{x^{\frac{5}{2}}}$

ⓑ $\dfrac{y^{\frac{5}{2}}}{y^{\frac{1}{2}}}$

Problem: ⓒ $\dfrac{r^{\frac{4}{5}}}{r^{\frac{9}{5}}}$

Exercise:

ⓐ $\dfrac{s^{\frac{11}{5}}}{s^{\frac{6}{5}}}$

ⓑ $\dfrac{z^{\frac{7}{3}}}{z^{\frac{1}{3}}}$

Problem: ⓒ $\dfrac{w^{\frac{2}{7}}}{w^{\frac{9}{7}}}$

Solution:

ⓐ s ⓑ z^2 ⓒ $\dfrac{1}{w}$

Exercise:

ⓐ $\dfrac{t^{\frac{12}{5}}}{t^{\frac{7}{5}}}$

ⓑ $\dfrac{x^{\frac{3}{2}}}{x^{\frac{1}{2}}}$

Problem: ⓒ $\dfrac{m^{\frac{13}{8}}}{m^{\frac{5}{8}}}$

Exercise:

Problem: ⓐ $\frac{u^{\frac{13}{9}}}{u^{\frac{4}{9}}}$ ⓑ $\frac{r^{\frac{15}{7}}}{r^{\frac{8}{7}}}$ ⓒ $\frac{n^{\frac{3}{5}}}{n^{\frac{8}{5}}}$

Solution:

ⓐ u ⓑ r ⓒ $\frac{1}{n}$

Exercise:

Problem: ⓐ $\left(9p^{\frac{2}{3}}\right)^{\frac{5}{2}}$ ⓑ $\left(27q^{\frac{3}{2}}\right)^{\frac{4}{3}}$

Exercise:

Problem: ⓐ $\left(81r^{\frac{4}{5}}\right)^{\frac{1}{4}}$ ⓑ $\left(64s^{\frac{3}{7}}\right)^{\frac{1}{6}}$

Solution:

ⓐ $3r^{\frac{1}{5}}$ ⓑ $2s^{\frac{1}{14}}$

Exercise:

ⓐ $\left(16u^{\frac{1}{3}}\right)^{\frac{3}{4}}$

Problem: ⓑ $\left(100v^{\frac{2}{5}}\right)^{\frac{3}{2}}$

Exercise:

ⓐ $\left(27m^{\frac{3}{4}}\right)^{\frac{2}{3}}$

Problem: ⓑ $\left(625n^{\frac{8}{3}}\right)^{\frac{3}{4}}$

Solution:

ⓐ $9m^{\frac{1}{2}}$ ⓑ $125n^2$

Exercise:

ⓐ $\left(x^8y^{10}\right)^{\frac{1}{2}}$

Problem: ⓑ $\left(a^9b^{12}\right)^{\frac{1}{3}}$

Exercise:

ⓐ $\left(r^8s^4\right)^{\frac{1}{4}}$

Problem: ⓑ $\left(u^{15}v^{20}\right)^{\frac{1}{5}}$

Solution:

ⓐ r^2s ⓑ u^3v^4

Exercise:

ⓐ $\left(a^6b^{16}\right)^{\frac{1}{2}}$

Problem: ⓑ $\left(j^9k^6\right)^{\frac{2}{3}}$

Exercise:

ⓐ $\left(r^{16}s^{10}\right)^{\frac{1}{2}}$

Problem: ⓑ $\left(u^{10}v^5\right)^{\frac{4}{5}}$

Solution:

ⓐ r^8s^5 ⓑ u^8v^4

Exercise:

ⓐ $\frac{r^{\frac{5}{2}}\cdot r^{-\frac{1}{2}}}{r^{-\frac{3}{2}}}$

Problem: ⓑ $\frac{s^{\frac{1}{5}}\cdot s}{s^{-\frac{9}{5}}}$

Exercise:

ⓐ $\frac{a^{\frac{3}{4}}\cdot a^{-\frac{1}{4}}}{a^{-\frac{10}{4}}}$

Problem: ⓑ $\frac{b^{\frac{2}{3}}\cdot b}{b^{-\frac{7}{3}}}$

Solution:

ⓐ a^3 ⓑ b^4

Exercise:

Problem: ⓐ $\frac{c^{\frac{5}{3}} \cdot c^{-\frac{1}{3}}}{c^{-\frac{2}{3}}}$ ⓑ $\frac{d^{\frac{3}{5}} \cdot d}{d^{-\frac{2}{5}}}$

Exercise:

Problem: ⓐ $\frac{m^{\frac{7}{4}} \cdot m^{-\frac{5}{4}}}{m^{-\frac{2}{4}}}$ ⓑ $\frac{n^{\frac{3}{7}} \cdot n}{n^{-\frac{4}{7}}}$

Solution:

ⓐ m ⓑ n^2

Exercise:

Problem: $4^{\frac{5}{2}} \cdot 4^{\frac{1}{2}}$

Exercise:

Problem: $n^{\frac{2}{6}} \cdot n^{\frac{4}{6}}$

Solution:

n

Exercise:

Problem: $\left(a^{24}\right)^{\frac{1}{6}}$

Exercise:

Problem: $\left(b^{10}\right)^{\frac{3}{5}}$

Solution:

b^{6}

Exercise:

Problem: $\frac{w^{\frac{2}{5}}}{w^{\frac{7}{5}}}$

Exercise:

Problem: $\frac{z^{\frac{2}{3}}}{z^{\frac{8}{3}}}$

Solution:

$\frac{1}{z^{2}}$

Exercise:

Problem: $\left(27r^{\frac{3}{5}}\right)^{\frac{1}{3}}$

Exercise:

Problem: $\left(64s^{\frac{3}{5}}\right)^{\frac{1}{6}}$

Solution:

$2s^{\frac{1}{10}}$

Exercise:

Problem: $\left(r^9 s^{12}\right)^{\frac{1}{3}}$

Exercise:

Problem: $\left(u^{12} v^{18}\right)^{\frac{1}{6}}$

Solution:

$u^2 v^3$

Everyday Math

Exercise:

Problem:

Landscaping Joe wants to have a square garden plot in his backyard. He has enough compost to cover an area of 144 square feet. Simplify $144^{\frac{1}{2}}$ to find the length of each side of his garden.

Exercise:

Problem:

Landscaping Elliott wants to make a square patio in his yard. He has enough concrete to pave an area of 242 square feet. Simplify $242^{\frac{1}{2}}$ to find the length of each side of his patio.Round to the nearest tenth of a foot.

Solution:

15.6 feet

Exercise:

Problem:

Gravity While putting up holiday decorations, Bob dropped a decoration from the top of a tree that is 12 feet tall. Simplify $\frac{12^{\frac{1}{2}}}{16^{\frac{1}{2}}}$ to find how many seconds it took for the decoration to reach the ground. Round to the nearest tenth of a second.

Exercise:

Problem:

Gravity An airplane dropped a flare from a height of 1024 feet above a lake. Simplify $\frac{1024^{\frac{1}{2}}}{16^{\frac{1}{2}}}$ to find how many seconds it took for the flare to reach the water.

Solution:

8 seconds

Writing Exercises

Exercise:

Problem:

Show two different algebraic methods to simplify $4^{\frac{3}{2}}$. Explain all your steps.

Exercise:

Problem: Explain why the expression $(-16)^{\frac{3}{2}}$ cannot be evaluated.

Chapter Review Exercises

Simplify and Use Square Roots

Simplify Expressions with Square Roots

In the following exercises, simplify.

Exercise:

Problem: $\sqrt{64}$

Exercise:

Problem: $\sqrt{144}$

Solution:

12

Exercise:

Problem: $-\sqrt{25}$

Exercise:

Problem: $-\sqrt{81}$

Solution:

-9

Exercise:

Problem: $\sqrt{-9}$

Exercise:

Problem: $\sqrt{-36}$

Solution:

not a real number

Exercise:

Problem: $\sqrt{64} + \sqrt{225}$

Exercise:

Problem: $\sqrt{64 + 225}$

Solution:

17

Estimate Square Roots

In the following exercises, estimate each square root between two consecutive whole numbers.

Exercise:

Problem: $\sqrt{28}$

Exercise:

Problem: $\sqrt{155}$

Solution:

$12 < \sqrt{155} < 13$

Approximate Square Roots

In the following exercises, approximate each square root and round to two decimal places.

Exercise:

Problem: $\sqrt{15}$

Exercise:

Problem: $\sqrt{57}$

Solution:

7.55

Simplify Variable Expressions with Square Roots

In the following exercises, simplify.

Exercise:

Problem: $\sqrt{q^2}$

Exercise:

Problem: $\sqrt{64b^2}$

Solution:

$8b$

Exercise:

Problem: $-\sqrt{121a^2}$

Exercise:

Problem: $\sqrt{225m^2n^2}$

Solution:

$15mn$

Exercise:

Problem: $-\sqrt{100q^2}$

Exercise:

Problem: $\sqrt{49y^2}$

Solution:

$7y$

Exercise:

Problem: $\sqrt{4a^2b^2}$

Exercise:

Problem: $\sqrt{121c^2d^2}$

Solution:

$11cd$

Simplify Square Roots

Use the Product Property to Simplify Square Roots

In the following exercises, simplify.

Exercise:

Problem: $\sqrt{300}$

Exercise:

Problem: $\sqrt{98}$

Solution:

$7\sqrt{2}$

Exercise:

Problem: $\sqrt{x^{13}}$

Exercise:

Problem: $\sqrt{y^{19}}$

Solution:

$y^9\sqrt{y}$

Exercise:

Problem: $\sqrt{16m^4}$

Exercise:

Problem: $\sqrt{36n^{13}}$

Solution:

$6n^6\sqrt{n}$

Exercise:

Problem: $\sqrt{288m^{21}}$

Exercise:

Problem: $\sqrt{150n^7}$

Solution:

$5n^3\sqrt{6n}$

Exercise:

Problem: $\sqrt{48r^5s^4}$

Exercise:

Problem: $\sqrt{108r^5s^3}$

Solution:

$6r^2s\sqrt{3rs}$

Exercise:

Problem: $\frac{10-\sqrt{50}}{5}$

Exercise:

Problem: $\frac{6+\sqrt{72}}{6}$

Solution:

$1+\sqrt{2}$

Use the Quotient Property to Simplify Square Roots

In the following exercises, simplify.

Exercise:

Problem: $\sqrt{\frac{16}{25}}$

Exercise:

Problem: $\sqrt{\frac{81}{36}}$

Solution:

$\frac{3}{2}$

Exercise:

Problem: $\sqrt{\frac{x^8}{x^4}}$

Exercise:

Problem: $\sqrt{\frac{y^6}{y^2}}$

Solution:

y^2

Exercise:

Problem: $\sqrt{\frac{98p^6}{2p^2}}$

Exercise:

Problem: $\sqrt{\frac{72q^8}{2q^4}}$

Solution:

$6q^2$

Exercise:

Problem: $\sqrt{\frac{65}{121}}$

Exercise:

Problem: $\sqrt{\frac{26}{169}}$

Solution:

$\frac{\sqrt{26}}{13}$

Exercise:

Problem: $\sqrt{\frac{64x^4}{25x^2}}$

Exercise:

Problem: $\sqrt{\frac{36r^{10}}{16r^5}}$

Solution:

$\frac{3r^2\sqrt{r}}{2}$

Exercise:

Problem: $\sqrt{\frac{48p^3q^5}{27pq}}$

Exercise:

Problem: $\sqrt{\frac{12r^5s^7}{75r^2s}}$

Solution:

$$\frac{2rs^3\sqrt{r}}{5}$$

Add and Subtract Square Roots

Add and Subtract Like Square Roots

In the following exercises, simplify.

Exercise:

Problem: $3\sqrt{2}+\sqrt{2}$

Exercise:

Problem: $5\sqrt{5}+7\sqrt{5}$

Solution:

$12\sqrt{5}$

Exercise:

Problem: $4\sqrt{y}+4\sqrt{y}$

Exercise:

Problem: $6\sqrt{m}-2\sqrt{m}$

Solution:

$4\sqrt{m}$

Exercise:

Problem: $-3\sqrt{7}+2\sqrt{7}-\sqrt{7}$

Exercise:

Problem: $8\sqrt{13}+2\sqrt{3}+3\sqrt{13}$

Solution:

$11\sqrt{13}+2\sqrt{3}$

Exercise:

Problem: $3\sqrt{5xy}-\sqrt{5xy}+3\sqrt{5xy}$

Exercise:

Problem: $2\sqrt{3rs}+\sqrt{3rs}-5\sqrt{rs}$

Solution:

$3\sqrt{3rs}-5\sqrt{rs}$

Add and Subtract Square Roots that Need Simplification

In the following exercises, simplify.

Exercise:

Problem: $\sqrt{32}+3\sqrt{2}$

Exercise:

Problem: $\sqrt{8}+3\sqrt{2}$

Solution:

$5\sqrt{2}$

Exercise:

Problem: $\sqrt{72} + \sqrt{50}$

Exercise:

Problem: $\sqrt{48} + \sqrt{75}$

Solution:

$9\sqrt{3}$

Exercise:

Problem: $3\sqrt{32} + \sqrt{98}$

Exercise:

Problem: $\frac{1}{3}\sqrt{27} - \frac{1}{8}\sqrt{192}$

Solution:

0

Exercise:

Problem: $\sqrt{50y^5} - \sqrt{72y^5}$

Exercise:

Problem: $6\sqrt{18n^4} - 3\sqrt{8n^4} + n^2\sqrt{50}$

Solution:

$17n^2\sqrt{2}$

Multiply Square Roots

Multiply Square Roots

In the following exercises, simplify.

Exercise:

Problem: $\sqrt{2} \cdot \sqrt{20}$

Exercise:

Problem: $2\sqrt{2} \cdot 6\sqrt{14}$

Solution:

$24\sqrt{7}$

Exercise:

Problem: $\sqrt{2m^2} \cdot \sqrt{20m^4}$

Exercise:

Problem: $\left(6\sqrt{2y}\right)\left(3\sqrt{50y^3}\right)$

Solution:

$180y^2$

Exercise:

Problem: $\left(6\sqrt{3v^4}\right)\left(5\sqrt{30v}\right)$

Exercise:

Problem: $\left(\sqrt{8}\right)^2$

Solution:

8

Exercise:

Problem: $\left(-\sqrt{10}\right)^2$

Exercise:

Problem: $\left(2\sqrt{5}\right)\left(5\sqrt{5}\right)$

Solution:

50

Exercise:

Problem: $\left(-3\sqrt{3}\right)\left(5\sqrt{18}\right)$

Use Polynomial Multiplication to Multiply Square Roots

In the following exercises, simplify.

Exercise:

Problem: $10(2-\sqrt{7})$

Solution:

$20-10\sqrt{7}$

Exercise:

Problem: $\sqrt{3}\left(4+\sqrt{12}\right)$

Exercise:

Problem: $\left(5+\sqrt{2}\right)\left(3-\sqrt{2}\right)$

Solution:

$13-2\sqrt{2}$

Exercise:

Problem: $\left(5-3\sqrt{7}\right)\left(1-2\sqrt{7}\right)$

Exercise:

Problem: $\left(1-3\sqrt{x}\right)\left(5+2\sqrt{x}\right)$

Solution:

$5-13\sqrt{x}-6x$

Exercise:

Problem: $\left(3+4\sqrt{y}\right)\left(10-\sqrt{y}\right)$

Exercise:

Problem: $\left(1+6\sqrt{p}\right)^{2}$

Solution:

$1+12\sqrt{p}+36p$

Exercise:

Problem: $\left(2-6\sqrt{5}\right)^{2}$

Exercise:

Problem: $\left(3+2\sqrt{7}\right)\left(3-2\sqrt{7}\right)$

Solution:

-19

Exercise:

Problem: $\left(6-\sqrt{11}\right)\left(6+\sqrt{11}\right)$

Divide Square Roots

Divide Square Roots

In the following exercises, simplify.

Exercise:

Problem: $\frac{\sqrt{75}}{10}$

Solution:

$\frac{\sqrt{3}}{2}$

Exercise:

Problem: $\frac{2-\sqrt{12}}{6}$

Exercise:

Problem: $\frac{\sqrt{48}}{\sqrt{27}}$

Solution:

$\frac{4}{3}$

Exercise:

Problem: $\frac{\sqrt{75x^7}}{\sqrt{3x^3}}$

Exercise:

Problem: $\frac{\sqrt{20y^5}}{\sqrt{2y}}$

Solution:

$y^2\sqrt{10}$

Exercise:

Problem: $\frac{\sqrt{98p^6q^4}}{\sqrt{2p^4q^8}}$

Rationalize a One Term Denominator

In the following exercises, rationalize the denominator.

Exercise:

Problem: $\frac{10}{\sqrt{15}}$

Solution:

$\frac{2\sqrt{15}}{3}$

Exercise:

Problem: $\frac{6}{\sqrt{6}}$

Exercise:

Problem: $\frac{5}{3\sqrt{5}}$

Solution:

$\frac{\sqrt{5}}{3}$

Exercise:

Problem: $\frac{10}{2\sqrt{6}}$

Exercise:

Problem: $\sqrt{\frac{3}{28}}$

Solution:

$\frac{\sqrt{21}}{14}$

Exercise:

Problem: $\sqrt{\frac{9}{75}}$

Rationalize a Two Term Denominator

In the following exercises, rationalize the denominator.

Exercise:

Problem: $\frac{4}{4+\sqrt{27}}$

Solution:

$\frac{16-12\sqrt{3}}{-11}$

Exercise:

Problem: $\frac{5}{2-\sqrt{10}}$

Exercise:

Problem: $\frac{4}{2-\sqrt{5}}$

Solution:

$-8-4\sqrt{5}$

Exercise:

Problem: $\frac{5}{4-\sqrt{8}}$

Exercise:

Problem: $\frac{\sqrt{2}}{\sqrt{p}+\sqrt{3}}$

Solution:

$\frac{\sqrt{2p}-\sqrt{6}}{p-3}$

Exercise:

Problem: $\frac{\sqrt{x}-\sqrt{2}}{\sqrt{x}+\sqrt{2}}$

Solve Equations with Square Roots

Solve Radical Equations

In the following exercises, solve the equation.

Exercise:

Problem: $\sqrt{7z+1}=6$

Solution:

5

Exercise:

Problem: $\sqrt{4u-2}-4=0$

Exercise:

Problem: $\sqrt{6m+4}-5=0$

Solution:

$\frac{7}{2}$

Exercise:

Problem: $\sqrt{2u-3}+2=0$

Exercise:

Problem: $\sqrt{u-4}+4=u$

Solution:

no solution

Exercise:

Problem: $\sqrt{v-9}+9=0$

Exercise:

Problem: $\sqrt{r-4} - r = -10$

Solution:

13

Exercise:

Problem: $\sqrt{s-9} - s = -9$

Exercise:

Problem: $2\sqrt{2x-7} - 4 = 8$

Solution:

$\frac{43}{2}$

Exercise:

Problem: $\sqrt{2-x} = \sqrt{2x-7}$

Exercise:

Problem: $\sqrt{a} + 3 = \sqrt{a+9}$

Solution:

0

Exercise:

Problem: $\sqrt{r} + 3 = \sqrt{r+4}$

Exercise:

Problem: $\sqrt{u} + 2 = \sqrt{u+5}$

Solution:

$\frac{1}{16}$

Exercise:

Problem: $\sqrt{n+11}-1=\sqrt{n+4}$

Exercise:

Problem: $\sqrt{y+5}+1=\sqrt{2y+3}$

Solution:

11

Use Square Roots in Applications

In the following exercises, solve. Round approximations to one decimal place.

Exercise:

Problem:

A pallet of sod will cover an area of about 600 square feet. Trinh wants to order a pallet of sod to make a square lawn in his backyard. Use the formula $s=\sqrt{A}$ to find the length of each side of his lawn.

Exercise:

Problem:

A helicopter dropped a package from a height of 900 feet above a stranded hiker. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the package to reach the hiker.

Solution:

7.5 seconds

Exercise:

Problem:

Officer Morales measured the skid marks of one of the cars involved in an accident. The length of the skid marks was 245 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied.

Higher Roots

Simplify Expressions with Higher Roots

In the following exercises, simplify.

Exercise:

Problem: ⓐ $\sqrt[6]{64}$ ⓑ $\sqrt[3]{64}$

Solution:

ⓐ 2 ⓑ 4

Exercise:

Problem: ⓐ $\sqrt[3]{-27}$ ⓑ $\sqrt[4]{-64}$

Exercise:

ⓐ $\sqrt[9]{d^9}$
Problem: ⓑ $\sqrt[8]{v^8}$

Solution:

ⓐ d ⓑ $|v|$

Exercise:

ⓐ $\sqrt[5]{a^{10}}$
Problem: ⓑ $\sqrt[3]{b^{27}}$

Exercise:

ⓐ $\sqrt[4]{16x^8}$
Problem: ⓑ $\sqrt[6]{64y^{12}}$

Solution:

ⓐ $2x^2$ ⓑ $2y^2$

Exercise:

ⓐ $\sqrt[7]{128r^{14}}$
Problem: ⓑ $\sqrt[4]{81s^{24}}$

Use the Product Property to Simplify Expressions with Higher Roots

In the following exercises, simplify.
Exercise:

Problem: ⓐ $\sqrt[9]{d^9}$ ⓑ

$\sqrt[11]{m^{17}}$

Solution:

ⓐ d ⓑ

$m\sqrt[11]{m^6}$

Exercise:

Problem: ⓐ $\sqrt[3]{54}$ ⓑ $\sqrt[4]{128}$

Exercise:

Problem: ⓐ $\sqrt[5]{64c^8}$ ⓑ $\sqrt[4]{48d^7}$

Solution:

ⓐ $2c\sqrt[5]{2c^3}$ ⓑ $2d\sqrt[4]{3d^3}$

Exercise:

Problem: ⓐ $\sqrt[3]{343q^7}$ ⓑ $\sqrt[6]{192r^9}$

Exercise:

Problem: ⓐ $\sqrt[3]{-500}$
ⓑ $\sqrt[4]{-16}$

Solution:

ⓐ $-5\sqrt[3]{4}$ ⓑ not a real number

Use the Quotient Property to Simplify Expressions with Higher Roots

In the following exercises, simplify.

Exercise:

Problem: $\sqrt[5]{\frac{r^{10}}{r^5}}$

Exercise:

Problem: $\sqrt[3]{\frac{w^{12}}{w^2}}$

Solution:

$w^3\sqrt[3]{w}$

Exercise:

Problem: $\sqrt[4]{\frac{64y^8}{4y^5}}$

Exercise:

Problem: $\sqrt[3]{\frac{54z^9}{2z^3}}$

Solution:

$3z^2$

Exercise:

Problem: $\sqrt[6]{\frac{64a^7}{b^2}}$

Add and Subtract Higher Roots

In the following exercises, simplify.

Exercise:

Problem: $4\sqrt[5]{20} - 2\sqrt[5]{20}$

Solution:

$2\sqrt[5]{20}$

Exercise:

Problem: $4\sqrt[3]{18} + 3\sqrt[3]{18}$

Exercise:

Problem: $\sqrt[4]{1250} - \sqrt[4]{162}$

Solution:

$2\sqrt[4]{2}$

Exercise:

Problem: $\sqrt[3]{640c^5} - \sqrt[3]{-80c^3}$

Exercise:

Problem: $\sqrt[5]{96t^8} + \sqrt[5]{486t^4}$

Solution:

$2t\sqrt[5]{3t^3} + 3\sqrt[5]{2t^4}$

Rational Exponents

Simplify Expressions with $a^{\frac{1}{n}}$

In the following exercises, write as a radical expression.

Exercise:

Problem: $r^{\frac{1}{8}}$

Exercise:

Problem: $s^{\frac{1}{10}}$

Solution:

$\sqrt[10]{s}$

In the following exercises, write with a rational exponent.

Exercise:

Problem: $\sqrt[5]{u}$

Exercise:

Problem: $\sqrt[6]{v}$

Solution:

$v^{\frac{1}{6}}$

Exercise:

Problem: $\sqrt[3]{9m}$

Exercise:

Problem: $\sqrt[6]{10z}$

Solution:

$(10z)^{\frac{1}{6}}$

In the following exercises, simplify.

Exercise:

Problem: $16^{\frac{1}{4}}$

Exercise:

Problem: $32^{\frac{1}{5}}$

Solution:

2

Exercise:

Problem: $(-125)^{\frac{1}{3}}$

Exercise:

Problem: $(125)^{-\frac{1}{3}}$

Solution:

$\frac{1}{5}$

Exercise:

Problem: $(-9)^{\frac{1}{2}}$

Exercise:

Problem: $(36)^{-\frac{1}{2}}$

Solution:

$\frac{1}{6}$

Simplify Expressions with $a^{\frac{m}{n}}$

In the following exercises, write with a rational exponent.

Exercise:

Problem: $\sqrt[3]{q^5}$

Exercise:

Problem: $\sqrt[5]{n^8}$

Solution:

$n^{\frac{8}{5}}$

In the following exercises, simplify.

Exercise:

Problem: $27^{-\frac{2}{3}}$

Exercise:

Problem: $64^{\frac{5}{2}}$

Solution:

32,768

Exercise:

Problem: $36^{\frac{3}{2}}$

Exercise:

Problem: $81^{-\frac{5}{2}}$

Solution:

$\frac{1}{59,049}$

Use the Laws of Exponents to Simplify Expressions with Rational Exponents

In the following exercises, simplify.

Exercise:

Problem: $3^{\frac{4}{5}} \cdot 3^{\frac{6}{5}}$

Exercise:

Problem: $\left(x^6\right)^{\frac{4}{3}}$

Solution:

x^8

Exercise:

Problem: $\frac{z^{\frac{5}{2}}}{z^{\frac{7}{5}}}$

Exercise:

Problem: $\left(16s^{\frac{9}{4}}\right)^{\frac{1}{4}}$

Solution:

$2s^{\frac{9}{16}}$

Exercise:

Problem: $\left(m^8n^{12}\right)^{\frac{1}{4}}$

Exercise:

Problem: $\frac{z^{\frac{2}{3}} \cdot z^{-\frac{1}{3}}}{z^{-\frac{5}{3}}}$

Solution:

z^2

Practice Test

In the following exercises, simplify.

Exercise:

Problem: $\sqrt{81+144}$

Exercise:

Problem: $\sqrt{169m^4n^2}$

Solution:

$13m^2\left|n\right|$

Exercise:

Problem: $\sqrt{36n^{13}}$

Exercise:

Problem: $3\sqrt{13}+5\sqrt{2}+\sqrt{13}$

Solution:

$4\sqrt{13}+5\sqrt{2}$

Exercise:

Problem: $5\sqrt{20}+2\sqrt{125}$

Exercise:

Problem: $\left(3\sqrt{6y}\right)\left(2\sqrt{50y^3}\right)$

Solution:

$180y^2\sqrt{3}$

Exercise:

Problem: $\left(2-5\sqrt{x}\right)\left(3+\sqrt{x}\right)$

Exercise:

Problem: $\left(1-2\sqrt{q}\right)^2$

Solution:

$1-4\sqrt{q}+4q$

Exercise:

Problem: ⓐ $\sqrt[4]{a^{12}}$ ⓑ $\sqrt[3]{b^{21}}$

Exercise:

Problem: ⓐ $\sqrt[4]{81x^{12}}$ ⓑ $\sqrt[6]{64y^{18}}$

Solution:

ⓐ $3x^3$ ⓑ $2y^3$

Exercise:

Problem: $\sqrt{\frac{64r^{12}}{25r^6}}$

Exercise:

Problem: $\sqrt{\frac{14y^3}{7y}}$

Solution:

$y\sqrt{2}$

Exercise:

Problem: $\frac{\sqrt[5]{256x^7}}{\sqrt[5]{4x^2}}$

Exercise:

Problem: $\sqrt[4]{512} - 2\sqrt[4]{32}$

Solution:

0

Exercise:

Problem: ⓐ $256^{\frac{1}{4}}$ ⓑ $243^{\frac{1}{5}}$

Exercise:

Problem: $49^{\frac{3}{2}}$

Solution:

343

Exercise:

Problem: $25^{-\frac{5}{2}}$

Exercise:

Problem: $\frac{w^{\frac{3}{4}}}{w^{\frac{7}{4}}}$

Solution:

$\frac{1}{w}$

Exercise:

Problem: $\left(27s^{\frac{3}{5}}\right)^{\frac{1}{3}}$

In the following exercises, rationalize the denominator.

Exercise:

Problem: $\frac{3}{2\sqrt{6}}$

Solution:

$\frac{\sqrt{6}}{4}$

Exercise:

Problem: $\frac{\sqrt{3}}{\sqrt{x}+\sqrt{5}}$

In the following exercises, solve.

Exercise:

Problem: $3\sqrt{2x-3}-20=7$

Solution:

42

Exercise:

Problem: $\sqrt{3u-2}=\sqrt{5u+1}$

In the following exercise, solve.

Exercise:

Problem:

A helicopter flying at an altitude of 600 feet dropped a package to a lifeboat. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the package to reach the hiker. Round your answer to the nearest tenth of a second.

Solution:

6.1 seconds

Glossary

rational exponents

- If $\sqrt[n]{a}$ is a real number and $n \geq 2$, $a^{\frac{1}{n}} = \sqrt[n]{a}$.
- For any positive integers m and n, $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$ and $a^{\frac{m}{n}} = \sqrt[n]{a^m}$.

Introduction
class="introduction"

Fireworks accompany festive celebrations around the world. (Credit: modification of work by tlc, Flickr)

The trajectories of fireworks are modeled by quadratic equations. The equations can be used to predict the maximum height of a firework and the number of seconds it will take from launch to explosion. In this chapter, we will study the properties of quadratic equations, solve them, graph them, and see how they are applied as models of various situations.

Solve Quadratic Equations Using the Square Root Property: ASE
By the end of this section, you will be able to:

- Solve quadratic equations of the form $ax^2 = k$ using the Square Root Property
- Solve quadratic equations of the form $a(x-h)^2 = k$ using the Square Root Property

Quadratic equations are equations of the form $ax^2 + bx + c = 0$, where $a \neq 0$. They differ from linear equations by including a term with the variable raised to the second power. We use different methods to solve **quadratic equation**s than linear equations, because just adding, subtracting, multiplying, and dividing terms will not isolate the variable.

We have seen that some quadratic equations can be solved by factoring. More exactly, we should say factoring over the integers. That means that we have required the numbers in the factored form to be integers. In this chapter, we will use three other methods to solve quadratic equations. They not only solve quadratic equations but they also show us how to factor those equations if we allow the numbers to be real numbers. In this case, we would say that we factored over the real numbers.

Solve Quadratic Equations of the Form $ax^2 = k$ Using the Square Root Property

We have already solved some quadratic equations by factoring. Let's review how we used factoring to solve the quadratic equation $x^2 = 9$.

Equation:

$$\begin{array}{lrcr} & x^2 &=& 9 \\ \text{Put the equation in standard form.} & x^2 - 9 &=& 0 \\ \text{Factor the left side.} & (x-3)(x+3) &=& 0 \\ \text{Use the Zero Product Property.} & (x-3)=0,\;\; (x+3) &=& 0 \\ \text{Solve each equation.} & x=3, \qquad\quad x &=& -3 \\ \text{Combine the two solutions into } \pm \text{ form.} & x &=& \pm 3 \\ \text{(The solution is read \textbackslash}x\text{ is equal to positive or negative 3.')} & & & \end{array}$$

We can easily use factoring to find the solutions of similar equations, like $x^2 = 16$ and $x^2 = 25$, because 16 and 25 are perfect squares. But what happens when we have an equation like $x^2 = 7$? Since 7 is not a perfect square, we cannot solve the equation by factoring over the integers.

These equations are all of the form $x^2 = k$.
We defined the square root of a number in this way:
Equation:

$$\text{If } n^2 = m, \text{ then } n \text{ is a square root of } m.$$

This leads to the **Square Root Property**.

Note:
Square Root Property

If $x^2 = k$, and $k \geq 0$, then $x = \sqrt{k}$ or $x = -\sqrt{k}$.

Notice that the Square Root Property gives two solutions to an equation of the form $x^2 = k$: the principal square root of k and its opposite. We could also write the solution as $x = \pm\sqrt{k}$.

Now, we will solve the equation $x^2 = 9$ again, this time using the Square Root Property.
Equation:

$$\begin{array}{lrcl} & x^2 & = & 9 \\ \text{Use the Square Root Property.} & x & = & \pm\sqrt{9} \\ \text{Simplify the radical.} & x & = & \pm 3 \\ \text{Rewrite to show the two solutions.} & x = 3, x & = & -3 \end{array}$$

What happens when the constant is not a perfect square? Let's use the Square Root Property to solve the equation $x^2 = 7$.
Equation:

$$\begin{array}{lc} & x^2 = 7 \\ \text{Use the Square Root Property.} & x = \pm\sqrt{7} \\ \text{Rewrite to show two solutions.} & x = \sqrt{7}, \quad x = -\sqrt{7} \\ \text{We cannot simplify } \sqrt{7}, \text{ so we leave the answer as a radical.} & \end{array}$$

We can write the equation in factored form where we factor over the reals but not over the integers.

$x^2 = 7$

$x^2 - 7 = 0$

$\left(x + \sqrt{7}\right)\left(x - \sqrt{7}\right) = 0$

Example:
Exercise:

Problem: Solve: $x^2 = 169$.

Solution:
Solution

	$x^2 = 169$
Use the Square Root Property.	$x = \pm\sqrt{169}$
Simplify the radical.	$x = \pm 13$
Rewrite to show two solutions.	$x = 13, \quad x = -13$

Note:
Exercise:

Problem: Solve: $x^2 = 81$.

Solution:

$x = 9, x = -9$

Note:
Exercise:

Problem: Solve: $y^2 = 121$.

Solution:

$y = 11, y = -11$

Example:
How to Solve a Quadratic Equation of the Form $ax^2 = k$ Using the Square Root Property
Exercise:

Problem: Solve: $x^2 - 48 = 0$.

Solution:
Solution

Step 1. Isolate the quadratic term and make its coefficient one.	Add 48 to both sides to get x^2 by itself.	$x^2 - 48 = 0$ $x^2 = 48$
Step 2. Use the Square Root Property.	Remember to add the ± symbol.	$x = \pm\sqrt{48}$

Step 3. Simplify the radical.		$x = \pm\sqrt{16} \cdot \sqrt{3}$ $x = \pm 4\sqrt{3}$ $x = 4\sqrt{3}, x = -4\sqrt{3}$

Step 4. Check the solutions.	Substitute in $x = 4\sqrt{3}$ and $x = -4\sqrt{3}$.	$x^2 - 48 = 0$ $(4\sqrt{3})^2 - 48 \stackrel{?}{=} 0$ $16 \cdot 3 - 48 \stackrel{?}{=} 0$ $0 = 0 \checkmark$ $x^2 - 48 = 0$ $(-4\sqrt{3})^2 - 48 \stackrel{?}{=} 0$ $16 \cdot 3 - 48 \stackrel{?}{=} 0$ $0 = 0 \checkmark$

Note:
Exercise:

Problem: Solve: $x^2 - 50 = 0$.

Solution:

$x = 5\sqrt{2}, x = -5\sqrt{2}$

Note:
Exercise:

Problem: Solve: $y^2 - 27 = 0$.

Solution:

$y = 3\sqrt{3}, y = -3\sqrt{3}$

Note:
Solve a quadratic equation using the Square Root Property.

Isolate the quadratic term and make its coefficient one.
Use Square Root Property.
Simplify the radical.

Check the solutions.

To use the Square Root Property, the coefficient of the variable term must equal 1. In the next example, we must divide both sides of the equation by 5 before using the Square Root Property.

Example:
Exercise:

Problem: Solve: $5m^2 = 80$.

Solution:
Solution

The quadratic term is isolated.	$5m^2 = 80$
Divide by 5 to make its cofficient 1.	$\frac{5m^2}{5} = \frac{80}{5}$
Simplify.	$m^2 = 16$
Use the Square Root Property.	$m = \pm\sqrt{16}$
Simplify the radical.	$m = \pm 4$
Rewrite to show two solutions.	$m = 4, m = -4$
Check the solutions. $5m^2 = 80$ $5m^2 = 80$ $5(4)^2 \stackrel{?}{=} 80$ $5(-4)^2 \stackrel{?}{=} 80$ $5 \cdot 16 \stackrel{?}{=} 80$ $5 \cdot 16 \stackrel{?}{=} 80$ $80 = 80$ ✓ $80 = 80$ ✓	

Note:
Exercise:

Problem: Solve: $2x^2 = 98$.

Solution:

$x = 7, x = -7$

Note:
Exercise:

Problem: Solve: $3z^2 = 108$.

Solution:

$z = 6, z = -6$

The Square Root Property started by stating, 'If $x^2 = k$, and $k \geq 0$'. What will happen if $k < 0$? This will be the case in the next example.

Example:
Exercise:

Problem: Solve: $q^2 + 24 = 0$.

Solution:
Solution

	$q^2 + 24 = 0$
Isolate the quadratic term.	$q^2 = -24$
Use the Square Root Property.	$q = \pm\sqrt{-24}$
The $\sqrt{-24}$ is not a real number.	There is no real solution.

Note:
Exercise:

Problem: Solve: $c^2 + 12 = 0$.

Solution:

no real solution

Note:
Exercise:

Problem: Solve: $d^2 + 81 = 0$.

Solution:

no real solution

Remember, we first isolate the quadratic term and then make the coefficient equal to one.

Example:
Exercise:

Problem: Solve: $\frac{2}{3}u^2 + 5 = 17$.

Solution:
Solution

	$\frac{2}{3}u^2 + 5 = 17$
Isolate the quadratic term.	$\frac{2}{3}u^2 = 12$
Multiply by $\frac{3}{2}$ to make the coefficient 1.	$\frac{3}{2} \cdot \frac{2}{3}u^2 = \frac{3}{2} \cdot 12$
Simplify.	$u^2 = 18$
Use the Square Root Property.	$u = \pm\sqrt{18}$
Simplify the radical.	$u = \pm\sqrt{9}\sqrt{2}$
Simplify.	$u = \pm 3\sqrt{2}$
Rewrite to show two solutions.	$u = 3\sqrt{2}, u = -3\sqrt{2}$

Check.

$$\frac{2}{3}u^2+5=17 \qquad\qquad \frac{2}{3}u^2+5=17$$

$$\frac{2}{3}(3\sqrt{2})^2+5\stackrel{?}{=}17 \qquad\qquad \frac{2}{3}(-3\sqrt{2})^2+5\stackrel{?}{=}17$$

$$\frac{2}{3}\cdot 18+5\stackrel{?}{=}17 \qquad\qquad \frac{2}{3}\cdot 18+5\stackrel{?}{=}17$$

$$12+5\stackrel{?}{=}17 \qquad\qquad 12+5\stackrel{?}{=}17$$

$$17=17\checkmark \qquad\qquad 17=17\checkmark$$

Note:
Exercise:

Problem: Solve: $\frac{1}{2}x^2+4=24$.

Solution:

$x=2\sqrt{10}, x=-2\sqrt{10}$

Note:
Exercise:

Problem: Solve: $\frac{3}{4}y^2-3=18$.

Solution:

$y=2\sqrt{7}, y=-2\sqrt{7}$

The solutions to some equations may have fractions inside the radicals. When this happens, we normally rationalize the denominator.

Example:
Exercise:

Problem: Solve: $2c^2-4=45$.

Solution:

Solution

	$2c^2 - 4 = 45$
Isolate the quadratic term.	$2c^2 = 49$
Divide by 2 to make the coefficient 1.	$\frac{2c^2}{2} = \frac{49}{2}$
Simplify.	$c^2 = \frac{49}{2}$
Use the Square Root Property.	$c = \pm\sqrt{\frac{49}{2}}$
Simplify the radical.	$c = \pm\frac{\sqrt{49}}{\sqrt{2}}$
Rationalize the denominator.	$c = \pm\frac{\sqrt{49}\cdot\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}}$
Simplify.	$c = \pm\frac{7\sqrt{2}}{2}$
Rewrite to show two solutions.	$c = \frac{7\sqrt{2}}{2}, \quad c = -\frac{7\sqrt{2}}{2}$
Check. We leave the check for you.	

Note:
Exercise:

Problem: Solve: $5r^2 - 2 = 34$.

Solution:

$r = \frac{6\sqrt{5}}{5}, r = -\frac{6\sqrt{5}}{5}$

Note:
Exercise:

Problem: Solve: $3t^2 + 6 = 70$.

Solution:

$t = \frac{8\sqrt{3}}{3}, t = -\frac{8\sqrt{3}}{3}$

Solve Quadratic Equations of the Form $a(x - h)^2 = k$ Using the Square Root Property

We can use the Square Root Property to solve an equation like $(x - 3)^2 = 16$, too. We will treat the whole binomial, $(x - 3)$, as the quadratic term.

Example:
Exercise:

Problem: Solve: $(x-3)^2 = 16$.

Solution:
Solution

	$(x-3)^2 = 16$
Use the Square Root Property.	$x - 3 = \pm\sqrt{16}$
Simplify.	$x - 3 = \pm 4$
Write as two equations.	$x - 3 = 4, x - 3 = -4$
Solve.	$x = 7, x = -1$
Check. $(7-3)^2 = 16$ $\quad$ $(-1-3)^2 = 16$ $(4)^2 = 16$ $\quad$ $(-4)^2 = 16$ $16 = 16$ ✓ $\quad$ $16 = 16$ ✓	

Note:
Exercise:

Problem: Solve: $(q+5)^2 = 1$.

Solution:

$q = -6, q = -4$

Note:
Exercise:

Problem: Solve: $(r-3)^2 = 25$.

Solution:

$r=8, r=-2$

Example:
Exercise:

Problem: Solve: $(y-7)^2=12$.

Solution:
Solution

	$(y-7)^2=12$
Use the Square Root Property.	$y-7=\pm\sqrt{12}$
Simplify the radical.	$y-7=\pm 2\sqrt{3}$
Solve for *y*.	$y=7\pm 2\sqrt{3}$
Rewrite to show two solutions.	$y=7+2\sqrt{3}, y=7-2\sqrt{3}$
Check. $(y-7)^2=12$ $(y-7)^2=12$ $(7+2\sqrt{3}-7)^2\stackrel{?}{=}12$ $(7-2\sqrt{3}-7)^2\stackrel{?}{=}12$ $(2\sqrt{3})^2\stackrel{?}{=}12$ $(-2\sqrt{3})^2\stackrel{?}{=}12$ $12=12\checkmark$ $12=12\checkmark$	

Note:
Exercise:

Problem: Solve: $(a-3)^2=18$.

Solution:

$a = 3 + 3\sqrt{2}, a = 3 - 3\sqrt{2}$

Note:
Exercise:

Problem: Solve: $(b + 2)^2 = 40$.

Solution:

$b = -2 + 2\sqrt{10}, b = -2 - 2\sqrt{10}$

Remember, when we take the square root of a fraction, we can take the square root of the numerator and denominator separately.

Example:
Exercise:

Problem: Solve: $\left(x - \frac{1}{2}\right)^2 = \frac{5}{4}$.

Solution:
Solution

	$\left(x - \frac{1}{2}\right)^2 = \frac{5}{4}$
Use the Square Root Property.	$x - \frac{1}{2} = \pm\sqrt{\frac{5}{4}}$
Rewrite the radical as a fraction of square roots.	$x - \frac{1}{2} = \pm\frac{\sqrt{5}}{\sqrt{4}}$
Simplify the radical.	$x - \frac{1}{2} = \pm\frac{\sqrt{5}}{2}$
Solve for x.	$x = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$
Rewrite to show two solutions.	$x = \frac{1}{2} + \frac{\sqrt{5}}{2}, \quad x = \frac{1}{2} - \frac{\sqrt{5}}{2}$
Check. We leave the check for you.	

Note:
Exercise:

Problem: Solve: $\left(x-\frac{1}{3}\right)^2=\frac{5}{9}$.

Solution:

$x=\frac{1}{3}+\frac{\sqrt{5}}{3}, x=\frac{1}{3}-\frac{\sqrt{5}}{3}$

Note:
Exercise:

Problem: Solve: $\left(y-\frac{3}{4}\right)^2=\frac{7}{16}$.

Solution:

$y=\frac{3}{4}+\frac{\sqrt{7}}{4}, y=\frac{3}{4}-\frac{\sqrt{7}}{4}$

We will start the solution to the next example by isolating the binomial.

Example:
Exercise:

Problem: Solve: $(x-2)^2+3=30$.

Solution:
Solution

	$(x-2)^2+3 = 30$
Isolate the binomial term.	$(x-2)^2 = 27$
Use the Square Root Property.	$x-2 = \pm\sqrt{27}$
Simplify the radical.	$x-2 = \pm 3\sqrt{3}$
Solve for x.	$x = 2\pm 3\sqrt{3}$
Rewrite to show two solutions.	$x=2+3\sqrt{3},\quad x=2-3\sqrt{3}$
Check. We leave the check for you.	

Note:
Exercise:

Problem: Solve: $(a-5)^2+4=24$.

Solution:

$a=5+2\sqrt{5}, a=5-2\sqrt{5}$

Note:
Exercise:

Problem: Solve: $(b-3)^2-8=24$.

Solution:

$b=3+4\sqrt{2}, b=3-4\sqrt{2}$

Example:
Exercise:

Problem: Solve: $(3v-7)^2=-12$.

Solution:
Solution

Use the Square Root Property.	$\begin{aligned}(3v-7)^2 &= -12\\ 3v-7 &= \pm\sqrt{-12}\end{aligned}$
The $\sqrt{-12}$ is not a real number.	There is no real solution.

Note:
Exercise:

Problem: Solve: $(3r+4)^2=-8$.

Solution:

no real solution

Note:
Exercise:

Problem: Solve: $(2t-8)^2=-10$.

Solution:

no real solution

The left sides of the equations in the next two examples do not seem to be of the form $a(x-h)^2$. But they are perfect square trinomials, so we will factor to put them in the form we need.

Example:
Exercise:

Problem: Solve: $p^2-10p+25=18$.

Solution:
Solution

The left side of the equation is a perfect square trinomial. We will factor it first.

	$p^2-10p+25=18$
Factor the perfect square trinomial.	$(p-5)^2=18$
Use the Square Root Property.	$p-5=\pm\sqrt{18}$
Simplify the radical.	$p-5=\pm 3\sqrt{2}$
Solve for p.	$p=5\pm 3\sqrt{2}$
Rewrite to show two solutions.	$p=5+3\sqrt{2},\quad p=5-3\sqrt{2}$
Check. We leave the check for you.	

Note:
Exercise:

Problem: Solve: $x^2-6x+9=12$.

Solution:

$x=3+2\sqrt{3}, x=3-2\sqrt{3}$

Note:
Exercise:

Problem: Solve: $y^2 + 12y + 36 = 32$.

Solution:

$y = -6 + 4\sqrt{2}, y = -6 - 4\sqrt{2}$

Example:
Exercise:

Problem: Solve: $4n^2 + 4n + 1 = 16$.

Solution:
Solution

Again, we notice the left side of the equation is a perfect square trinomial. We will factor it first.

	$4n^2 + 4n + 1 = 16$
Factor the perfect square trinomial.	$(2n+1)^2 = 16$
Use the Square Root Property.	$2n + 1 = \pm\sqrt{16}$
Simplify the radical.	$2n + 1 = \pm 4$
Solve for *n*.	$2n = -1 \pm 4$
Divide each side by 2.	$\frac{2n}{2} = \frac{-1\pm4}{2}$ $n = \frac{-1\pm4}{2}$
Rewrite to show two solutions.	$n = \frac{-1+4}{2}, n = \frac{-1-4}{2}$
Simplify each equation.	$n = \frac{3}{2}, n = -\frac{5}{2}$
Check.	

$$4n^2 + 4n + 1 = 16 \qquad 4n^2 + 4n + 1 = 16$$

$$4\left(\frac{3}{2}\right)^2 + 4\left(\frac{3}{2}\right) + 1 \stackrel{?}{=} 16 \qquad 4\left(-\frac{5}{2}\right)^2 + 4\left(-\frac{5}{2}\right) + 1 \stackrel{?}{=} 16$$

$$4\left(\frac{9}{4}\right) + 4\left(\frac{3}{2}\right) + 1 \stackrel{?}{=} 16 \qquad 4\left(\frac{25}{4}\right) + 4\left(-\frac{5}{2}\right) + 1 \stackrel{?}{=} 16$$

$$9 + 6 + 1 \stackrel{?}{=} 16 \qquad 25 - 10 + 1 \stackrel{?}{=} 16$$

$$16 = 16\checkmark \qquad 16 = 16\checkmark$$

Note:
Exercise:

Problem: Solve: $9m^2 - 12m + 4 = 25$.

Solution:

$m = 7, m = -3$

Note:
Exercise:

Problem: Solve: $16n^2 + 40n + 25 = 4$.

Solution:

$n = -\frac{3}{4}, n = -\frac{7}{4}$

Note:
Access these online resources for additional instruction and practice with solving quadratic equations:

- Solving Quadratic Equations: Solving by Taking Square Roots
- Using Square Roots to Solve Quadratic Equations
- Solving Quadratic Equations: The Square Root Method

Key Concepts

- Square Root Property
  If $x^2 = k$, and $k \geq 0$, then $x = \sqrt{k}$ or $x = -\sqrt{k}$.

Practice Makes Perfect

Solve Quadratic Equations of the form $\mathbf{ax^2 = k}$ Using the Square Root Property

In the following exercises, solve the following quadratic equations.
Exercise:

Problem: $a^2 = 49$

Solution:

$a = \pm 7$

Exercise:

Problem: $b^2 = 144$

Exercise:

Problem: $r^2 - 24 = 0$

Solution:

$r = \pm 2\sqrt{6}$

Exercise:

Problem: $t^2 - 75 = 0$

Exercise:

Problem: $u^2 - 300 = 0$

Solution:

$u = \pm 10\sqrt{3}$

Exercise:

Problem: $v^2 - 80 = 0$

Exercise:

Problem: $4m^2 = 36$

Solution:

$m = \pm 3$

Exercise:

Problem: $3n^2 = 48$

Exercise:

Problem: $x^2 + 20 = 0$

Solution:

no real solution

Exercise:

Problem: $y^2 + 64 = 0$

Exercise:

Problem: $\frac{2}{5}a^2 + 3 = 11$

Solution:

$a = \pm\, 2\sqrt{5}$

Exercise:

Problem: $\frac{3}{2}b^2 - 7 = 41$

Exercise:

Problem: $7p^2 + 10 = 26$

Solution:

$p = \pm\, \frac{4\sqrt{7}}{7}$

Exercise:

Problem: $2q^2 + 5 = 30$

Solve Quadratic Equations of the Form $\mathbf{a(x - h)^2 = k}$ Using the Square Root Property

In the following exercises, solve the following quadratic equations.

Exercise:

Problem: $(x + 2)^2 = 9$

Solution:

$x = 1, x = -5$

Exercise:

Problem: $(y - 5)^2 = 36$

Exercise:

Problem: $(u-6)^2=64$

Solution:

$u=14, u=-2$

Exercise:

Problem: $(v+10)^2=121$

Exercise:

Problem: $(m-6)^2=20$

Solution:

$m=6\pm 2\sqrt{5}$

Exercise:

Problem: $(n+5)^2=32$

Exercise:

Problem: $\left(r-\frac{1}{2}\right)^2=\frac{3}{4}$

Solution:

$r=\frac{1}{2}\pm\frac{\sqrt{3}}{2}$

Exercise:

Problem: $\left(t-\frac{5}{6}\right)^2=\frac{11}{25}$

Exercise:

Problem: $(a-7)^2+5=55$

Solution:

$a=7\pm 5\sqrt{2}$

Exercise:

Problem: $(b-1)^2-9=39$

Exercise:

Problem: $(5c+1)^2 = -27$

Solution:

no real solution

Exercise:

Problem: $(8d-6)^2 = -24$

Exercise:

Problem: $m^2 - 4m + 4 = 8$

Solution:

$m = 2 \pm 2\sqrt{2}$

Exercise:

Problem: $n^2 + 8n + 16 = 27$

Exercise:

Problem: $25x^2 - 30x + 9 = 36$

Solution:

$x = -\frac{3}{5}, x = \frac{9}{5}$

Exercise:

Problem: $9y^2 + 12y + 4 = 9$

Mixed Practice

In the following exercises, solve using the Square Root Property.

Exercise:

Problem: $2r^2 = 32$

Solution:

$r = \pm 4$

Exercise:

Problem: $4t^2 = 16$

Exercise:

Problem: $(a-4)^2=28$

Solution:

$a=4\pm 2\sqrt{7}$

Exercise:

Problem: $(b+7)^2=8$

Exercise:

Problem: $9w^2-24w+16=1$

Solution:

$w=1, w=\frac{5}{3}$

Exercise:

Problem: $4z^2+4z+1=49$

Exercise:

Problem: $a^2-18=0$

Solution:

$a=\pm 3\sqrt{2}$

Exercise:

Problem: $b^2-108=0$

Exercise:

Problem: $\left(p-\frac{1}{3}\right)^2=\frac{7}{9}$

Solution:

$p=\frac{1}{3}\pm\frac{\sqrt{7}}{3}$

Exercise:

Problem: $\left(q-\frac{3}{5}\right)^2=\frac{3}{4}$

Exercise:

Problem: $m^2+12=0$

Solution:

no real solution

Exercise:

Problem: $n^2 + 48 = 0$

Exercise:

Problem: $u^2 - 14u + 49 = 72$

Solution:

$u = 7 \pm 6\sqrt{2}$

Exercise:

Problem: $v^2 + 18v + 81 = 50$

Exercise:

Problem: $(m - 4)^2 + 3 = 15$

Solution:

$m = 4 \pm 2\sqrt{3}$

Exercise:

Problem: $(n - 7)^2 - 8 = 64$

Exercise:

Problem: $(x + 5)^2 = 4$

Solution:

$x = -3, x = -7$

Exercise:

Problem: $(y - 4)^2 = 64$

Exercise:

Problem: $6c^2 + 4 = 29$

Solution:

$c = \pm \frac{5\sqrt{6}}{6}$

Exercise:

Problem: $2d^2 - 4 = 77$

Exercise:

Problem: $(x-6)^2 + 7 = 3$

Solution:

no real solution

Exercise:

Problem: $(y-4)^2 + 10 = 9$

Everyday Math

Exercise:

Problem:

Paola has enough mulch to cover 48 square feet. She wants to use it to make three square vegetable gardens of equal sizes. Solve the equation $3s^2 = 48$ to find s, the length of each garden side.

Solution:

4 feet

Exercise:

Problem:

Kathy is drawing up the blueprints for a house she is designing. She wants to have four square windows of equal size in the living room, with a total area of 64 square feet. Solve the equation $4s^2 = 64$ to find s, the length of the sides of the windows.

Writing Exercises

Exercise:

Problem: Explain why the equation $x^2 + 12 = 8$ has no solution.

Solution:

Answers will vary.

Exercise:

Problem: Explain why the equation $y^2 + 8 = 12$ has two solutions.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
solve quadratic equations of the form $ax^2 = k$ using the square root property.			
solve quadratic equations of the form $a(x - h)^2 = k$ using the square root property.			

ⓑ If most of your checks were:

…confidently: Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help: This must be addressed quickly because topics you do not master become potholes in your road to success. In math, every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no-I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

Glossary

quadratic equation
: A quadratic equation is an equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$.

Square Root Property
: The Square Root Property states that, if $x^2 = k$ and $k \geq 0$, then $x = \sqrt{k}$ or $x = -\sqrt{k}$.

Solve Quadratic Equations by Completing the Square: ASE
By the end of this section, you will be able to:

- Complete the square of a binomial expression
- Solve quadratic equations of the form $x^2+bx+c=0$ by completing the square
- Solve quadratic equations of the form $ax^2+bx+c=0$ by completing the square

So far, we have solved quadratic equations by factoring and using the Square Root Property. In this section, we will solve quadratic equations by a process called ‘completing the square.’

Complete The Square of a Binomial Expression

In the last section, we were able to use the Square Root Property to solve the equation $(y-7)^2=12$ because the left side was a perfect square.
Equation:

$$\begin{aligned}(y-7)^2 &= 12\\ y-7 &= \pm\sqrt{12}\\ y-7 &= \pm 2\sqrt{3}\\ y &= 7\pm 2\sqrt{3}\end{aligned}$$

We also solved an equation in which the left side was a perfect square trinomial, but we had to rewrite it the form $(x-k)^2$ in order to use the square root property.
Equation:

$$\begin{aligned}x^2-10x+25 &= 18\\ (x-5)^2 &= 18\end{aligned}$$

What happens if the variable is not part of a perfect square? Can we use algebra to make a perfect square?

Let’s study the binomial square pattern we have used many times. We will look at two examples.

Equation:

$$\begin{array}{ll} (x+9)^2 & (y-7)^2 \\ (x+9)(x+9) & (y-7)\ (y-7) \\ x^2+9x+9x+81 & y^2-7y-7y+49 \\ x^2+18x+81 & y^2-14y+49 \end{array}$$

Note:

Binomial Squares Pattern

If a, b are real numbers,

Equation:

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$\underbrace{(a+b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} + \underbrace{2ab}_{2\times(\text{product of terms})} + \underbrace{b^2}_{(\text{second term})^2}$$

Equation:

$$(a-b)^2 = a^2 - 2ab + b^2$$

$$\underbrace{(a-b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} - \underbrace{2ab}_{2\times(\text{product of terms})} + \underbrace{b^2}_{(\text{second term})^2}$$

We can use this pattern to “make” a perfect square.

We will start with the expression $x^2 + 6x$. Since there is a plus sign between the two terms, we will use the $(a + b)^2$ pattern.

Equation:

$$a^2 + 2ab + b^2 = (a + b)^2$$

Notice that the first term of $x^2 + 6x$ is a square, x^2.

We now know $a = x$.

What number can we add to $x^2 + 6x$ to make a perfect square trinomial?

$$\begin{array}{l} a^2 + 2ab + b^2 \\ x^2 + 6x + \_\_ \end{array}$$

The middle term of the Binomial Squares Pattern, $2ab$, is twice the product of the two terms of the binomial. This means twice the product of x and some number is $6x$. So, two times some number must be six. The number we need is $\frac{1}{2} \cdot 6 = 3$. The second term in the binomial, b, must be 3.

$$\begin{array}{l} a^2 + \quad 2ab \quad + b^2 \\ x^2 + 2 \cdot 3 \cdot x + \_\_ \end{array}$$

We now know $b = 3$.

Now, we just square the second term of the binomial to get the last term of the perfect square trinomial, so we square three to get the last term, nine.

$$\begin{array}{l} a^2 + 2ab + b^2 \\ x^2 + 6x + 9 \end{array}$$

We can now factor to

$$\begin{array}{l} (a + b)^2 \\ (x + 3)^2 \end{array}$$

So, we found that adding nine to $x^2 + 6x$ ‘completes the square,’ and we write it as $(x+3)^2$.

Note:
Complete a square.
To complete the square of $x^2 + bx$:

Identify b, the coefficient of x.
Find $\left(\frac{1}{2}b\right)^2$, the number to complete the square.
Add the $\left(\frac{1}{2}b\right)^2$ to $x^2 + bx$.

Example:
Exercise:

Problem:

Complete the square to make a perfect square trinomial. Then, write the result as a binomial square.

$x^2 + 14x$

Solution:
Solution

The coefficient of *x* is 14.	$x^2 + bx$ $x^2 + 14x$

Find $\left(\frac{1}{2}b\right)^2$. $\left(\frac{1}{2}\cdot 14\right)^2$ $(7)^2$ 49	
Add 49 to the binomial to complete the square.	$x^2+14x+49$
Rewrite as a binomial square.	$(x+7)^2$

Note:
Exercise:

Problem:

Complete the square to make a perfect square trinomial. Write the result as a binomial square.

y^2+12y

Solution:

$(y+6)^2$

Note:
Exercise:

Problem:

Complete the square to make a perfect square trinomial. Write the result as a binomial square.

$z^2 + 8z$

Solution:

$(z + 4)^2$

Example:
Exercise:

Problem:

Complete the square to make a perfect square trinomial. Then, write the result as a binomial squared. $m^2 - 26m$

Solution:
Solution

The coefficient of m is -26.	$\overset{x^2 - bx}{m^2 - 26m}$

Find $\left(\frac{1}{2}b\right)^2$. $\left(\frac{1}{2}\cdot(-26)\right)^2$ $(-13)^2$ 169	
Add 169 to the binomial to complete the square.	$m^2-26m+169$
Rewrite as a binomial square.	$(m-13)^2$

Note:
Exercise:

Problem:

Complete the square to make a perfect square trinomial. Write the result as a binomial square.

a^2-20a

Solution:

$(a-10)^2$

Note:
Exercise:

Problem:

Complete the square to make a perfect square trinomial. Write the result as a binomial square.

$b^2 - 4b$

Solution:

$(b-2)^2$

Example:
Exercise:

Problem:

Complete the square to make a perfect square trinomial. Then, write the result as a binomial squared.

$u^2 - 9u$

Solution:
Solution

The coefficient of u is −9.	$x^2 + bx$ $u^2 - 9u$

Find $\left(\frac{1}{2}b\right)^2$. $\left(\frac{1}{2}\cdot(-9)\right)^2$ $\left(-\frac{9}{2}\right)^2$ $\frac{81}{4}$	
Add $\frac{81}{4}$ to the binomial to complete the square.	$u^2 - 9u + \frac{81}{4}$
Rewrite as a binomial square.	$\left(u - \frac{9}{2}\right)^2$

Note:
Exercise:

Problem:

Complete the square to make a perfect square trinomial. Write the result as a binomial square.

$m^2 - 5m$

Solution:

$\left(m - \frac{5}{2}\right)^2$

Note:
Exercise:

Problem:

Complete the square to make a perfect square trinomial. Write the result as a binomial square.

$n^2 + 13n$

Solution:

$\left(n + \frac{13}{2}\right)^2$

Example:
Exercise:

Problem:

Complete the square to make a perfect square trinomial. Then, write the result as a binomial squared.

$p^2 + \frac{1}{2}p$

Solution:
Solution

The coefficient of p is $\frac{1}{2}$.	$\overset{x^2}{p^2} \overset{+}{+} \overset{bx}{\frac{1}{2}p}$

Find $\left(\frac{1}{2}b\right)^2$. $\left(\frac{1}{2}\cdot\frac{1}{2}\right)^2$ $\left(\frac{1}{4}\right)^2$ $\frac{1}{16}$	
Add $\frac{1}{16}$ to the binomial to complete the square.	$p^2+\frac{1}{2}p+\frac{1}{16}$
Rewrite as a binomial square.	$\left(p+\frac{1}{4}\right)^2$

Note:
Exercise:

Problem:

Complete the square to make a perfect square trinomial. Write the result as a binomial square.

$p^2+\frac{1}{4}p$

Solution:

$\left(p+\frac{1}{8}\right)^2$

Note:
Exercise:

Problem:

Complete the square to make a perfect square trinomial. Write the result as a binomial square.

$q^2 - \frac{2}{3}q$

Solution:

$\left(q - \frac{1}{3}\right)^2$

Solve Quadratic Equations of the Form $x^2 + bx + c = 0$ by completing the square

In solving equations, we must always do the same thing to both sides of the equation. This is true, of course, when we solve a quadratic equation by completing the square, too. When we add a term to one side of the equation to make a perfect square trinomial, we must also add the same term to the other side of the equation.

For example, if we start with the equation $x^2 + 6x = 40$ and we want to complete the square on the left, we will add nine to both sides of the equation.

$$\begin{aligned} x^2 + 6x &= 40 \\ x^2 + 6x + \_\_ &= 40 + \_\_ \\ x^2 + 6x \color{red}{+ 9} &= 40 \color{red}{+ 9} \end{aligned}$$

Then, we factor on the left and simplify on the right.
Equation:

$$(x + 3)^2 = 49$$

Now the equation is in the form to solve using the Square Root Property. Completing the square is a way to transform an equation into the form we need to be able to use the Square Root Property.

Example:
How To Solve a Quadratic Equation of the Form $x^2 + bx + c = 0$ by Completing the Square
Exercise:

Problem: Solve $x^2 + 8x = 48$ by completing the square.

Solution:
Solution

Step 1. Isolate the variable terms on one side and the constant terms on the other.	This equation has all the variables on the left.	$\underbrace{x^2 + 8x}_{x^2+bx} = \underbrace{48}_{c}$
Step 2. Find $\left(\frac{1}{2} \cdot b\right)^2$, the number to complete the square. Add it to both sides of the equation.	Take half of 8 and square it. $4^2 = 16$ Add 16 to BOTH sides of the equation.	$x^2 + 8x + \underset{\left(\frac{1}{2}\cdot 8\right)^2}{\underline{\quad}} = 48$ $x^2 + 8x + 16 = 48 + 16$
Step 3. Factor the perfect square trinomial as a binomial square.	$x^2 + 8x + 16 = (x + 4)^2$ Add the terms on the right.	$(x + 4)^2 = 64$
Step 4. Use the Square Root Property.		$x + 4 = \pm\sqrt{64}$
Step 5. Simplify the radical and then solve the two resulting equations.		$x + 4 = \pm 8$ $x + 4 = 8 \quad x + 4 = -8$ $x = 4 \quad x = -12$

Step 6. Check the solutions.	Put each answer in the original equation to check. Substitute $x = 4$.	$x^2 + 8x = 48$ $(4)^2 + 8(4) \stackrel{?}{=} 48$ $16 + 32 \stackrel{?}{=} 48$ $48 = 48 \checkmark$
	Substitute $x = -12$.	$x^2 + 8x = 48$ $(-12)^2 + 8(-12) \stackrel{?}{=} 48$ $144 - 96 \stackrel{?}{=} 48$ $48 = 48 \checkmark$

Note:
Exercise:

Problem: Solve $c^2 + 4c = 5$ by completing the square.

Solution:

$c = -5, c = 1$

Note:
Exercise:

Problem: Solve $d^2 + 10d = -9$ by completing the square.

Solution:

$d = -9, d = -1$

Note:
Solve a quadratic equation of the form $x^2 + bx + c = 0$ by completing the square.

Isolate the variable terms on one side and the constant terms on the other.
Find $\left(\frac{1}{2} \cdot b\right)^2$, the number to complete the square. Add it to both sides of the equation.
Factor the perfect square trinomial as a binomial square.
Use the Square Root Property.
Simplify the radical and then solve the two resulting equations.
Check the solutions.

Example:
Exercise:

Problem: Solve $y^2 - 6y = 16$ by completing the square.

Solution:
Solution

The variable terms are on the left side.	$\overset{x^2 - bx}{y^2 - 6y} = \overset{c}{16}$
Take half of -6 and square it. $\left(\frac{1}{2}(-6)\right)^2 = 9$	$y^2 - 6y + \underset{\left(\frac{1}{2} \cdot (-6)\right)^2}{\underline{\qquad}} = 16$
Add 9 to both sides.	$y^2 - 6y + 9 = 16 + 9$

Factor the perfect square trinomial as a binomial square.	$(y-3)^2 = 25$
Use the Square Root Property.	$y-3 = \pm\sqrt{25}$
Simplify the radical.	$y-3 = \pm 5$
Solve for *y*.	$y = 3 \pm 5$
Rewrite to show two solutions.	$y = 3+5,\ y = 3-5$
Solve the equations.	$y = 8,\ y = -2$
Check. $$\begin{array}{rcl} y^2 - 6y &=& 16 \\ 8^2 - 6 \cdot 8 &\stackrel{?}{=}& 16 \\ 64 - 48 &\stackrel{?}{=}& 16 \\ 16 &=& 16\ \checkmark \end{array} \qquad \begin{array}{rcl} y^2 - 6y &=& 16 \\ (-2)^2 - 6(-2) &\stackrel{?}{=}& 16 \\ 4 + 12 &\stackrel{?}{=}& 16 \\ 16 &=& 16\ \checkmark \end{array}$$	

Note:
Exercise:

Problem: Solve $r^2 - 4r = 12$ by completing the square.

Solution:

$r = -2, r = 6$

Note:
Exercise:

Problem: Solve $t^2 - 10t = 11$ by completing the square.

Solution:

$t = -1, t = 11$

Example:
Exercise:

Problem: Solve $x^2 + 4x = -21$ by completing the square.

Solution:
Solution

The variable terms are on the left side.	$\overset{x^2+bx}{x^2 + 4x} = \overset{c}{-21}$

Take half of 4 and square it. $\left(\frac{1}{2}(4)\right)^2 = 4$	$x^2 + 4x + \underset{\left(\frac{1}{2}\cdot 4\right)^2}{\underline{\qquad}} = -21$
Add 4 to both sides.	$x^2 + 4x + 4 = -21 + 4$
Factor the perfect square trinomial as a binomial square.	$(x+2)^2 = -17$
Use the Square Root Property.	$x + 2 = \pm\sqrt{-17}$
We cannot take the square root of a negative number.	There is no real solution.

Note:
Exercise:

Problem: Solve $y^2 - 10y = -35$ by completing the square.

Solution:

no real solution

Note:
Exercise:

Problem: Solve $z^2 + 8z = -19$ by completing the square.

Solution:

no real solution

In the previous example, there was no real solution because $(x+k)^2$ was equal to a negative number.

Example:
Exercise:

Problem: Solve $p^2 - 18p = -6$ by completing the square.

Solution:
Solution

The variable terms are on the left side.	$\underbrace{P^2 - 18P}_{x^2+bx} = \underbrace{-6}_{c}$
Take half of -18 and square it. $\left(\frac{1}{2}(-18)\right)^2 = 81$	$P^2 - 18p + \underbrace{\_\_\_\_\_\_}_{\left(\frac{1}{2}\cdot(-18)\right)^2} = -6$
Add 81 to both sides.	$p^2 - 18p + 81 = -6 + 81$
Factor the perfect square trinomial as a binomial square.	$(p-9)^2 = 75$

Use the Square Root Property.	$p - 9 = \pm\sqrt{75}$
Simplify the radical.	$p - 9 = \pm 5\sqrt{3}$
Solve for *p*.	$p = 9 \pm 5\sqrt{3}$
Rewrite to show two solutions.	$p = 9 + 5\sqrt{3},\ p = 9 - 5\sqrt{3}$
Check. $p^2 - 18p = -6$; $(9 + 5\sqrt{3})^2 - 18(9 + 5\sqrt{3}) \stackrel{?}{=} -6$; $81 + 90\sqrt{3} + 75 - 162 - 90\sqrt{3} \stackrel{?}{=} -6$; $-6 = -6\checkmark$ — $p^2 - 18p = -6$; $(9 - 5\sqrt{3})^2 - 18(9 - 5\sqrt{3}) \stackrel{?}{=} -6$; $81 - 90\sqrt{3} + 75 - 162 + 90\sqrt{3} \stackrel{?}{=} -6$; $-6 = -6\checkmark$	

Another way to check this would be to use a calculator. Evaluate $p^2 - 18p$ for both of the solutions. The answer should be -6.

Note:
Exercise:

Problem: Solve $x^2 - 16x = -16$ by completing the square.

Solution:

$x = 8 \pm 4\sqrt{3}$

Note:
Exercise:

Problem: Solve $y^2+8y=11$ by completing the square.

Solution:

$y=-4\pm 3\sqrt{3}$

We will start the next example by isolating the variable terms on the left side of the equation.

Example:
Exercise:

Problem: Solve $x^2+10x+4=15$ by completing the square.

Solution:
Solution

The variable terms are on the left side.	$x^2+10x+4=15$
Subtract 4 to get the constant terms on the right side.	$x^2+10x \quad =11$
Take half of 10 and square it.	

$\left(\frac{1}{2}(10)\right)^2 = 25$	$x^2 + 10x + \underline{\qquad} = 11$ $\left(\frac{1}{2} \cdot (10)\right)^2$
Add 25 to both sides.	$x^2 + 10x + 25 = 11 + 25$
Factor the perfect square trinomial as a binomial square.	$(x + 5)^2 = 36$
Use the Square Root Property.	$x + 5 = \pm\sqrt{36}$
Simplify the radical.	$x + 5 = \pm\sqrt{36}$
Solve for *x*.	$x = -5 \pm 6$
Rewrite to show two equations.	$x = -5 + 6, x = -5 - 6$
Solve the equations.	$x = 1, x = -11$
Check. $x^2 + 10x + 4 = 15$ $(1)^2 + 10(1) + 4 \stackrel{?}{=} 15$ $1 + 10 + 4 \stackrel{?}{=} 15$ $15 = 15 \checkmark$ $x^2 + 10x + 4 = 15$ $(-11)^2 + 10(-11) + 4 \stackrel{?}{=} 15$ $121 - 110 + 4 \stackrel{?}{=} 15$ $15 = 15 \checkmark$	

Note:
Exercise:

Problem: Solve $a^2 + 4a + 9 = 30$ by completing the square.

Solution:

$a = -7, a = 3$

Note:
Exercise:

Problem: Solve $b^2 + 8b - 4 = 16$ by completing the square.

Solution:

$b = -10, b = -2$

To solve the next equation, we must first collect all the variable terms to the left side of the equation. Then, we proceed as we did in the previous examples.

Example:
Exercise:

Problem: Solve $n^2 = 3n + 11$ by completing the square.

Solution:
Solution

	$n^2 = 3n + 11$
Subtract $3n$ to get the variable terms on the left side.	$n^2 - 3n \quad = 11$
Take half of -3 and square it. $\left(\frac{1}{2}(-3)\right)^2 = \frac{9}{4}$	$n^2 - 3n + \underset{\left(\frac{1}{2}\cdot(-3)\right)^2}{\underline{\qquad}} = 11$
Add $\frac{9}{4}$ to both sides.	$n^2 - 3n + \frac{9}{4} = 11 + \frac{9}{4}$
Factor the perfect square trinomial as a binomial square.	$\left(n - \frac{3}{2}\right)^2 = \frac{44}{4} + \frac{9}{4}$
Add the fractions on the right side.	$\left(n - \frac{3}{2}\right)^2 = \frac{53}{4}$
Use the Square Root Property.	$n - \frac{3}{2} = \pm\sqrt{\frac{53}{4}}$
Simplify the radical.	$n - \frac{3}{2} = \pm\frac{\sqrt{53}}{2}$
Solve for n.	$n = \frac{3}{2} \pm \frac{\sqrt{53}}{2}$
Rewrite to show two equations.	$n = \frac{3}{2} + \frac{\sqrt{53}}{2}, n = \frac{3}{2} - \frac{\sqrt{53}}{2}$

Check. We leave the check for you!	

Note:
Exercise:

Problem: Solve $p^2 = 5p + 9$ by completing the square.

Solution:

$p = \frac{5}{2} \pm \frac{\sqrt{61}}{2}$

Note:
Exercise:

Problem: Solve $q^2 = 7q - 3$ by completing the square.

Solution:

$q = \frac{7}{2} \pm \frac{\sqrt{37}}{2}$

Notice that the left side of the next equation is in factored form. But the right side is not zero, so we cannot use the Zero Product Property. Instead, we multiply the factors and then put the equation into the standard form to solve by completing the square.

Example:

Exercise:

Problem: Solve $(x-3)(x+5)=9$ by completing the square.

Solution:
Solution

	$(x-3)(x+5)=9$
We multiply binomials on the left.	$x^2+2x-15=9$
Add 15 to get the variable terms on the left side.	$x^2+2x \quad =24$
Take half of 2 and square it. $\left(\frac{1}{2}(2)\right)^2=1$	$x^2+2x+\underline{\qquad}=24$ $\left(\frac{1}{2}\cdot(2)\right)^2$
Add 1 to both sides.	$x^2+2x+1=24+1$
Factor the perfect square trinomial as a binomial square.	$(x+1)^2=25$
Use the Square Root Property.	$x+1=\pm\sqrt{25}$
Solve for *x*.	$x=-1\pm 5$

Rewite to show two solutions.	$x = -1 + 5, x = -1 - 5$
Simplify.	$x = 4, x = -6$
Check. We leave the check for you!	

Note:
Exercise:

Problem: Solve $(c-2)(c+8)=7$ by completing the square.

Solution:

$c=-3 \pm 4\sqrt{2}$

Note:
Exercise:

Problem: Solve $(d-7)(d+3)=56$ by completing the square.

Solution:

$d=-7, d=11$

Solve Quadratic Equations of the form $ax^2 + bx + c = 0$ by completing the square

The process of completing the square works best when the leading coefficient is one, so the left side of the equation is of the form $x^2 + bx + c$. If the x^2 term has a coefficient, we take some preliminary steps to make the coefficient equal to one.

Sometimes the coefficient can be factored from all three terms of the trinomial. This will be our strategy in the next example.

Example:
Exercise:

Problem: Solve $3x^2 - 12x - 15 = 0$ by completing the square.

Solution:
Solution

To complete the square, we need the coefficient of x^2 to be one. If we factor out the coefficient of x^2 as a common factor, we can continue with solving the equation by completing the square.

	$3x^2 - 12x - 15 = 0$
Factor out the greatest common factor.	$3(x^2 - 4x - 5) = 0$
Divide both sides by 3 to isolate the	

trinomial.	$\frac{3(x^2 - 4x - 5)}{3} = \frac{0}{3}$
Simplify.	$x^2 - 4x - 5 = 0$
Subtract 5 to get the constant terms on the right.	$x^2 - 4x \qquad = 5$
Take half of 4 and square it. $\left(\frac{1}{2}(4)\right)^2 = 4$	$x^2 - 4x + \underset{\left(\frac{1}{2}\cdot(4)\right)^2}{\underline{\qquad}} = 5$
Add 4 to both sides.	$x^2 - 4x + 4 = 5 + 4$
Factor the perfect square trinomial as a binomial square.	$(x - 2)^2 = 9$
Use the Square Root Property.	$x - 2 = \pm\sqrt{9}$
Solve for *x*.	$x - 2 = \pm 3$
Rewrite to show 2 solutions.	$x = 2 + 3, x = 2 - 3$
Simplify.	$x = 5, x = -1$
Check.	

$x = 5$	$x = -1$
$3x^2 - 12x - 15 = 0$	$3x^2 - 12x - 15 = 0$
$3(5)^2 - 12(5) - 15 \stackrel{?}{=} 0$	$3(-1)^2 - 12(-1) - 15 \stackrel{?}{=} 0$
$75 - 60 - 15 \stackrel{?}{=} 0$	$3 + 12 - 15 \stackrel{?}{=} 0$
$0 = 0 ✓$	$0 = 0 ✓$

Note:
Exercise:

Problem: Solve $2m^2 + 16m - 8 = 0$ by completing the square.

Solution:

$m = -4 \pm 2\sqrt{5}$

Note:
Exercise:

Problem: Solve $4n^2 - 24n - 56 = 8$ by completing the square.

Solution:

$n = -2, 8$

To complete the square, the leading coefficient must be one. When the leading coefficient is not a factor of all the terms, we will divide both sides of the equation by the leading coefficient. This will give us a fraction for

the second coefficient. We have already seen how to complete the square with fractions in this section.

Example:
Exercise:

Problem: Solve $2x^2 - 3x = 20$ by completing the square.

Solution:
Solution

Again, our first step will be to make the coefficient of x^2 be one. By dividing both sides of the equation by the coefficient of x^2, we can then continue with solving the equation by completing the square.

	$2x^2 - 3x = 20$
Divide both sides by 2 to get the coefficient of x^2 to be 1.	$\frac{2x^2 - 3x}{2} = \frac{20}{2}$
Simplify.	$x^2 - \frac{3}{2}x = 10$
Take half of $-\frac{3}{2}$ and square it. $\left(\frac{1}{2}\left(-\frac{3}{2}\right)\right)^2 = \frac{9}{16}$	$x^2 - \frac{3}{2}x + \underset{\left(\frac{1}{2}\cdot\left(-\frac{3}{2}\right)\right)^2}{\underline{\qquad}} = 10$

Add $\frac{9}{16}$ to both sides.	$x^2 - \frac{3}{2}x + \frac{9}{16} = 10 + \frac{9}{16}$
Factor the perfect square trinomial as a binomial square.	$\left(x - \frac{3}{4}\right)^2 = \frac{160}{16} + \frac{9}{16}$
Add the fractions on the right side.	$\left(x - \frac{3}{4}\right)^2 = \frac{169}{16}$
Use the Square Root Property.	$x - \frac{3}{4} = \pm\sqrt{\frac{169}{16}}$
Simplify the radical.	$x - \frac{3}{4} = \pm\frac{13}{4}$
Solve for *x*.	$x = \frac{3}{4} \pm \frac{13}{4}$
Rewrite to show 2 solutions.	$x = \frac{3}{4} + \frac{13}{4}, x = \frac{3}{4} - \frac{13}{4}$
Simplify.	$x = 4, x = -\frac{5}{2}$
Check. We leave the check for you.	

Note:
Exercise:

Problem: Solve $3r^2 - 2r = 21$ by completing the square.

Solution:

$r = -\frac{7}{3}, r = 3$

Note:
Exercise:

Problem: Solve $4t^2 + 2t = 20$ by completing the square.

Solution:

$t = -\frac{5}{2}, t = 2$

Example:
Exercise:

Problem: Solve $3x^2 + 2x = 4$ by completing the square.

Solution:
Solution

Again, our first step will be to make the coefficient of x^2 be one. By dividing both sides of the equation by the coefficient of x^2, we can then continue with solving the equation by completing the square.

	$3x^2+2x=4$
Divide both sides by 3 to make the coefficient of x^2 equal 1.	$\frac{3x^2+2x}{3}=\frac{4}{3}$
Simplify.	$x^2+\frac{2}{3}x=\frac{4}{3}$
Take half of $\frac{2}{3}$ and square it. $\left(\frac{1}{2}\cdot\frac{2}{3}\right)^2=\frac{1}{9}$	$x^2+\frac{2}{3}x+\underset{\left(\frac{1}{2}\cdot\frac{2}{3}\right)^2}{\underline{\qquad}}=\frac{4}{3}$
Add $\frac{1}{9}$ to both sides.	$x^2+\frac{2}{3}x+\frac{1}{9}=\frac{4}{3}+\frac{1}{9}$
Factor the perfect square trinomial as a binomial square.	$\left(x+\frac{1}{3}\right)^2=\frac{12}{9}+\frac{1}{9}$
Use the Square Root Property.	$x+\frac{1}{3}=\pm\sqrt{\frac{13}{9}}$
Simplify the radical.	$x+\frac{1}{3}=\pm\frac{\sqrt{13}}{3}$
Solve for x.	$x=-\frac{1}{3}\pm\frac{\sqrt{13}}{3}$
Rewrite to show 2 solutions.	$x=-\frac{1}{3}+\frac{\sqrt{13}}{3},\ x=-\frac{1}{3}-\frac{\sqrt{13}}{3}$

Check. We leave the check for you.	

Note:
Exercise:

Problem: Solve $4x^2+3x=12$ by completing the square.

Solution:

$x=-\frac{3}{8}\pm\frac{\sqrt{201}}{8}$

Note:
Exercise:

Problem: Solve $5y^2+3y=10$ by completing the square.

Solution:

$y=-\frac{3}{10}\pm\frac{\sqrt{209}}{10}$

Note:
Access these online resources for additional instruction and practice with solving quadratic equations by completing the square:

- Introduction to the method of completing the square
- How to Solve By Completing the Square

Key Concepts

- Binomial Squares Pattern If a, b are real numbers,
  $(a+b)^2 = a^2 + 2ab + b^2$

  $$\underbrace{(a+b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} + \underbrace{2ab}_{2\times(\text{product of terms})} + \underbrace{b^2}_{(\text{second term})^2}$$

  $(a-b)^2 = a^2 - 2ab + b^2$

  $$\underbrace{(a-b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} - \underbrace{2ab}_{2\times(\text{product of terms})} + \underbrace{b^2}_{(\text{second term})^2}$$

- Complete a Square
  To complete the square of $x^2 + bx$:

  Identify$b$, the coefficient of$x$.
  Find$\left(\frac{1}{2}b\right)^2$, the number to complete the square.
  Add the$\left(\frac{1}{2}b\right)^2tox^2 + bx$.

Practice Makes Perfect

Complete the Square of a Binomial Expression

In the following exercises, complete the square to make a perfect square trinomial. Then, write the result as a binomial squared.
Exercise:

Problem: $a^2 + 10a$

Solution:

$(a+5)^2$

Exercise:

Problem: b^2+12b

Exercise:

Problem: m^2+18m

Solution:

$(m+9)^2$

Exercise:

Problem: n^2+16n

Exercise:

Problem: m^2-24m

Solution:

$(m-12)^2$

Exercise:

Problem: n^2-16n

Exercise:

Problem: p^2-22p

Solution:

$(p-11)^2$

Exercise:

Problem: $q^2 - 6q$

Exercise:

Problem: $x^2 - 9x$

Solution:

$\left(x - \frac{9}{2}\right)^2$

Exercise:

Problem: $y^2 + 11y$

Exercise:

Problem: $p^2 - \frac{1}{3}p$

Solution:

$\left(p - \frac{1}{6}\right)^2$

Exercise:

Problem: $q^2 + \frac{3}{4}q$

Solve Quadratic Equations of the Form $\mathbf{x^2 + bx + c = 0}$ by Completing the Square

In the following exercises, solve by completing the square.

Exercise:

Problem: $v^2 + 6v = 40$

Solution:

$v = -10, v = 4$

Exercise:

Problem: $w^2 + 8w = 65$

Exercise:

Problem: $u^2 + 2u = 3$

Solution:

$u = -3, u = 1$

Exercise:

Problem: $z^2 + 12z = -11$

Exercise:

Problem: $c^2 - 12c = 13$

Solution:

$c = -1, c = 13$

Exercise:

Problem: $d^2 - 8d = 9$

Exercise:

Problem: $x^2 - 20x = 21$

Solution:

$x = -1, x = 21$

Exercise:

Problem: $y^2 - 2y = 8$

Exercise:

Problem: $m^2 + 4m = -44$

Solution:

no real solution

Exercise:

Problem: $n^2 - 2n = -3$

Exercise:

Problem: $r^2 + 6r = -11$

Solution:

no real solution

Exercise:

Problem: $t^2 - 14t = -50$

Exercise:

Problem: $a^2 - 10a = -5$

Solution:

$a = 5 \pm 2\sqrt{5}$

Exercise:

Problem: $b^2 + 6b = 41$

Exercise:

Problem: $u^2 - 14u + 12 = -1$

Solution:

$u = 1, u = 13$

Exercise:

Problem: $z^2 + 2z - 5 = 2$

Exercise:

Problem: $v^2 = 9v + 2$

Solution:

$v = \frac{9}{2} \pm \frac{\sqrt{89}}{2}$

Exercise:

Problem: $w^2 = 5w - 1$

Exercise:

Problem: $(x + 6)(x - 2) = 9$

Solution:

$x = -7, x = 3$

Exercise:

Problem: $(y + 9)(y + 7) = 79$

Solve Quadratic Equations of the Form $\mathbf{ax^2 + bx + c = 0}$ by Completing the Square

In the following exercises, solve by completing the square.

Exercise:

Problem: $3m^2 + 30m - 27 = 6$

Solution:

$m = -11, m = 1$

Exercise:

Problem: $2n^2 + 4n - 26 = 0$

Exercise:

Problem: $2c^2 + c = 6$

Solution:

$c = -2, c = \frac{3}{2}$

Exercise:

Problem: $3d^2 - 4d = 15$

Exercise:

Problem: $2p^2 + 7p = 14$

Solution:

$p = -\frac{7}{4} \pm \frac{\sqrt{161}}{4}$

Exercise:

Problem: $3q^2 - 5q = 9$

Everyday Math

Exercise:

Problem:

Rafi is designing a rectangular playground to have an area of 320 square feet. He wants one side of the playground to be four feet longer than the other side. Solve the equation $p^2 + 4p = 320$ for p, the length of one side of the playground. What is the length of the other side?

Solution:

16 feet, 20 feet

Exercise:

Problem:

Yvette wants to put a square swimming pool in the corner of her backyard. She will have a 3 foot deck on the south side of the pool and a 9 foot deck on the west side of the pool. She has a total area of 1080 square feet for the pool and two decks. Solve the equation $(s+3)(s+9) = 1080$ for s, the length of a side of the pool.

Writing Exercises

Exercise:

Problem:

Solve the equation $x^2 + 10x = -25$ ⓐ by using the Square Root Property and ⓑ by completing the square. ⓒ Which method do you prefer? Why?

Solution:

ⓐ -5 ⓑ -5 ⓒ Answers will vary.

Exercise:

Problem:

Solve the equation $y^2 + 8y = 48$ by completing the square and explain all your steps.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
complete the square of a binomial expression.			
solve quadratic equations of the form $x^2 + bx + c = 0$ by completing the square.			
solve quadratic equations of the form $ax^2 + bx + c = 0$ by completing the square.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

Glossary

completing the square
Completing the square is a method used to solve quadratic equations.

Solve Quadratic Equations Using the Quadratic Formula: ASE
By the end of this section, you will be able to:

- Solve quadratic equations using the quadratic formula
- Use the discriminant to predict the number of solutions of a quadratic equation
- Identify the most appropriate method to use to solve a quadratic equation

When we solved quadratic equations in the last section by completing the square, we took the same steps every time. By the end of the exercise set, you may have been wondering ‘isn’t there an easier way to do this?’ The answer is ‘yes.’ In this section, we will derive and use a formula to find the solution of a quadratic equation.

We have already seen how to solve a formula for a specific variable ‘in general’ so that we would do the algebraic steps only once and then use the new formula to find the value of the specific variable. Now, we will go through the steps of completing the square in general to solve a quadratic equation for *x*. It may be helpful to look at one of the examples at the end of the last section where we solved an equation of the form $ax^2+bx+c=0$ as you read through the algebraic steps below, so you see them with numbers as well as ‘in general.’

We start with the standard form of a quadratic equation and solve it for x by completing the square.	$ax^2+bx+c = 0 \qquad a\neq 0$
Isolate the variable terms on one side.	$ax^2+bx = -c$
Make leading coefficient 1, by dividing by a.	$\frac{ax^2}{a}+\frac{b}{a}x = -\frac{c}{a}$
Simplify.	$x^2+\frac{b}{a}x = -\frac{c}{a}$
To complete the square, find $\left(\frac{1}{2}\cdot\frac{b}{a}\right)^2$ and add it to both sides of the equation. $\left(\frac{1}{2}\ \frac{b}{a}\right)^2=\frac{b^2}{4a^2}$	$x^2+\frac{b}{a}x+\frac{b^2}{4a^2} = -\frac{c}{a}+\frac{b^2}{4a^2}$
The left side is a perfect square, factor it.	$\left(x+\frac{b}{2a}\right)^2 = -\frac{c}{a}+\frac{b^2}{4a^2}$
Find the common denominator of the right side and write equivalent fractions with the common denominator.	$\left(x+\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}-\frac{c\cdot 4a}{a\cdot 4a}$
Simplify.	$\left(x+\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}-\frac{4ac}{4a^2}$
Combine to one fraction.	$\left(x+\frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2}$
Use the square root property.	$x+\frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$
Simplify.	$x+\frac{b}{2a} = \pm\frac{\sqrt{b^2-4ac}}{2a}$
Add $-\frac{b}{2a}$ to both sides of the equation.	$x = -\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$
Combine the terms on the right side.	$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

This last equation is the Quadratic Formula.

Note:
Quadratic Formula
The solutions to a quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$ are given by the formula:
Equation:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

To use the Quadratic Formula, we substitute the values of $a, b,$ and c into the expression on the right side of the formula. Then, we do all the math to simplify the expression. The result gives the solution(s) to the quadratic equation.

Example:
How to Solve a Quadratic Equation Using the Quadratic Formula
Exercise:

Problem: Solve $2x^2 + 9x - 5 = 0$ by using the Quadratic Formula.

Solution:
Solution

Step 1. Write the quadratic equation in standard form. Identify the *a*, *b*, *c* values.	This equation is in standard form.	$ax^2 + bx + c = 0$ $2x^2 + 9x - 5 = 0$ $a = 2, b = 9, c = -5$
Step 2. Write the quadratic formula. Then substitute in the values of *a*, *b*, *c*.	Substitute in $a = 2$, $b = 9$, $c = -5$	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $x = \frac{-9 \pm \sqrt{9^2 - 4 \cdot 2 \cdot (-5)}}{2 \cdot 2}$
Step 3. Simplify the fraction, and solve for *x*.		$x = \frac{-9 \pm \sqrt{81 - (-40)}}{4}$ $x = \frac{-9 \pm \sqrt{121}}{4}$ $x = \frac{-9 \pm 11}{4}$ $x = \frac{-9 + 11}{4}$ $\quad x = \frac{-9 - 11}{4}$ $x = \frac{2}{4}$ $\quad x = \frac{-20}{4}$ $x = \frac{1}{2}$ $\quad x = -5$

Step 4. Check the solutions.	Put each answer in the original equation to check. Substitute $x = \frac{1}{2}$.	$2x^2 + 9x - 5 = 0$ $2\left(\frac{1}{2}\right)^2 + 9 \cdot \frac{1}{2} - 5 \stackrel{?}{=} 0$ $2 \cdot \frac{1}{4} + 9 \cdot \frac{1}{2} - 5 \stackrel{?}{=} 0$ $2 \cdot \frac{1}{4} + 9 \cdot \frac{1}{2} - 5 \stackrel{?}{=} 0$ $\frac{1}{2} + \frac{9}{2} - 5 \stackrel{?}{=} 0$ $\frac{10}{2} - 5 \stackrel{?}{=} 0$ $5 - 5 \stackrel{?}{=} 0$ $0 = 0 \checkmark$
	Substitute $x = -5$.	$2x^2 + 9x - 5 = 0$ $2(-5)^2 + 9(-5) - 5 \stackrel{?}{=} 0$ $2 \cdot 25 - 45 - 5 \stackrel{?}{=} 0$ $50 - 45 - 5 \stackrel{?}{=} 0$ $0 = 0 \checkmark$

Note:
Exercise:

Problem: Solve $3y^2 - 5y + 2 = 0$ by using the Quadratic Formula.

Solution:

$y = \frac{2}{3}, y = 1$

Note:
Exercise:

Problem: Solve $4z^2 + 2z - 6 = 0$ by using the Quadratic Formula.

Solution:

$z = -\frac{3}{2}, z = 1$

Note:
Solve a quadratic equation using the Quadratic Formula.

Write the Quadratic Formula in standard form. Identify the a,b, and c values.
Write the Quadratic Formula. Then substitute in the values of a,b, and c.
Simplify.
Check the solutions.

If you say the formula as you write it in each problem, you'll have it memorized in no time. And remember, the Quadratic Formula is an equation. Be sure you start with '$x =$'.

Example:
Exercise:

Problem: Solve $x^2 - 6x + 5 = 0$ by using the Quadratic Formula.

Solution:
Solution

	$x^2 - 6x + 5 = 0$
This equation is in standard form.	$\begin{matrix} ax^2 + bx + c = 0 \\ x^2 - 6x + 5 = 0 \end{matrix}$
Identify the *a*, *b*, *c* values.	$a = 1,\ b = -6,\ c = 5$
Write the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of *a*, *b*, *c*.	$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 1 \cdot (5)}}{2 \cdot 1}$
Simplify.	$x = \frac{6 \pm \sqrt{36 - 20}}{2}$ $x = \frac{6 \pm \sqrt{16}}{2}$

	$x = \frac{6 \pm 4}{2}$
Rewrite to show two solutions.	$x = \frac{6+4}{2}, x = \frac{6-4}{2}$
Simplify.	$x = \frac{10}{2}, x = \frac{2}{2}$ $x = 5,\ x = 1$
Check. $x^2 - 6x + 5 = 0$ $\quad$ $x^2 - 6x + 5 = 0$ $5^2 - 6 \cdot 5 + 5 \stackrel{?}{=} 0$ $\quad$ $1^2 - 6 \cdot 1 + 5 \stackrel{?}{=} 0$ $25 - 30 + 5 \stackrel{?}{=} 0$ $\quad$ $1 - 6 + 5 \stackrel{?}{=} 0$ $0 = 0 ✓$ $\quad$ $0 = 0 ✓$	

Note:
Exercise:

Problem: Solve $a^2 - 2a - 15 = 0$ by using the Quadratic Formula.

Solution:

$a = -3, a = 5$

Note:
Exercise:

Problem: Solve $b^2 + 10b + 24 = 0$ by using the Quadratic Formula.

Solution:

$b = -6, b = -4$

When we solved quadratic equations by using the Square Root Property, we sometimes got answers that had radicals. That can happen, too, when using the Quadratic Formula. If we get a radical as a solution, the final answer must have the radical in its simplified form.

Example:
Exercise:

Problem: Solve $4y^2 - 5y - 3 = 0$ by using the Quadratic Formula.

Solution:
Solution

We can use the Quadratic Formula to solve for the variable in a quadratic equation, whether or not it is named ‘*x*’.

	$4y^2 - 5y - 3 = 0$
This equation is in standard form.	$ax^2 + bx + c = 0$ $4y^2 - 5y - 3 = 0$
Identify the *a*, *b*, *c* values.	$a = 4,\ b = -5,\ c = -3$
Write the Quadratic Formula.	$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of *a*, *b*, *c*.	$y = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \cdot 4 \cdot (-3)}}{2 \cdot 4}$
Simplify.	$y = \frac{5 \pm \sqrt{25 + 48}}{8}$ $y = \frac{5 \pm \sqrt{73}}{8}$
Rewrite to show two solutions.	$y = \frac{5 + \sqrt{73}}{8},\ y = \frac{5 - \sqrt{73}}{8}$

Check. We leave the check to you.	

Note:
Exercise:

Problem: Solve $2p^2 + 8p + 5 = 0$ by using the Quadratic Formula.

Solution:

$p = \frac{-4 \pm \sqrt{6}}{2}$

Note:
Exercise:

Problem: Solve $5q^2 - 11q + 3 = 0$ by using the Quadratic Formula.

Solution:

$q = \frac{11 \pm \sqrt{61}}{10}$

Example:
Exercise:

Problem: Solve $2x^2 + 10x + 11 = 0$ by using the Quadratic Formula.

Solution:
Solution

	$2x^2 + 10x + 11 = 0$
This equation is in standard form.	$\begin{array}{l} ax^2 + bx + c = 0 \\ 2x^2 + 10x + 11 = 0 \end{array}$
Identify the *a*, *b*, *c* values.	$a = 2, b = 10, c = 11$

Write the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of *a, b, c.*	$x = \frac{-(10) \pm \sqrt{(10)^2 - 4 \cdot 2 \cdot (11)}}{2 \cdot 2}$
Simplify.	$x = \frac{-10 \pm \sqrt{100 - 88}}{4}$ $x = \frac{-10 \pm \sqrt{12}}{4}$
Simplify the radical.	$x = \frac{-10 \pm 4\sqrt{3}}{4}$
Factor out the common factor in the numerator.	$x = \frac{2(-5 \pm 2\sqrt{3})}{4}$
Remove the common factors.	$x = \frac{-5 \pm 2\sqrt{3}}{2}$
Rewrite to show two solutions.	$x = \frac{-5 + 2\sqrt{3}}{2}, x = \frac{-5 - 2\sqrt{3}}{2}$
Check. We leave the check to you.	

Note:
Exercise:

Problem: Solve $3m^2 + 12m + 7 = 0$ by using the Quadratic Formula.

Solution:

$m = \frac{-6 \pm \sqrt{15}}{3}$

Note:
Exercise:

Problem: Solve $5n^2 + 4n - 4 = 0$ by using the Quadratic Formula.

Solution:

$n = \frac{-2 \pm 2\sqrt{6}}{5}$

We cannot take the square root of a negative number. So, when we substitute a, b, and c into the Quadratic Formula, if the quantity inside the radical is negative, the quadratic equation has no real solution. We will see this in the next example.

Example:
Exercise:

Problem: Solve $3p^2 + 2p + 9 = 0$ by using the Quadratic Formula.

Solution:
Solution

This equation is in standard form.	$\begin{matrix} ax^2 + bx + c = 0 \\ 3p^2 + 2p + 9 = 0 \end{matrix}$
Identify the a, b, c values.	$a = 3, b = 2, c = 9$
Write the Quadratic Formula.	$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, c.	$p = \frac{-(2) \pm \sqrt{(2)^2 - 4 \cdot 3 \cdot (9)}}{2 \cdot 3}$
Simplify.	$p = \frac{-2 \pm \sqrt{4 - 108}}{6}$
Simplify the radical.	

	$p = \frac{-2 \pm \sqrt{-104}}{6}$
We cannot take the square root of a negative number.	There is no real solution.

Note:
Exercise:

Problem: Solve $4a^2 - 3a + 8 = 0$ by using the Quadratic Formula.

Solution:

no real solution

Note:
Exercise:

Problem: Solve $5b^2 + 2b + 4 = 0$ by using the Quadratic Formula.

Solution:

no real solution

The quadratic equations we have solved so far in this section were all written in standard form, $ax^2 + bx + c = 0$. Sometimes, we will need to do some algebra to get the equation into standard form before we can use the Quadratic Formula.

Example:
Exercise:

Problem: Solve $x\left(x + 6\right) + 4 = 0$ by using the Quadratic Formula.

Solution:
Solution

	$x(x + 6) + 4 = 0$

Distribute to get the equation in standard form.	$x^2 + 6x + 4 = 0$
This equation is now in standard form.	$ax^2 + bx + c = 0$ $x^2 + 6x + 4 = 0$
Identify the *a*, *b*, *c* values.	$a = 1, b = 6, c = 4$
Write the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of *a*, *b*, *c*.	$x = \frac{-(6) \pm \sqrt{(6)^2 - 4 \cdot 1 \cdot (4)}}{2 \cdot 1}$
Simplify.	$x = \frac{-6 \pm \sqrt{36 - 16}}{2}$
Simplify inside the radical.	$x = \frac{-6 \pm \sqrt{20}}{2}$
Simplify the radical.	$x = \frac{-6 \pm 2\sqrt{5}}{2}$
Factor out the common factor in the numerator.	$x = \frac{2(-3 \pm 2\sqrt{5})}{2}$
Remove the common factors.	$x = -3 \pm 2\sqrt{5}$
Rewrite to show two solutions.	$x = -3 + 2\sqrt{5}, \quad x = -3 - 2\sqrt{5}$
Check. We leave the check to you.	

Note:

Exercise:

Problem: Solve $x\left(x+2\right)-5=0$ by using the Quadratic Formula.

Solution:

$x=-1\pm\sqrt{6}$

Note:
Exercise:

Problem: Solve $y\left(3y-1\right)-2=0$ by using the Quadratic Formula.

Solution:

$y=-\frac{2}{3},y=1$

When we solved linear equations, if an equation had too many fractions we 'cleared the fractions' by multiplying both sides of the equation by the LCD. This gave us an equivalent equation—without fractions—to solve. We can use the same strategy with quadratic equations.

Example:
Exercise:

Problem: Solve $\frac{1}{2}u^2+\frac{2}{3}u=\frac{1}{3}$ by using the Quadratic Formula.

Solution:
Solution

	$\frac{1}{2}u^2+\frac{2}{3}u=\frac{1}{3}$
Multiply both sides by the LCD, 6, to clear the fractions.	$6\left(\frac{1}{2}u^2+\frac{2}{3}u\right)=6\left(\frac{1}{3}\right)$
Multiply.	$3u^2+4u=2$

Subtract 2 to get the equation in standard form.	$\begin{array}{l} ax^2 + bx + c = 0 \\ 3u^2 + 4u - 2 = 0 \end{array}$
Identify the *a*, *b*, *c* values.	$a = 3,\ b = 4,\ c = -2$
Write the Quadratic Formula.	$u = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of *a, b, c*.	$u = \dfrac{-(4) \pm \sqrt{(4)^2 - 4 \cdot 3 \cdot (-2)}}{2 \cdot 3}$
Simplify.	$u = \dfrac{-4 \pm \sqrt{16 + 24}}{6}$ $u = \dfrac{-4 \pm \sqrt{40}}{6}$
Simplify the radical.	$u = \dfrac{-4 \pm 2\sqrt{10}}{6}$
Factor out the common factor in the numerator.	$u = \dfrac{2(-2 \pm \sqrt{10})}{6}$
Remove the common factors.	$u = \dfrac{-2 \pm \sqrt{10}}{3}$
Rewrite to show two solutions.	$u = \dfrac{-2 \pm \sqrt{10}}{3},\ u = \dfrac{-2 - \sqrt{10}}{3}$
Check. We leave the check to you.	

Note:
Exercise:

Problem: Solve $\frac{1}{4}c^2 - \frac{1}{3}c = \frac{1}{12}$ by using the Quadratic Formula.

Solution:

$c = \frac{2 \pm \sqrt{7}}{3}$

Note:
Exercise:

Problem: Solve $\frac{1}{9}d^2 - \frac{1}{2}d = -\frac{1}{2}$ by using the Quadratic Formula.

Solution:

$d = \frac{2}{3}, d = 0$

Think about the equation $(x-3)^2 = 0$. We know from the Zero Products Principle that this equation has only one solution: $x = 3$.

We will see in the next example how using the Quadratic Formula to solve an equation with a perfect square also gives just one solution.

Example:
Exercise:

Problem: Solve $4x^2 - 20x = -25$ by using the Quadratic Formula.

Solution:
Solution

	$4x^2 - 20x = -25$
Add 25 to get the equation in standard form.	$\begin{array}{l} ax^2 + bx + c = 0 \\ 4x^2 - 20x + 25 = 0 \end{array}$
Identify the *a*, *b*, *c* values.	$a = 4, b = -20, c = 25$

Write the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of *a, b, c*.	$x = \frac{-(-20) \pm \sqrt{(-20)^2 - 4 \cdot 4 \cdot (25)}}{2 \cdot 4}$
Simplify.	$x = \frac{20 \pm \sqrt{400 - 400}}{8}$ $x = \frac{20 \pm \sqrt{0}}{8}$
Simplify the radical.	$x = \frac{20}{8}$
Simplify the fraction.	$x = \frac{5}{2}$
Check. We leave the check to you.	

Did you recognize that $4x^2 - 20x + 25$ is a perfect square?

Note:
Exercise:

Problem: Solve $r^2 + 10r + 25 = 0$ by using the Quadratic Formula.

Solution:

$r = -5$

Note:
Exercise:

Problem: Solve $25t^2 - 40t = -16$ by using the Quadratic Formula.

Solution:

$t = \frac{4}{5}$

Use the Discriminant to Predict the Number of Solutions of a Quadratic Equation

When we solved the quadratic equations in the previous examples, sometimes we got two solutions, sometimes one solution, sometimes no real solutions. Is there a way to predict the number of solutions to a quadratic equation without actually solving the equation?

Yes, the quantity inside the radical of the Quadratic Formula makes it easy for us to determine the number of solutions. This quantity is called the discriminant.

Note:
Discriminant
In the Quadratic Formula $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, the quantity b^2-4ac is called the **discriminant**.

Let's look at the discriminant of the equations in [link], [link], and [link], and the number of solutions to those quadratic equations.

	Quadratic Equation (in standard form)	**Discriminant** b^2-4ac	**Sign of the Discriminant**	**Number of real solutions**
[link]	$2x^2+9x-5=0$	$9^2-4\cdot 2\left(-5\right)=121$	+	2
[link]	$4x^2-20x+25=0$	$\left(-20\right)^2-4\cdot 4\cdot 25=0$	0	1
[link]	$3p^2+2p+9=0$	$2^2-4\cdot 3\cdot 9=-104$	−	0

When the discriminant is **positive** $\left(x=\frac{-b\pm\sqrt{+}}{2a}\right)$ the quadratic equation has **two solutions**.

When the discriminant is **zero** $\left(x=\frac{-b\pm\sqrt{0}}{2a}\right)$ the quadratic equation has **one solution**.

When the discriminant is **negative** $\left(x=\frac{-b\pm\sqrt{-}}{2a}\right)$ the quadratic equation has **no real solutions**.

Note:
Use the discriminant, b^2-4ac, to determine the number of solutions of a Quadratic Equation.
For a quadratic equation of the form $ax^2+bx+c=0$, $a\neq 0$,

- if $b^2-4ac>0$, the equation has two solutions.

- if $b^2 - 4ac = 0$, the equation has one solution.
- if $b^2 - 4ac < 0$, the equation has no real solutions.

Example:
Exercise:

Problem: Determine the number of solutions to each quadratic equation:

ⓐ $2v^2 - 3v + 6 = 0$ ⓑ $3x^2 + 7x - 9 = 0$ ⓒ $5n^2 + n + 4 = 0$ ⓓ $9y^2 - 6y + 1 = 0$

Solution:
Solution

To determine the number of solutions of each quadratic equation, we will look at its discriminant.

ⓐ

	$2v^2 - 3v + 6 = 0$
The equation is in standard form, identify a, b, c.	$a = 2, b = -3, c = 6$
Write the discriminant.	$b^2 - 4ac$
Substitute in the values of a, b, c.	$(3)^2 - 4 \cdot 2 \cdot 6$
Simplify.	$9 - 48$
	-39

Because the discriminant is negative, there are no real solutions to the equation.

ⓑ

	$3x^2 + 7x - 9 = 0$
The equation is in standard form, identify a, b, c.	$a = 3, b = 7, c = -9$
Write the discriminant.	$b^2 - 4ac$
Substitute in the values of a, b, c.	$(7)^2 - 4 \cdot 3 \cdot (-9)$
Simplify.	$49 + 108$
	157

Because the discriminant is positive, there are two solutions to the equation.

ⓒ

	$5n^2 + n + 4 = 0$
The equation is in standard form, identify $a, b,$ and c.	$a = 5, b = 1, c = 4$
Write the discriminant.	$b^2 - 4ac$
Substitute in the values of a, b, c.	$(1)^2 - 4 \cdot 5 \cdot 4$
Simplify.	$1 - 80$
	-79

Because the discriminant is negative, there are no real solutions to the equation.

ⓓ

$$9y^2 - 6y + 1 = 0$$

The equation is in standard form, identify a, b, c. $\quad a = 9, b = -6, c = 1$

Write the discriminant. $\quad b^2 - 4ac$

Substitute in the values of a, b, c. $\quad (-6)^2 - 4 \cdot 9 \cdot 1$

Simplify. $\quad 36 - 36$; $\quad 0$

Because the discriminant is 0, there is one solution to the equation.

Note:
Exercise:

Problem: Determine the number of solutions to each quadratic equation:

ⓐ $8m^2 - 3m + 6 = 0$ ⓑ $5z^2 + 6z - 2 = 0$ ⓒ $9w^2 + 24w + 16 = 0$ ⓓ $9u^2 - 2u + 4 = 0$

Solution:

ⓐ no real solutions ⓑ 2 ⓒ 1 ⓓ no real solutions

Note:
Exercise:

Problem: Determine the number of solutions to each quadratic equation:

ⓐ $b^2 + 7b - 13 = 0$ ⓑ $5a^2 - 6a + 10 = 0$ ⓒ $4r^2 - 20r + 25 = 0$ ⓓ $7t^2 - 11t + 3 = 0$

Solution:

ⓐ 2 ⓑ no real solutions ⓒ 1 ⓓ 2

Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

We have used four methods to solve quadratic equations:

- Factoring
- Square Root Property
- Completing the Square
- Quadratic Formula

You can solve any quadratic equation by using the Quadratic Formula, but that is not always the easiest method to use.

Note:
Identify the most appropriate method to solve a Quadratic Equation.

Try **Factoring** first. If the quadratic factors easily, this method is very quick.
Try the **Square Root Property** next. If the equation fits the form $ax^2 = k$ or $a(x-h)^2 = k$, it can easily be solved by using the Square Root Property.
Use the **Quadratic Formula**. Any quadratic equation can be solved by using the Quadratic Formula.

What about the method of completing the square? Most people find that method cumbersome and prefer not to use it. We needed to include it in this chapter because we completed the square in general to derive the Quadratic Formula. You will also use the process of completing the square in other areas of algebra.

Example:
Exercise:

Problem: Identify the most appropriate method to use to solve each quadratic equation:

ⓐ $5z^2 = 17$ ⓑ $4x^2 - 12x + 9 = 0$ ⓒ $8u^2 + 6u = 11$

Solution:
Solution

ⓐ $5z^2 = 17$

Since the equation is in the $ax^2 = k$, the most appropriate method is to use the Square Root Property.

ⓑ $4x^2 - 12x + 9 = 0$

We recognize that the left side of the equation is a perfect square trinomial, and so Factoring will be the most appropriate method.

ⓒ $8u^2 + 6u = 11$

Put the equation in standard form. $8u^2 + 6u - 11 = 0$

While our first thought may be to try Factoring, thinking about all the possibilities for trial and error leads us to choose the Quadratic Formula as the most appropriate method

Note:
Exercise:

Problem: Identify the most appropriate method to use to solve each quadratic equation:

ⓐ $x^2 + 6x + 8 = 0$ ⓑ $(n-3)^2 = 16$ ⓒ $5p^2 - 6p = 9$

Solution:

ⓐ factor ⓑ Square Root Property ⓒ Quadratic Formula

Note:
Exercise:

Problem: Identify the most appropriate method to use to solve each quadratic equation:

ⓐ $8a^2 + 3a - 9 = 0$ ⓑ $4b^2 + 4b + 1 = 0$ ⓒ $5c^2 = 125$

Solution:

ⓐ Quadratic Formula ⓑ factoring ⓒ Square Root Property

Note:
Access these online resources for additional instruction and practice with using the Quadratic Formula:

- Solving Quadratic Equations: Solving with the Quadratic Formula
- How to solve a quadratic equation in standard form using the Quadratic Formula (example)
- Solving Quadratic Equations using the Quadratic Formula—Example 3
- Solve Quadratic Equations using Quadratic Formula

Key Concepts

- **Quadratic Formula** The solutions to a quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$ are given by the formula:
  Equation:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- **Solve a Quadratic Equation Using the Quadratic Formula**
  To solve a quadratic equation using the Quadratic Formula.

  Write the quadratic formula in standard form. Identify thea, b, cvalues.
  Write the quadratic formula. Then substitute in the values of$a, b, c.$
  Simplify.
  Check the solutions.

- **Using the Discriminant, $b^2 - 4ac$, to Determine the Number of Solutions of a Quadratic Equation**
  For a quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$,

  - if $b^2 - 4ac > 0$, the equation has 2 solutions.

- if $b^2 - 4ac = 0$, the equation has 1 solution.
- if $b^2 - 4ac < 0$, the equation has no real solutions.

- **To identify the most appropriate method to solve a quadratic equation:**

  Try Factoring first. If the quadratic factors easily this method is very quick.
  Try the Square Root Property next. If the equation fits the form $ax^2 = k$ or $a(x-h)^2 = k$, it can easily be solved by using the Square Root Property.
  Use the Quadratic Formula. Any other quadratic equation is best solved by using the Quadratic Formula.

Practice Makes Perfect

Solve Quadratic Equations Using the Quadratic Formula

In the following exercises, solve by using the Quadratic Formula.
Exercise:

Problem: $4m^2 + m - 3 = 0$

Solution:

$m = -1, m = \frac{3}{4}$

Exercise:

Problem: $4n^2 - 9n + 5 = 0$

Exercise:

Problem: $2p^2 - 7p + 3 = 0$

Solution:

$p = \frac{1}{2}, p = 3$

Exercise:

Problem: $3q^2 + 8q - 3 = 0$

Exercise:

Problem: $p^2 + 7p + 12 = 0$

Solution:

$p = -4, p = -3$

Exercise:

Problem: $q^2 + 3q - 18 = 0$

Exercise:

Problem: $r^2 - 8r - 33 = 0$

Solution:

$r = -3, r = 11$

Exercise:

Problem: $t^2 + 13t + 40 = 0$

Exercise:

Problem: $3u^2 + 7u - 2 = 0$

Solution:

$u = \frac{-7 \pm \sqrt{73}}{6}$

Exercise:

Problem: $6z^2 - 9z + 1 = 0$

Exercise:

Problem: $2a^2 - 6a + 3 = 0$

Solution:

$a = \frac{3 \pm \sqrt{3}}{2}$

Exercise:

Problem: $5b^2 + 2b - 4 = 0$

Exercise:

Problem: $2x^2 + 3x + 9 = 0$

Solution:

no real solution

Exercise:

Problem: $6y^2 - 5y + 2 = 0$

Exercise:

Problem: $v\left(v + 5\right) - 10 = 0$

Solution:

$v = \frac{-5 \pm \sqrt{65}}{2}$

Exercise:

Problem: $3w\left(w-2\right)-8=0$

Exercise:

Problem: $\frac{1}{3}m^{2}+\frac{1}{12}m=\frac{1}{4}$

Solution:

$m=-1,m=\frac{3}{4}$

Exercise:

Problem: $\frac{1}{3}n^{2}+n=-\frac{1}{2}$

Exercise:

Problem: $16c^{2}+24c+9=0$

Solution:

$c=-\frac{3}{4}$

Exercise:

Problem: $25d^{2}-60d+36=0$

Exercise:

Problem: $5m^{2}+2m-7=0$

Solution:

$m=-\frac{7}{5},m=1$

Exercise:

Problem: $8n^{2}-3n+3=0$

Exercise:

Problem: $p^{2}-6p-27=0$

Solution:

$p=-3,p=9$

Exercise:

Problem: $25q^{2}+30q+9=0$

Exercise:

Problem: $4r^2+3r-5=0$

Solution:

$r=\frac{-3\pm\sqrt{89}}{8}$

Exercise:

Problem: $3t\left(t-2\right)=2$

Exercise:

Problem: $2a^2+12a+5=0$

Solution:

$a=\frac{-6\pm\sqrt{26}}{2}$

Exercise:

Problem: $4d^2-7d+2=0$

Exercise:

Problem: $\frac{3}{4}b^2+\frac{1}{2}b=\frac{3}{8}$

Solution:

$b=\frac{-2\pm\sqrt{11}}{6}$

Exercise:

Problem: $\frac{1}{9}c^2+\frac{2}{3}c=3$

Exercise:

Problem: $2x^2+12x-3=0$

Solution:

$x=\frac{-6\pm\sqrt{42}}{4}$

Exercise:

Problem: $16y^2+8y+1=0$

Use the Discriminant to Predict the Number of Solutions of a Quadratic Equation

In the following exercises, determine the number of solutions to each quadratic equation.

Exercise:

ⓐ $4x^2 - 5x + 16 = 0$
ⓑ $36y^2 + 36y + 9 = 0$
ⓒ $6m^2 + 3m - 5 = 0$
Problem: ⓓ $18n^2 - 7n + 3 = 0$

Solution:

ⓐ no real solutions ⓑ 1
ⓒ 2 ⓓ no real solutions

Exercise:

ⓐ $9v^2 - 15v + 25 = 0$
ⓑ $100w^2 + 60w + 9 = 0$
ⓒ $5c^2 + 7c - 10 = 0$
Problem: ⓓ $15d^2 - 4d + 8 = 0$

Exercise:

ⓐ $r^2 + 12r + 36 = 0$
ⓑ $8t^2 - 11t + 5 = 0$
ⓒ $4u^2 - 12u + 9 = 0$
Problem: ⓓ $3v^2 - 5v - 1 = 0$

Solution:

ⓐ 1 ⓑ no real solutions
ⓒ 1 ⓓ 2

Exercise:

ⓐ $25p^2 + 10p + 1 = 0$
ⓑ $7q^2 - 3q - 6 = 0$
ⓒ $7y^2 + 2y + 8 = 0$
Problem: ⓓ $25z^2 - 60z + 36 = 0$

Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

In the following exercises, identify the most appropriate method (Factoring, Square Root, or Quadratic Formula) to use to solve each quadratic equation. Do not solve.
Exercise:

ⓐ $x^2 - 5x - 24 = 0$
ⓑ $(y + 5)^2 = 12$
Problem: ⓒ $14m^2 + 3m = 11$

Solution:

ⓐ factor ⓑ square root
ⓒ Quadratic Formula

Exercise:

Problem: ⓐ $(8v+3)^2 = 81$
ⓑ $w^2 - 9w - 22 = 0$
ⓒ $4n^2 - 10 = 6$

Exercise:

Problem: ⓐ $6a^2 + 14 = 20$
ⓑ $\left(x - \frac{1}{4}\right)^2 = \frac{5}{16}$
ⓒ $y^2 - 2y = 8$

Solution:

ⓐ factor ⓑ square root
ⓒ factor

Exercise:

Problem: ⓐ $8b^2 + 15b = 4$
ⓑ $\frac{5}{9}v^2 - \frac{2}{3}v = 1$
ⓒ $\left(w + \frac{4}{3}\right)^2 = \frac{2}{9}$

Everyday Math

Exercise:

Problem:

A flare is fired straight up from a ship at sea. Solve the equation $16\left(t^2 - 13t + 40\right) = 0$ for t, the number of seconds it will take for the flare to be at an altitude of 640 feet.

Solution:

5 seconds, 8 seconds

Exercise:

Problem:

An architect is designing a hotel lobby. She wants to have a triangular window looking out to an atrium, with the width of the window 6 feet more than the height. Due to energy restrictions, the area of the window must be 140 square feet. Solve the equation $\frac{1}{2}h^2 + 3h = 140$ for h, the height of the window.

Writing Exercises

Exercise:

Problem: Solve the equation $x^2 + 10x = 200$
ⓐ by completing the square
ⓑ using the Quadratic Formula
ⓒ Which method do you prefer? Why?

Solution:

ⓐ $-20, 10$ ⓑ $-20, 10$
ⓒ answers will vary

Exercise:

Problem: Solve the equation $12y^2 + 23y = 24$
ⓐ by completing the square
ⓑ using the Quadratic Formula
ⓒ Which method do you prefer? Why?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve quadratic equations using the quadratic formula.			
use the discriminant to predict the number of solutions of a quadratic equation.			
identify the most appropriate method to use to solve a quadratic equation.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Glossary

discriminant
: In the Quadratic Formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ the quantity $b^2 - 4ac$ is called the discriminant.

Solve Applications Modeled by Quadratic Equations: ASE
By the end of this section, you will be able to:

- Solve applications modeled by Quadratic Equations

Solve Applications of the Quadratic Formula

We solved some applications that are modeled by quadratic equations earlier, when the only method we had to solve them was factoring. Now that we have more methods to solve quadratic equations, we will take another look at applications. To get us started, we will copy our usual Problem Solving Strategy here so we can follow the steps.

Note:
Use the problem solving strategy.

Readthe problem. Make sure all the words and ideas are understood.
Identifywhat we are looking for.
Namewhat we are looking for. Choose a variable to represent that quantity.
Translateinto an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.
Solvethe equation using good algebra techniques.
Checkthe answer in the problem and make sure it makes sense.
Answerthe question with a complete sentence.

We have solved number applications that involved **consecutive even integers** and **consecutive odd integers** by modeling the situation with linear equations. Remember, we noticed each even integer is 2 more than the number preceding it. If we call the first one n, then the next one is $n+2$. The next one would be $n+2+2$ or $n+4$. This is also true when we use odd integers. One set of even integers and one set of odd integers are shown below.
Equation:

	Consecutive even integers		**Consecutive odd integers**
	$64, 66, 68$		$77, 79, 81$
n	1^{st} even integer	n	1^{st} odd integer
$n+2$	2^{nd} consecutive even integer	$n+2$	2^{nd} consecutive odd integer
$n+4$	3^{rd} consecutive even integer	$n+4$	3^{rd} consecutive odd integer

Some applications of consecutive odd integers or consecutive even integers are modeled by quadratic equations. The notation above will be helpful as you name the variables.

Example:

Exercise:

Problem: The product of two consecutive odd integers is 195. Find the integers.

Solution:
Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for two consecutive odd integers.
Step 3. Name what we are looking for.	Let $n =$ the first odd integer. $n + 2 =$ the next odd integer
Step 4. Translate into an equation. State the problem in one sentence.	"The product of two consecutive odd integers is 195." The product of the first odd integer and the second odd integer is 195.
Translate into an equation	$n(n+2) = 195$
Step 5. Solve the equation. Distribute.	$n^2 + 2n = 195$
Subtract 195 to get the equation in standard form.	$\begin{array}{l} ax^2 + bx + c = 0 \\ n^2 + 2n - 195 = 0 \end{array}$
Identify the *a*, *b*, *c* values.	$a = 1, b = 2, c = -195$
Write the quadratic equation.	$n = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of *a*, *b*, *c*..	$n = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot (-195)}}{2 \cdot 1}$
Simplify.	

	$n = \frac{-2 \pm \sqrt{4 + 780}}{2}$ $n = \frac{-2 \pm \sqrt{784}}{2}$
Simplify the radical.	$n = \frac{-2 \pm 28}{2}$
Rewrite to show two solutions.	$n = \frac{-2 + 28}{2}, n = \frac{-2 - 28}{2}$
Solve each equation.	$n = \frac{26}{2}, n = \frac{-30}{2}$ $n = 13 \quad n = -15$
There are two values of *n* that are solutions. This will give us two pairs of consecutive odd integers for our solution.	First odd integer $n = 13$ next odd integer $n + 2$ $13 + 2$ 15
	First odd integer $n = -15$ next odd integer $n + 2$ $-15 + 2$ -13
Step 6. Check the answer. Do these pairs work? Are they consecutive odd integers? Is their product 195?	$13, 15, \text{ yes} \qquad -13, -15, \text{ yes}$ $13 \cdot 15 = 195, \text{ yes} \qquad -13(-15) = 195, \text{ yes}$
Step 7. Answer the question.	The two consecutive odd integers whose product is 195 are 13, 15, and −13, −15.

Note:
Exercise:

Problem: The product of two consecutive odd integers is 99. Find the integers.

Solution:

Two consecutive odd numbers whose product is 99 are 9 and 11, and -9 and -11.

Note:
Exercise:

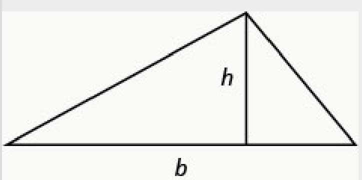
Problem: The product of two consecutive odd integers is 168. Find the integers.

Solution:

Two consecutive even numbers whose product is 168 are 12 and 14,and -12 and -14.

We will use the formula for the area of a triangle to solve the next example.

Note:
Area of a Triangle
For a triangle with base b and height h, the area, A, is given by the formula $A = \frac{1}{2}bh$.



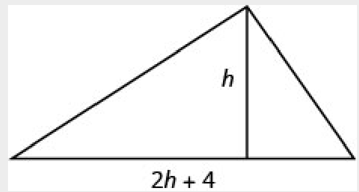
Recall that, when we solve geometry applications, it is helpful to draw the figure.

Example:
Exercise:

Problem:

An architect is designing the entryway of a restaurant. She wants to put a triangular window above the doorway. Due to energy restrictions, the window can have an area of 120 square feet and the architect wants the width to be 4 feet more than twice the height. Find the height and width of the window.

Solution:
Solution

Step 1. Read the problem. Draw a picture.	
Step 2. Identify what we are looking for.	We are looking for the height and width.
Step 3. Name what we are looking for.	Let $h =$ the height of the triangle. $2h + 4 =$ the width of the triangle
Step 4. Translate.	We know the area. Write the formula for the area of a triangle. $A = \frac{1}{2}bh$
Step 5. Solve the equation. Substitute in the values.	$120 = \frac{1}{2}(2h + 4)h$
Distribute.	$120 = h^2 + 2h$
This is a quadratic equation, rewrite it in standard form.	$\begin{array}{l} ax^2 + bx + c = 0 \\ h^2 + 2h - 120 = 0 \end{array}$
Solve the equation using the Quadratic Formula. Identify the *a*, *b*, *c* values.	$a = 1, b = 2, c = -120$
Write the quadratic equation.	$h = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of *a*, *b*, *c*..	

	$h = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot (-120)}}{2 \cdot 1}$
Simplify.	$h = \frac{-2 \pm \sqrt{4 + 480}}{2}$ $h = \frac{-2 \pm \sqrt{484}}{2}$
Simplify the radical.	$h = \frac{-2 \pm 22}{2}$
Rewrite to show two solutions.	$h = \frac{-2 + 22}{2}, \; h = \frac{-2 - 22}{2}$
Simplify.	$h = \frac{20}{2}, \; h = \frac{-24}{2}$
Since h is the height of a window, a value of $h = -12$ does not make sense.	$h = 10 \quad \cancel{h = -12}$
	The height of the triangle: $h = 10$ The width of the triangle: $2h + 4$ $2 \cdot 10 + 4$ 24
Step 6. Check the answer. Does a triangle with a height 10 and width 24 have area 120? Yes.	
Step 7. Answer the question.	The height of the triangular window is 10 feet and the width is 24 feet.

Notice that the solutions were integers. That tells us that we could have solved the equation by factoring.

When we wrote the equation in standard form, $h^2 + 2h - 120 = 0$, we could have factored it. If we did, we would have solved the equation $(h + 12)(h - 10) = 0$.

Note:
Exercise:

Problem:

Find the dimensions of a triangle whose width is four more than six times its height and has an area of 208 square inches.

Solution:

The height of the triangle is 8 inches and the width is 52 inches.

Note:
Exercise:

Problem:

If a triangle that has an area of 110 square feet has a height that is two feet less than twice the width, what are its dimensions?

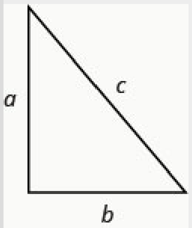
Solution:

The height of the triangle is 20 feet and the width is 11 feet.

In the two preceding examples, the number in the radical in the Quadratic Formula was a perfect square and so the solutions were rational numbers. If we get an irrational number as a solution to an application problem, we will use a calculator to get an approximate value.

The Pythagorean Theorem gives the relation between the legs and hypotenuse of a right triangle. We will use the Pythagorean Theorem to solve the next example.

Note:
Pythagorean Theorem
In any right triangle, where a and b are the lengths of the legs and c is the length of the hypotenuse, $a^2 + b^2 = c^2$

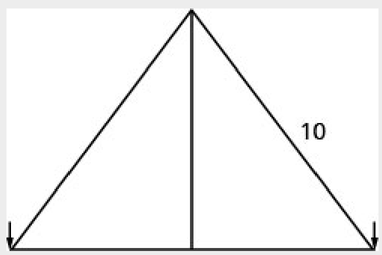
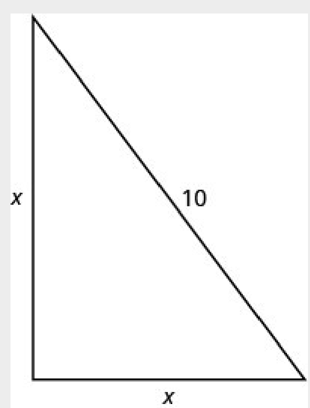


Example:
Exercise:

Problem:

Rene is setting up a holiday light display. He wants to make a ‘tree’ in the shape of two right triangles, as shown below, and has two 10-foot strings of lights to use for the sides. He will attach the lights to the top of a pole and to two stakes on the ground. He wants the height of the pole to be the same as the distance from the base of the pole to each stake. How tall should the pole be?

Solution:
Solution

Step 1. Read the problem. Draw a picture.	
Step 2. Identify what we are looking for.	We are looking for the height of the pole.
Step 3. Name what we are looking for.	The distance from the base of the pole to either stake is the same as the height of the pole. Let $x =$ the height of the pole. $x =$ the distance from the pole to stake
Each side is a right triangle. We draw a picture of one of them.	

Step 4. Translate into an equation. We can use the Pythagorean Theorem to solve for *x*.	
Write the Pythagorean Theorem.	$a^2 + b^2 = c^2$
Step 5. Solve the equation. Substitute.	$x^2 + x^2 = 10^2$
Simplify.	$2x^2 = 100$
Divide by 2 to isolate the variable.	$\frac{2x^2}{2} = \frac{100}{2}$
Simplify.	$x^2 = 50$
Use the Square Root Property.	$x = \pm\sqrt{50}$
Simplify the radical.	$x = \pm 5\sqrt{2}$
Rewrite to show two solutions.	$x = 5\sqrt{2}$ $\cancel{x = -5\sqrt{2}}$
Approximate this number to the nearest tenth with a calculator.	$x \approx 7.1$
Step 6. Check the answer. Check on your own in the Pythagorean Theorem.	
Step 7. Answer the question.	The pole should be about 7.1 feet tall.

Note:
Exercise:

Problem:

The sun casts a shadow from a flag pole. The height of the flag pole is three times the length of its shadow. The distance between the end of the shadow and the top of the flag pole is 20 feet. Find the length of the shadow and the length of the flag pole. Round to the nearest tenth of a foot.

Solution:

The length of the shadow is 6.3 feet and the length of the flag pole is 18.9 ft.

Note:
Exercise:

Problem:

The distance between opposite corners of a rectangular field is four more than the width of the field. The length of the field is twice its width. Find the distance between the opposite corners. Round to the nearest tenth.

Solution:

The distance to the opposite corner is 3.2.

Example:
Exercise:

Problem:

Mike wants to put 150 square feet of artificial turf in his front yard. This is the maximum area of artificial turf allowed by his homeowners association. He wants to have a rectangular area of turf with length one foot less than three times the width. Find the length and width. Round to the nearest tenth of a foot.

Solution:
Solution

Step 1. Read the problem. Draw a picture.	
Step 2. Identify what we are looking for.	We are looking for the length and width.
Step 3. Name what we are looking for.	Let $w =$ the width of the rectangle. $3w - 1 =$ the length of the rectangle
Step 4. Translate into an equation. We know the area. Write the formula for the area of a rectangle.	$A = L \cdot W$

Step 5. Solve the equation. Substitute in the values.	$150 = (3w - 1)w$
Distribute.	$150 = 3w^2 - w$
This is a quadratic equation, rewrite it in standard form.	$\underset{3w^2 - w - 150 = 0}{ax^2 + bx + c = 0}$
Solve the equation using the Quadratic Formula.	
Identify the *a*, *b*, *c* values.	$a = 3,\ b = -1,\ c = -150$
Write the Quadratic Formula.	$w = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of *a*, *b*, *c*.	$w = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 3 \cdot (-150)}}{2 \cdot 3}$
Simplify.	$w = \frac{1 \pm \sqrt{1 + 1800}}{6}$ $w = \frac{1 \pm \sqrt{1801}}{6}$
Rewrite to show two solutions.	$w = \frac{1 + \sqrt{1801}}{6},\ w = \frac{1 - \sqrt{1801}}{6}$
Approximate the answers using a calculator. We eliminate the negative solution for the width.	

	$w \approx 7.2, \quad \cancel{w \approx -6.9}$ Width $w \approx 7.2$ Length $\approx 3w - 1$ $\approx 3(7.2) - 1$ ≈ 20.6
Step 6. Check the answer. Make sure that the answers make sense.	
Step 7. Answer the question.	The width of the rectangle is approximately 7.2 feet and the length 20.6 feet.

Note:
Exercise:

Problem:

The length of a 200 square foot rectangular vegetable garden is four feet less than twice the width. Find the length and width of the garden. Round to the nearest tenth of a foot.

Solution:

The width of the garden is 11 feet and the length is 18 feet.

Note:
Exercise:

Problem:

A rectangular tablecloth has an area of 80 square feet. The width is 5 feet shorter than the length. What are the length and width of the tablecloth? Round to the nearest tenth of a foot.

Solution:

The width of the tablecloth is 6.8 feet and the length is 11.8 feet.

The height of a projectile shot upwards is modeled by a quadratic equation. The initial velocity, v_0, propels the object up until gravity causes the object to fall back down.

Note:
Projectile Motion
The height in feet, h, of an object shot upwards into the air with initial velocity, v_0, after t seconds is given by the formula:
Equation:

$$h = -16t^2 + v_0 t$$

We can use the formula for projectile motion to find how many seconds it will take for a firework to reach a specific height.

Example:
Exercise:

Problem:

A firework is shot upwards with initial velocity 130 feet per second. How many seconds will it take to reach a height of 260 feet? Round to the nearest tenth of a second.

Solution:
Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for the number of seconds, which is time.
Step 3. Name what we are looking for.	Let $t =$ the number of seconds.
Step 4. Translate into an equation.	Use the formula.
	$h = -16t^2 + v_0 t$
Step 5. Solve the equation. We know the velocity v_0 is 130 feet per second.	
The height is 260 feet. Substitute the values.	$260 = -16t^2 + 130t$
This is a quadratic equation, rewrite it in	

standard form.	$\overset{ax^2}{16t^2} \overset{+}{+} \overset{bx}{130t} \overset{+}{+} \overset{c}{260} \overset{=}{=} \overset{0}{0}$
Solve the equation using the Quadratic Formula.	
Identify the *a*, *b*, *c* values.	$a = 16,\ b = -130,\ c = 260$
Write the Quadratic Formula.	$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of *a*, *b*, *c*.	$t = \frac{-(-130) \pm \sqrt{(-130)^2 - 4 \cdot 16 \cdot (260)}}{2 \cdot 16}$
Simplify.	$t = \frac{130 \pm \sqrt{16{,}900 - 16{,}640}}{32}$ $t = \frac{130 \pm \sqrt{260}}{32}$
Rewrite to show two solutions.	$t = \frac{130 + \sqrt{260}}{32},\ t = \frac{130 - \sqrt{260}}{32}$
Approximate the answers with a calculator.	$t \approx 4.6$ seconds, $t \approx 3.6$
Step 6. Check the answer. The check is left to you.	
Step 7. Answer the question.	The firework will go up and then fall back down. As the firework goes up, it will reach 260 feet after approximately 3.6 seconds. It will also pass that height on the way down at 4.6 seconds.

Note:
Exercise:

Problem:

An arrow is shot from the ground into the air at an initial speed of 108 ft/sec. Use the formula $h = -16t^2 + v_0 t$ to determine when the arrow will be 180 feet from the ground. Round the nearest tenth of a second.

Solution:

The arrow will reach 180 on its way up in 3 seconds, and on the way down in 3.8 seconds.

Note:
Exercise:

Problem:

A man throws a ball into the air with a velocity of 96 ft/sec. Use the formula $h = -16t^2 + v_0 t$ to determine when the height of the ball will be 48 feet. Round to the nearest tenth of a second.

Solution:

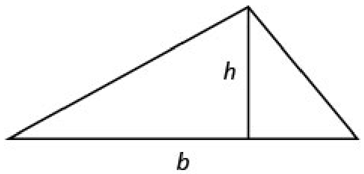
The ball will reach 48 feet on its way up in .6 seconds and on the way down in 5.5 seconds.

Note:
Access these online resources for additional instruction and practice with solving word problems using the quadratic equation:

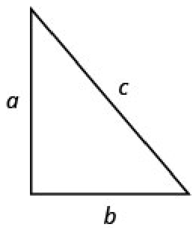
- General Quadratic Word Problems
- Word problem: Solve a projectile problem using a quadratic equation

Key Concepts

- **Area of a Triangle** For a triangle with base, b, and height, h, the area, A, is given by the formula: $A = \frac{1}{2}bh$



- **Pythagorean Theorem** In any right triangle, where a and b are the lengths of the legs, and c is the length of the hypothenuse, $a^2 + b^2 = c^2$



- **Projectile motion** The height in feet, h, of an object shot upwards into the air with initial velocity, v_0, after t seconds can be modeled by the formula:
  Equation:

$$h = -16t^2 + v_0 t$$

Practice Makes Perfect

Solve Applications of the Quadratic Formula

In the following exercises, solve by using methods of factoring, the square root principle, or the Quadratic Formula. Round your answers to the nearest tenth.

Exercise:

Problem: The product of two consecutive odd numbers is 255. Find the numbers.

Solution:

Two consecutive odd numbers whose product is 255 are 15 and 17, and -15 and -17.

Exercise:

Problem: The product of two consecutive even numbers is 360. Find the numbers.

Exercise:

Problem: The product of two consecutive even numbers is 624. Find the numbers.

Solution:

Two consecutive even numbers whose product is 624 are 24 and 26, and -26 and -24.

Exercise:

Problem: The product of two consecutive odd numbers is 1023. Find the numbers.

Exercise:

Problem: The product of two consecutive odd numbers is 483. Find the numbers.

Solution:

Two consecutive odd numbers whose product is 483 are 21 and 23, and -21 and -23.

Exercise:

Problem: The product of two consecutive even numbers is 528. Find the numbers.

Exercise:

Problem:

A triangle with area 45 square inches has a height that is two less than four times the width. Find the height and width of the triangle.

Solution:

The width of the triangle is 5 inches and the height is 18 inches.

Exercise:

Problem:

The width of a triangle is six more than twice the height. The area of the triangle is 88 square yards. Find the height and width of the triangle.

Exercise:

Problem:

The hypotenuse of a right triangle is twice the length of one of its legs. The length of the other leg is three feet. Find the lengths of the three sides of the triangle.

Solution:

The leg of the right triangle is 1.7 feet and the hypotenuse is 3.4 feet.

Exercise:

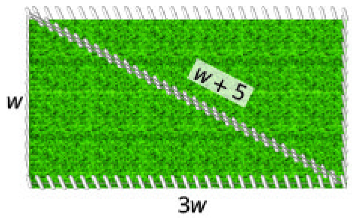
Problem:

The hypotenuse of a right triangle is 10 cm long. One of the triangle's legs is three times the length of the other leg. Round to the nearest tenth. Find the lengths of the three sides of the triangle.

Exercise:

Problem:

A farmer plans to fence off sections of a rectangular corral. The diagonal distance from one corner of the corral to the opposite corner is five yards longer than the width of the corral. The length of the corral is three times the width. Find the length of the diagonal of the corral.



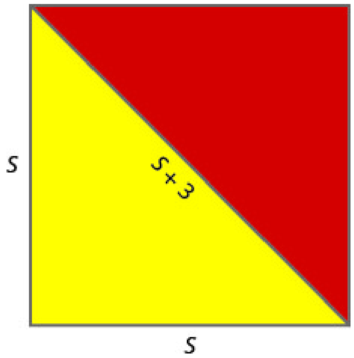
Solution:

The length of the fence is 7.1 units.

Exercise:

Problem:

Nautical flags are used to represent letters of the alphabet. The flag for the letter O consists of a yellow right triangle and a red right triangle which are sewn together along their hypotenuse to form a square. The adjoining side of the two triangles is three inches longer than a side of the flag. Find the length of the side of the flag.



Exercise:

Problem:

The length of a rectangular driveway is five feet more than three times the width. The area is 350 square feet. Find the length and width of the driveway.

Solution:

The width of the driveway is 10 feet and its length is 35 feet.

Exercise:

Problem:

A rectangular lawn has area 140 square yards. Its width that is six less than twice the length. What are the length and width of the lawn?

Exercise:

Problem:

A firework rocket is shot upward at a rate of 640 ft/sec. Use the projectile formula $h = -16t^2 + v_0 t$ to determine when the height of the firework rocket will be 1200 feet.

Solution:

The rocket will reach 1,200 feet on its way up in 2 seconds and on the way down in 38 seconds.

Exercise:

Problem:

An arrow is shot vertically upward at a rate of 220 feet per second. Use the projectile formula $h = -16t^2 + v_0 t$ to determine when height of the arrow will be 400 feet.

Everyday Math

Exercise:

Problem:

A bullet is fired straight up from a BB gun with initial velocity 1120 feet per second at an initial height of 8 feet. Use the formula $h = -16t^2 + v_0 t + 8$ to determine how many seconds it will take for the bullet to hit the ground. (That is, when will $h = 0$?)

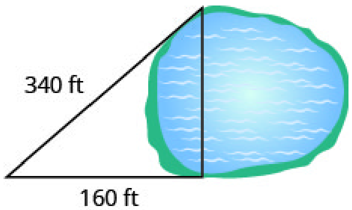
Solution:

70 seconds

Exercise:

Problem:

A city planner wants to build a bridge across a lake in a park. To find the length of the bridge, he makes a right triangle with one leg and the hypotenuse on land and the bridge as the other leg. The length of the hypotenuse is 340 feet and the leg is 160 feet. Find the length of the bridge.



Writing Exercises

Exercise:

Problem:

Make up a problem involving the product of two consecutive odd integers. Start by choosing two consecutive odd integers. ⓐ What are your integers? ⓑ What is the product of your integers? ⓒ Solve the equation $n\,(n+2)=p$, where p is the product you found in part (b). ⓓ Did you get the numbers you started with?

Solution:

ⓐ answers will vary
ⓑ answers will vary ⓒ answers will vary ⓓ answers will vary

Exercise:

Problem:

Make up a problem involving the product of two consecutive even integers. Start by choosing two consecutive even integers. ⓐ What are your integers? ⓑ What is the product of your integers? ⓒ Solve the equation $n\,(n+2)=p$, where p is the product you found in part (b). ⓓ Did you get the numbers you started with?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve applications of the quadratic formula.			

ⓑ On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Glossary

consecutive even integers

Consecutive even integers are even integers that follow right after one another. If an even integer is represented by n, the next consecutive even integer is $n + 2$, and the next after that is $n + 4$.

consecutive odd integers

Consecutive odd integers are odd integers that follow right after one another. If an odd integer is represented by n, the next consecutive odd integer is $n + 2$, and the next after that is $n + 4$.

Graphing Quadratic Equations: ASE
By the end of this section, you will be able to:

- Recognize the graph of a quadratic equation in two variables
- Find the axis of symmetry and vertex of a parabola
- Find the intercepts of a parabola
- Graph quadratic equations in two variables
- Solve maximum and minimum applications

Recognize the Graph of a Quadratic Equation in Two Variables

We have graphed equations of the form $Ax+By=C$. We called equations like this linear equations because their graphs are straight lines.

Now, we will graph equations of the form $y=ax^2+bx+c$. We call this kind of equation a quadratic equation in two variables.

Note:
Quadratic Equation in Two Variables
A **quadratic equation in two variables**, where $a, b,$ and c are real numbers and $a \neq 0$, is an equation of the form
Equation:

$$y=ax^2+bx+c$$

Just like we started graphing linear equations by plotting points, we will do the same for quadratic equations.

Let's look first at graphing the quadratic equation $y=x^2$. We will choose integer values of x between -2 and 2 and find their y values. See [link].

$y=x^2$	
x	y
0	0
1	1
-1	1
2	4

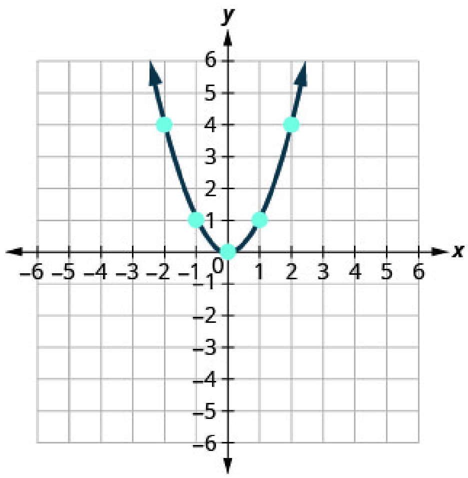
-2	4

Notice when we let $x = 1$ and $x = -1$, we got the same value for y.
Equation:

$$\begin{array}{ll} y = x^2 & y = x^2 \\ y = 1^2 & y = (-1)^2 \\ y = 1 & y = 1 \end{array}$$

The same thing happened when we let $x = 2$ and $x = -2$.

Now, we will plot the points to show the graph of $y = x^2$. See [link].



The graph is not a line. This figure is called a **parabola**. Every quadratic equation has a graph that looks like this.

In [link] you will practice graphing a parabola by plotting a few points.

Example:
Exercise:

Problem: Graph $y = x^2 - 1$.

Solution:
Solution

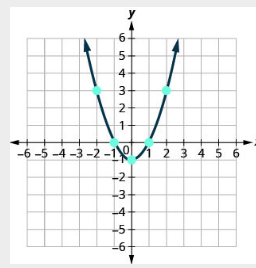
We will graph the equation by plotting points.

Choose integers values for *x*, substitute them into the equation and solve for *y*.

Record the values of the ordered pairs in the chart.

$y = x^2 - 1$	
x	y
0	–1
1	0
–1	0
2	3
–2	3

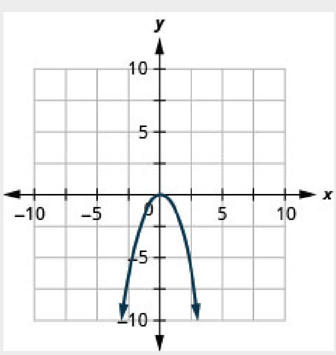
Plot the points, and then connect them with a smooth curve. The result will be the graph of the equation$y = x^2 - 1$.



Note:
Exercise:

Problem: Graph $y = -x^2$.

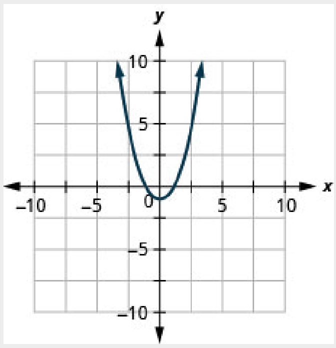
Solution:



Note:
Exercise:

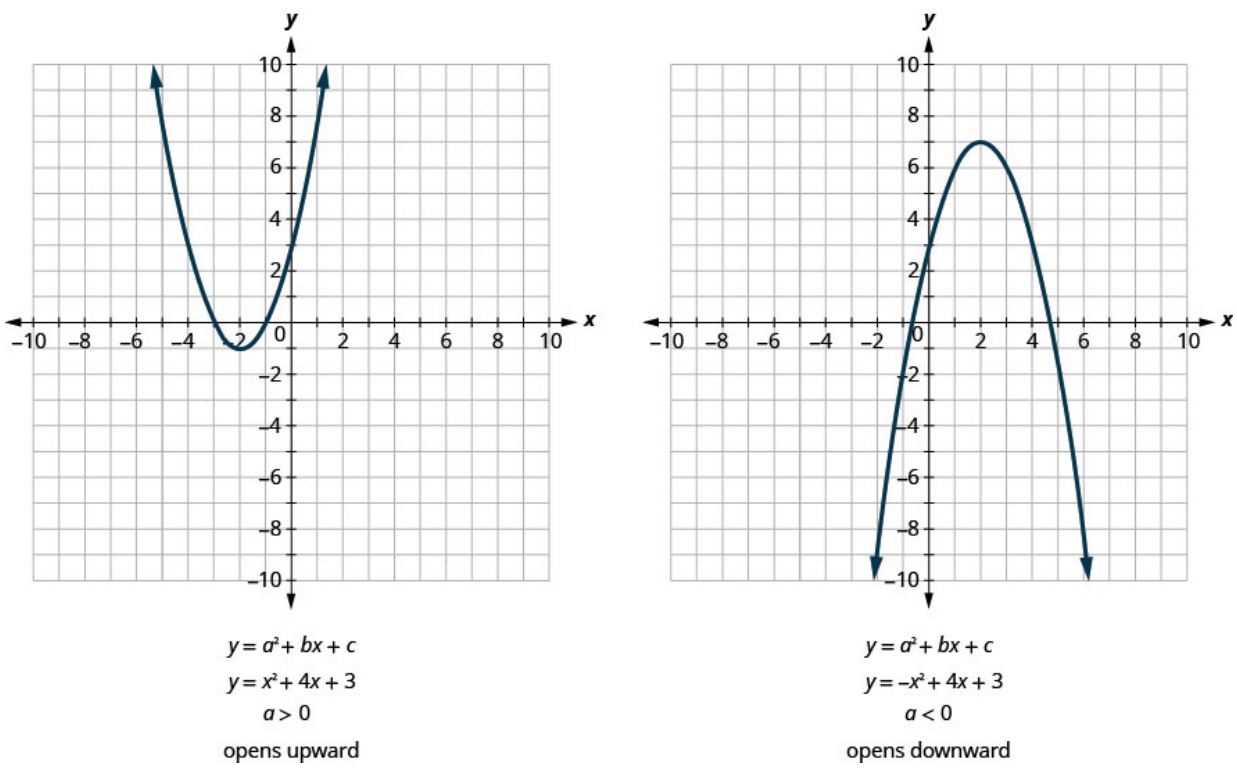
Problem: Graph $y = x^2 + 1$.

Solution:



How do the equations $y = x^2$ and $y = x^2 - 1$ differ? What is the difference between their graphs? How are their graphs the same?

All parabolas of the form $y = ax^2 + bx + c$ open upwards or downwards. See [link].



Notice that the only difference in the two equations is the negative sign before the x^2 in the equation of the second graph in [link]. When the x^2 term is positive, the parabola opens upward, and when the x^2

term is negative, the parabola opens downward.

Note:
Parabola Orientation
For the quadratic equation $y = ax^2 + bx + c$, if:

- $a > 0$, the parabola opens upward 
- $a < 0$, the parabola opens downward 

Example:
Exercise:

Problem: Determine whether each parabola opens upward or downward:

ⓐ $y = -3x^2 + 2x - 4$ ⓑ $y = 6x^2 + 7x - 9$

Solution:
Solution

ⓐ Find the value of "*a*".	$y = ax^2 + bx + c$ $y = -3x^2 + 2x - 4$ $a = -3$ Since the "a" is negative, the parabola will open downward.
ⓑ Find the value of "*a*".	$y = ax^2 + bx + c$ $y = 6x^2 + 7x - 9$ $a = 6$ Since the "a" is positive, the parabola will open upward.

Note:
Exercise:

Problem: Determine whether each parabola opens upward or downward:

ⓐ $y = 2x^2 + 5x - 2$ ⓑ $y = -3x^2 - 4x + 7$

Solution:

ⓐ up ⓑ down

Note:
Exercise:

Problem: Determine whether each parabola opens upward or downward:

ⓐ $y = -2x^2 - 2x - 3$ ⓑ $y = 5x^2 - 2x - 1$

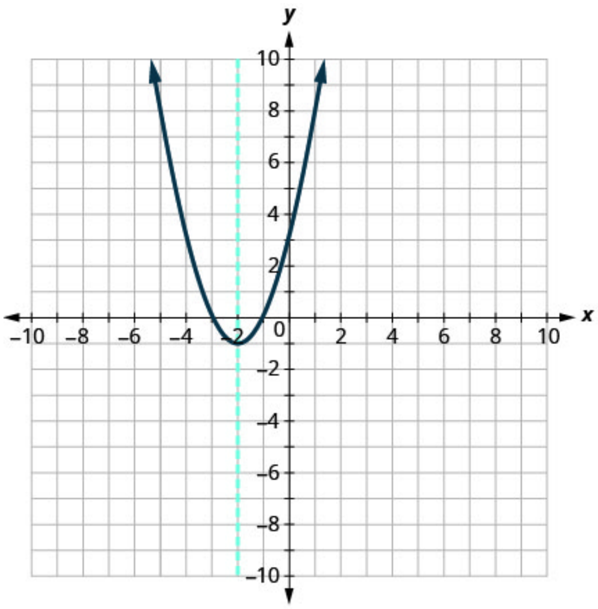
Solution:

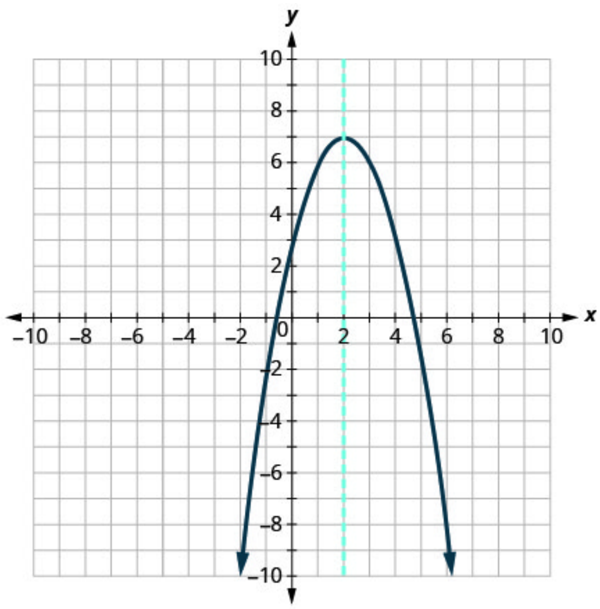
ⓐ down ⓑ up

Find the Axis of Symmetry and Vertex of a Parabola

Look again at [link]. Do you see that we could fold each parabola in half and that one side would lie on top of the other? The 'fold line' is a line of symmetry. We call it the **axis of symmetry** of the parabola.

We show the same two graphs again with the axis of symmetry in red. See [link].





The equation of the axis of symmetry can be derived by using the Quadratic Formula. We will omit the derivation here and proceed directly to using the result. The equation of the axis of symmetry of the graph of $y = ax^2 + bx + c$ is $x = -\frac{b}{2a}$.

So, to find the equation of symmetry of each of the parabolas we graphed above, we will substitute into the formula $x = -\frac{b}{2a}$.

$ax^2 + bx + c$	$ax^2 + bx + c$
$y = x^2 + 4x + 3$	$y = -x^2 + 4x + 3$
axis of symmetry	axis of symmetry
$x = -\frac{b}{2a}$	$x = -\frac{b}{2a}$
$x = -\frac{4}{2 \cdot 1}$	$x = -\frac{4}{2(-1)}$
$x = -2$	$x = 2$

Look back at [link]. Are these the equations of the dashed red lines?

The point on the parabola that is on the axis of symmetry is the lowest or highest point on the parabola, depending on whether the parabola opens upwards or downwards. This point is called the **vertex** of the parabola.

We can easily find the coordinates of the vertex, because we know it is on the axis of symmetry. This means its *x*-coordinate is $-\frac{b}{2a}$. To find the *y*-coordinate of the vertex, we substitute the value of the *x*-coordinate into the quadratic equation.

$y = x^2 + 4x + 3$	$y = -x^2 + 4x + 3$
axis of symmetry is $x = -2$	axis of symmetry is $x = 2$
vertex is $(-2, \underline{\quad})$	vertex is $(2, \underline{\quad})$
$y = x^2 + 4x + 3$	$y = -x^2 + 4x + 3$
$y = (-2)^2 + 4(-2) + 3$	$y = -(2)^2 + 4(2) + 3$
$y = -1$	$y = 7$
vertex is $(-2, -1)$	vertex is $(2, 7)$

Note:
Axis of Symmetry and Vertex of a Parabola
For a parabola with equation $y = ax^2 + bx + c$:

- The axis of symmetry of a parabola is the line $x = -\frac{b}{2a}$.
- The vertex is on the axis of symmetry, so its *x*-coordinate is $-\frac{b}{2a}$.

To find the *y*-coordinate of the vertex, we substitute $x = -\frac{b}{2a}$ into the quadratic equation.

Example:

Exercise:

Problem: For the parabola $y = 3x^2 - 6x + 2$ find: ⓐ the axis of symmetry and ⓑ the vertex.

Solution:
Solution

ⓐ	$y = ax^2 + bx + c$ $y = 3x^2 - 6x + 2$
The axis of symmetry is the line $x = -\frac{b}{2a}$.	$x = -\frac{b}{2a}$
Substitute the values of *a*, *b* into the equation.	$x = -\frac{-6}{2 \cdot 3}$
Simplify.	$x = 1$
	The axis of symmetry is the line $x = 1$.
ⓑ	$y = 3x^2 - 6x + 2$
The vertex is on the line of symmetry, so its *x*-coordinate will be $x = 1$.	
Substitute$x = 1$ into the equation and solve for *y*.	$y = 3(1)^2 - 6(1) + 2$
Simplify.	$y = 3 \cdot 1 - 6 + 2$
This is the *y*-coordinate.	$y = -1$ The vertex is $(1, -1)$.

Note:

Exercise:

Problem: For the parabola $y=2x^2-8x+1$ find: ⓐ the axis of symmetry and ⓑ the vertex.

Solution:

ⓐ $x=2$ ⓑ $(2,-7)$

Note:
Exercise:

Problem: For the parabola $y=2x^2-4x-3$ find: ⓐ the axis of symmetry and ⓑ the vertex.

Solution:

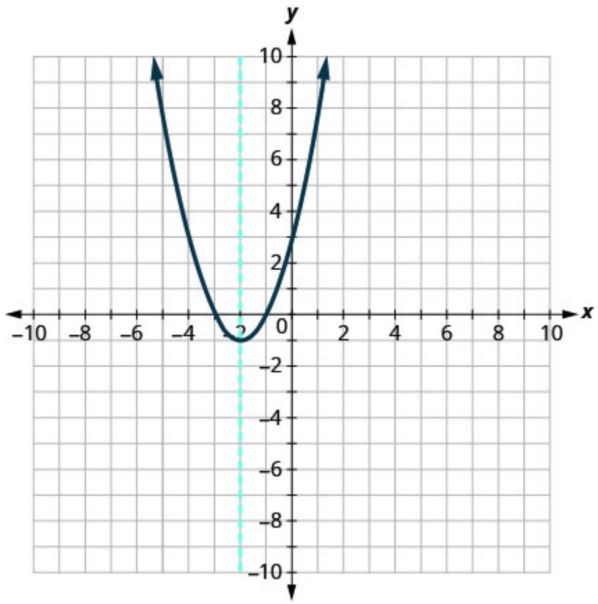
ⓐ $x=1$ ⓑ $(1,-5)$

Find the Intercepts of a Parabola

When we graphed linear equations, we often used the *x*- and *y*-intercepts to help us graph the lines. Finding the coordinates of the intercepts will help us to graph parabolas, too.

Remember, at the ***y*-intercept** the value of x is zero. So, to find the *y*-intercept, we substitute $x=0$ into the equation.

Let's find the *y*-intercepts of the two parabolas shown in the figure below.

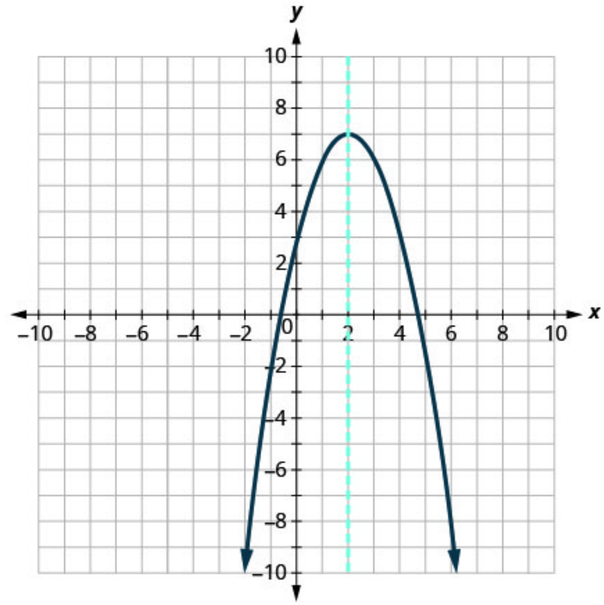


$y = x^2 + 4x + 3$

$x = 0 \quad y = 0^2 + 4 \cdot 0 + 3$

$y = 3$

y-intercept (0, 3)



$y = -x^2 + 4x + 3$

$x = 0 \quad y = -0^2 + 4 \cdot 0 + 3$

$y = 3$

y-intercept (0, 3)

At an ***x*-intercept**, the value of y is zero. To find an x-intercept, we substitute $y = 0$ into the equation. In other words, we will need to solve the equation $0 = ax^2 + bx + c$ for x.
Equation:

$$y = ax^2 + bx + c$$
$$0 = ax^2 + bx + c$$

But solving quadratic equations like this is exactly what we have done earlier in this chapter.

We can now find the *x*-intercepts of the two parabolas shown in [link].

First, we will find the *x*-intercepts of a parabola with equation $y = x^2 + 4x + 3$.

	$y = x^2 + 4x + 3$
Let $y = 0$.	$0 = x^2 + 4x + 3$
Factor.	

	$0 = (x+1)(x+3)$
Use the zero product property.	$x+1=0, x+3=0$
Solve.	$x=-1, x=-3$
	The *x* intercepts are $(-1,0)$ and $(-3,0)$.

Now, we will find the *x*-intercepts of the parabola with equation $y=-x^2+4x+3$.

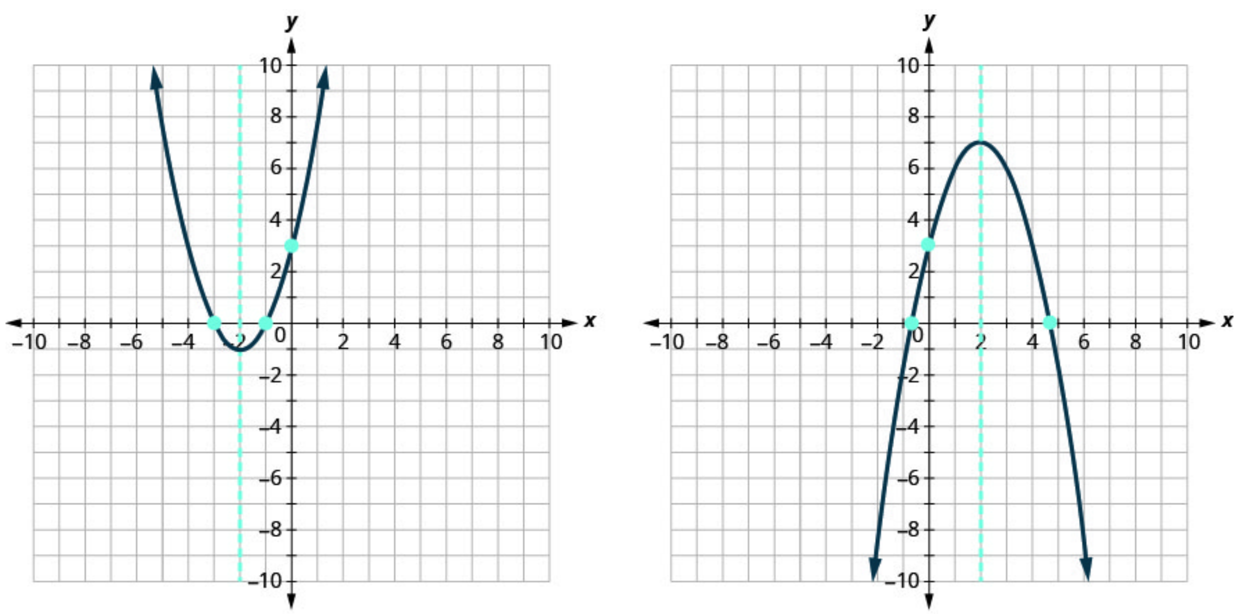
	$y=-x^2+4x+3$
Let $y=0$.	$0=-x^2+4x+3$
This quadratic does not factor, so we use the Quadratic Formula.	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
$a=-1, b=4, c=3$	$x=\frac{-4\pm\sqrt{4^2-4(-1)(3)}}{2(-1)}$
Simplify.	$x=\frac{-4\pm\sqrt{28}}{-2}$ $x=\frac{-4\pm 2\sqrt{7}}{-2}$ $x=\frac{-2(2\pm\sqrt{7})}{-2}$ $x=2\pm\sqrt{7}$
	The *x* intercepts are $\left(2+\sqrt{7},0\right)$ and

	$\left(2-\sqrt{7},0\right)$.

We will use the decimal approximations of the x-intercepts, so that we can locate these points on the graph.
Equation:

$$\left(2+\sqrt{7},0\right)\approx(4.6,0) \qquad \left(2-\sqrt{7},0\right)\approx(-0.6,0)$$

Do these results agree with our graphs? See [link].



$y=x^2+4x+3$
y-intercept (0, 3)
x-intercepts (–1, 0) and (–3, 0)

$y=-x^2+4x+3$
y-intercept (0, 3)
x-intercepts $(2+\sqrt{7},0)\approx(4.6,0)$
$(2-\sqrt{7},0)\approx(-0.6,0)$

Note:
Find the intercepts of a parabola.
To find the intercepts of a parabola with equation $y=ax^2+bx+c$:
Equation:

y-intercept	x-intercepts
Let $x=0$ and solve for y.	Let $y=0$ and solve for x.

Example:
Exercise:

Problem: Find the intercepts of the parabola $y = x^2 - 2x - 8$.

Solution:
Solution

	$y = x^2 - 2x - 8$
To find the *y*-intercept, let $x = 0$ and solve for *y*.	$y = 0^2 - 2 \cdot 0 - 8$ $y = -8$
	When $x = 0$, then $y = -8$. The *y*-intercept is the point $(0, -8)$.
	$y = x^2 - 2x - 8$
To find the *x*-intercept, let $y = 0$ and solve for *x*.	$0 = x^2 - 2x - 8$
Solve by factoring.	$0 = (x - 4)(x + 2)$
	$0 = x - 4 \quad 0 = x + 2$ $4 = x \quad -2 = x$

When $y = 0$, then $x = 4$ or $x = -2$. The *x*-intercepts are the points $(4, 0)$ and $(-2, 0)$.

Note:
Exercise:

Problem: Find the intercepts of the parabola $y = x^2 + 2x - 8$.

Solution:

y: $(0, -8)$; x: $(-4, 0), (2, 0)$

Note:
Exercise:

Problem: Find the intercepts of the parabola $y = x^2 - 4x - 12$.

Solution:

y: $(0, -12)$; x: $(6, 0), (-2, 0)$

In this chapter, we have been solving quadratic equations of the form $ax^2 + bx + c = 0$. We solved for x and the results were the solutions to the equation.

We are now looking at quadratic equations in two variables of the form $y = ax^2 + bx + c$. The graphs of these equations are parabolas. The *x*-intercepts of the parabolas occur where $y = 0$.

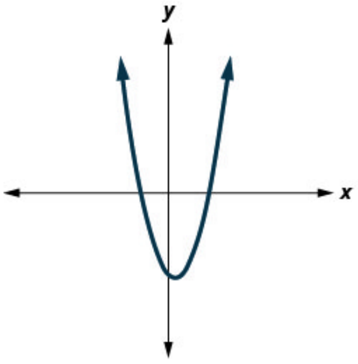
For example:
Equation:

Quadratic equation	**Quadratic equation in two variables**
	$y = x^2 - 2x - 15$
$\begin{aligned} x^2 - 2x - 15 &= 0 \\ (x-5)(x+3) &= 0 \end{aligned}$	let $y = 0$ $\quad \begin{aligned} 0 &= x^2 - 2x - 15 \\ 0 &= (x-5)(x+3) \end{aligned}$
$\begin{aligned} x - 5 &= 0 \\ x &= 5 \end{aligned} \qquad \begin{aligned} x + 3 &= 0 \\ x &= -3 \end{aligned}$	$\begin{aligned} x - 5 &= 0 \\ x &= 5 \end{aligned} \qquad \begin{aligned} x + 3 &= 0 \\ x &= -3 \end{aligned}$
	$(5, 0)$ and $(-3, 0)$
	x-intercepts

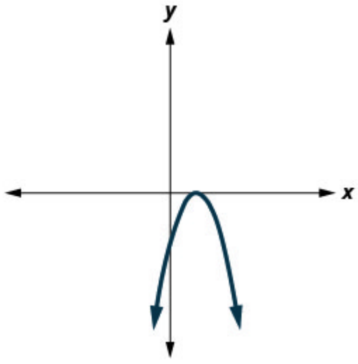
The solutions of the quadratic equation are the x values of the *x*-intercepts.

Earlier, we saw that quadratic equations have 2, 1, or 0 solutions. The graphs below show examples of parabolas for these three cases. Since the solutions of the equations give the *x*-intercepts of the graphs, the number of *x*-intercepts is the same as the number of solutions.

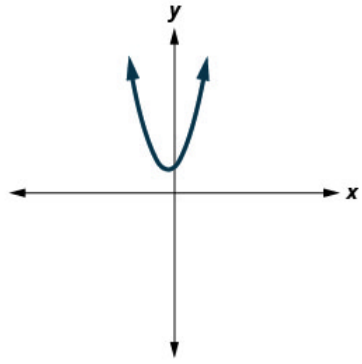
Previously, we used the discriminant to determine the number of solutions of a quadratic equation of the form $ax^2 + bx + c = 0$. Now, we can use the discriminant to tell us how many *x*-intercepts there are on the graph.



$b^2 - 4ac > 0$
Two solutions
Two x-intercepts



$b^2 - 4ac = 0$
One solution
One x-intercept



$b^2 - 4ac < 0$
No real solution
No x-intercept

Before you start solving the quadratic equation to find the values of the *x*-intercepts, you may want to evaluate the discriminant so you know how many solutions to expect.

Example:
Exercise:

Problem: Find the intercepts of the parabola $y = 5x^2 + x + 4$.

Solution:
Solution

	$y = 5x^2 + x + 4$
To find the *y*-intercept, let $x = 0$ and solve for *y*.	$y = 5 \cdot 0^2 + 0 + 4$ $y = 4$ When $x = 0$, then $y = 4$. The *y*-intercept is the point $(0, 4)$.
	$y = 5x^2 + x + 4$

To find the *x*-intercept, let $y = 0$ and solve for *x*.	$0 = 5x^2 + x + 4$
Find the value of the discriminant to predict the number of solutions and so *x*-intercepts.	$\begin{array}{ccc} b^2 & - & 4ac \\ 1^2 & - & 4 \cdot 5 \cdot 4 \\ 1 & - & 80 \\ & -79 & \end{array}$
Since the value of the discriminant is negative, there is no real solution to the equation.	There are no *x*-intercepts.

Note:
Exercise:

Problem: Find the intercepts of the parabola $y = 3x^2 + 4x + 4$.

Solution:

y: $(0, 4)$; x: none

Note:
Exercise:

Problem: Find the intercepts of the parabola $y = x^2 - 4x - 5$.

Solution:

y: $(0, -5)$; x: $(5, 0)$ $(-1, 0)$

Example:
Exercise:

Problem: Find the intercepts of the parabola $y = 4x^2 - 12x + 9$.

Solution:
Solution

	$y = 4x^2 - 12x + 9$
To find the *y*-intercept, let $x = 0$ and solve for *y*.	$y = 4 \cdot 0^2 - 12 \cdot 0 + 9$ $y = 9$
	When $x = 0$, then $y = 9$. The *y*-intercept is the point $(0, 9)$.
	$y = 4x^2 - 12x + 9$
To find the *x*-intercept, let $y = 0$ and solve for *x*.	$0 = 4x^2 - 12x + 9$
Find the value of the discriminant to predict the number of solutions and so *x*-intercepts.	$b^2 - 4ac$ $2^2 - 4 \cdot 4 \cdot 9$ $144 - 144$ 0
	Since the value of the discriminant is 0, there is no real solution to the equation. So there is one *x*-intercept.
Solve the equation by factoring the perfect square trinomial.	$0 = (2x - 3)^2$
Use the Zero Product Property.	$0 = 2x - 3$
Solve for *x*.	$3 = 2x$ $\frac{3}{2} = x$
	When $y = 0$, then $\frac{3}{2} = x$.
	The *x*-intercept is the point $\left(\frac{3}{2}, 0\right)$.

Note:
Exercise:

Problem: Find the intercepts of the parabola $y=-x^2-12x-36$.

Solution:

y: $(0,-36)$; x: $(-6,0)$

Note:
Exercise:

Problem: Find the intercepts of the parabola $y=9x^2+12x+4$.

Solution:

y: $(0,4)$; x: $\left(-\frac{2}{3},0\right)$

Graph Quadratic Equations in Two Variables

Now, we have all the pieces we need in order to graph a quadratic equation in two variables. We just need to put them together. In the next example, we will see how to do this.

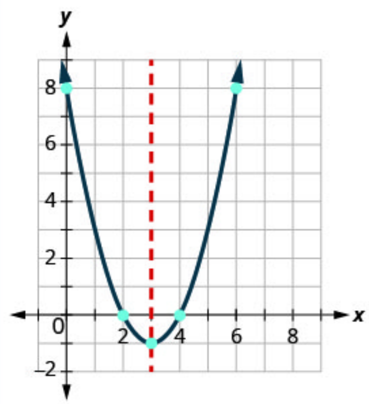
Example:
How To Graph a Quadratic Equation in Two Variables
Exercise:

Problem: Graph $y=x^2-6x+8$.

Solution:
Solution

Step 1. Write the quadratic equation with y on one side.	This equation has y on one side.	$y=x^2-6x+8$ $a=1$, $b=-6$, $c=8$
Step 2. Determine whether the parabola opens upward or downward.	Look at a in the equation. $y=x^2-6x+8$ Since a is positive, the parabola opens upward.	 **The parabola opens upward.**

Step 3. Find the axis of symmetry.	$y = x^2 - 6x + 8$ The axis of symmetry is the line $x = -\frac{b}{2a}$.	Axis of Symmetry $x = -\frac{b}{2a}$ $x = -\frac{(-6)}{2 \cdot 1}$ $x = 3$ **The axis of symmetry is the line $x = 3$.**
Step 4. Find the vertex.	The vertex is on the axis of symmetry. Substitute $x = 3$ into the equation and solve for y.	Vertex $y = x^2 - 6x + 8$ $y = (3)^2 - 6(3) + 8$ $y = -1$ **The vertex is (3, –1).**
Step 5. Find the y-intercept. Find the point symmetric to the y-intercept across the axis of symmetry.	We substitute $x = 0$ into the equation. We use the axis of symmetry to find a point symmetric to the y-intercept. The y-intercept is 3 units left of the axis of symmetry, $x = 3$. A point 3 units to the right of the axis of symmetry has $x = 6$.	y-intercept $y = x^2 - 6x + 8$ $y = (0)^2 - 6(0) + 8$ $y = 8$ **The y-intercept is (0, 8).** Point symmetric to y-intercept **The point is (6, 8).**
Step 6. Find the x-intercepts.	We substitute $y = 0$ into the equation. We can solve this quadratic equation by factoring.	x-intercept $y = x^2 - 6x + 8$ $0 = x^2 - 6x + 8$ $0 = (x - 2)(x - 4)$ $x = 2$ or $x = 4$ **The x-intercepts are (2, 0) and (4, 0).**

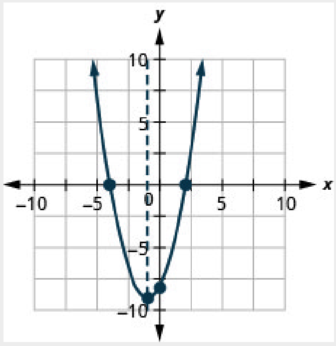
Step 7. Graph the parabola.	We graph the vertex, intercepts, and the point symmetric to the *y*-intercept. We connect these 5 points to sketch the parabola.	

Note:
Exercise:

Problem: Graph the parabola $y = x^2 + 2x - 8$.

Solution:

y: $(0, -8)$; x: $(2, 0), (-4, 0)$;
axis: $x = -1$; vertex: $(-1, -9)$;

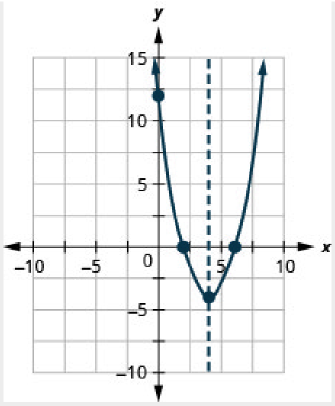


Note:
Exercise:

Problem: Graph the parabola $y = x^2 - 8x + 12$.

Solution:

y: $(0, 12)$; x: $(2, 0), (6, 0)$;
axis: $x = 4$; vertex: $(4, -4)$;



Note:
Graph a quadratic equation in two variables.

Write the quadratic equation with y on one side.
Determine whether the parabola opens upward or downward.
Find the axis of symmetry.
Find the vertex.
Find the *y*-intercept. Find the point symmetric to the *y*-intercept across the axis of symmetry.
Find the *x*-intercepts.
Graph the parabola.

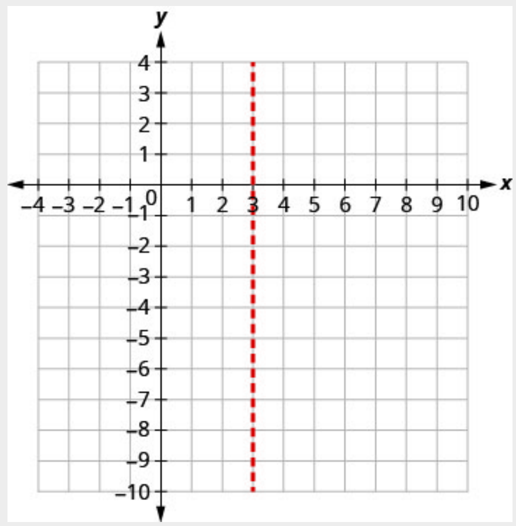
We were able to find the *x*-intercepts in the last example by factoring. We find the *x*-intercepts in the next example by factoring, too.

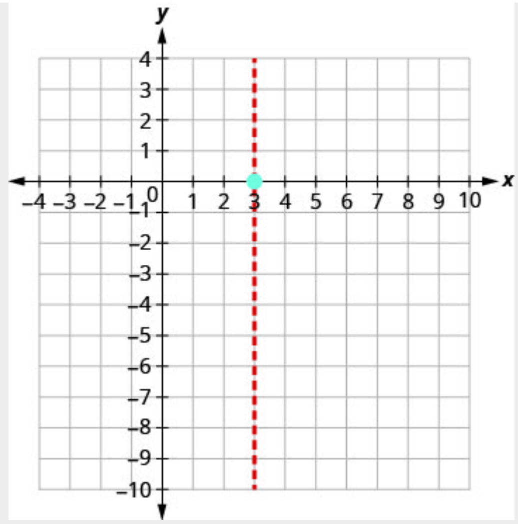
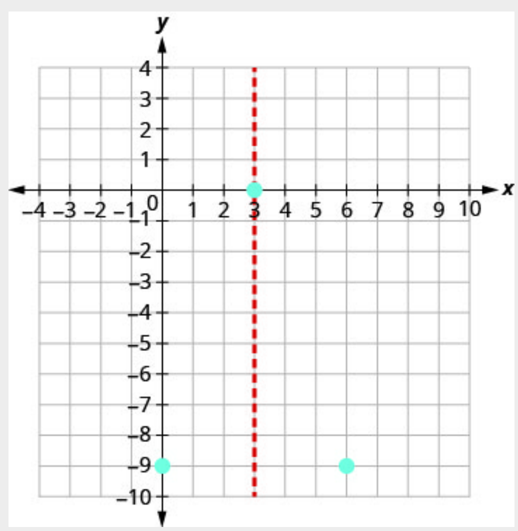
Example:
Exercise:

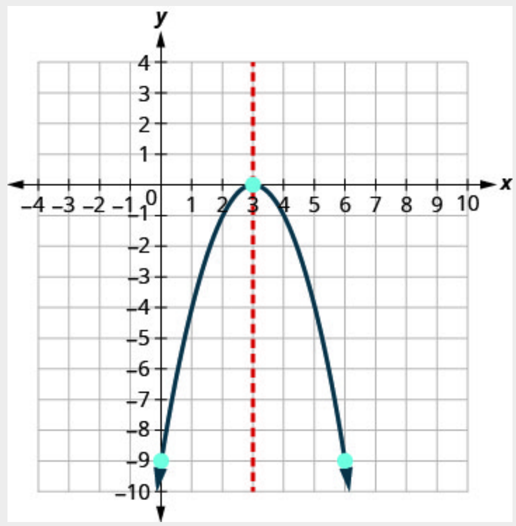
Problem: Graph $y = -x^2 + 6x - 9$.

Solution:
Solution

The equation *y* has on one side.	$y = ax^2 + bx + c$ $y = -x^2 + 6x - 9$

Since a is -1, the parabola opens downward. To find the axis of symmetry, find $x=-\frac{b}{2a}$.		$x=-\frac{b}{2a}$ $x=-\frac{6}{2(-1)}$ $x=3$ The axis of symmetry is $x=3$. The vertex is on the line $x=3$. 
Find y when $x=3$.		$y=-x^2+6x-9$ $y=-3^2+6\cdot 3-9$ $y=-9+18-9$ $y=0$ The vertex is $(3,0)$.

	
The y-intercept occurs when $x = 0$. Substitute $x = 0$. Simplify. The point $(0, -9)$ is three units to the left of the line of symmetry. The point three units to the right of the line of symmetry is $(6, -9)$. Point symmetric to the y-intercept is $(6, -9)$	$y = -x^2 + 6x - 9$ $y = -0^2 + 6 \cdot 0 - 9$ $y = -9$ The y-intercept is $(0, -9)$. 
The x-intercept occurs when $y = 0$.	$y = -x^2 + 6x - 9$
Substitute $y = 0$.	

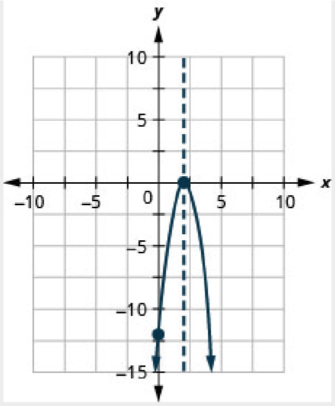
	$0 = -x^2 + 6x - 9$
Factor the GCF.	$0 = -(x^2 - 6x + 9)$
Factor the trinomial.	$0 = -(x - 3)^2$
Solve for *x*.	$x = 3$
Connect the points to graph the parabola.	

Note:
Exercise:

Problem: Graph the parabola $y = -3x^2 + 12x - 12$.

Solution:

y: $(0, -12)$; x: $(2, 0)$;
axis: $x = 2$; vertex: $(2, 0)$;

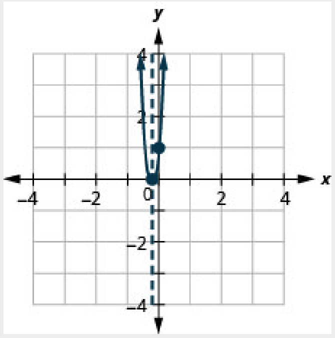


Note:
Exercise:

Problem: Graph the parabola $y = 25x^2 + 10x + 1$.

Solution:

y: $(0, 1)$; x: $\left(-\frac{1}{5}, 0\right)$;
axis: $x = -\frac{1}{5}$; vertex: $\left(-\frac{1}{5}, 0\right)$;



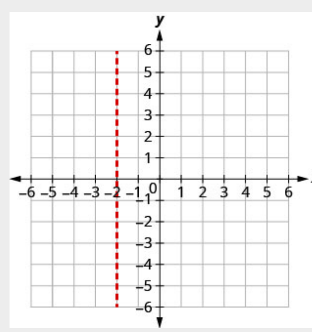
For the graph of $y = -x^2 + 6x - 9$, the vertex and the *x*-intercept were the same point. Remember how the discriminant determines the number of solutions of a quadratic equation? The discriminant of the equation $0 = -x^2 + 6x - 9$ is 0, so there is only one solution. That means there is only one *x*-intercept, and it is the vertex of the parabola.

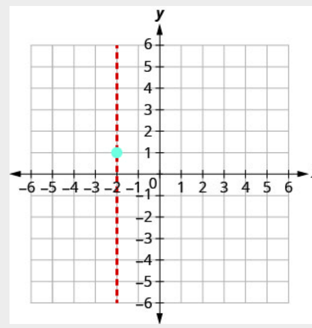
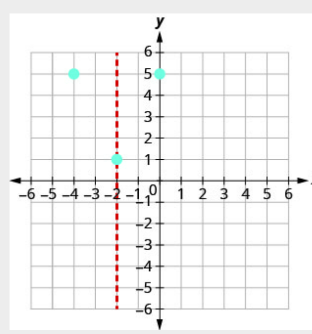
How many *x*-intercepts would you expect to see on the graph of $y = x^2 + 4x + 5$?

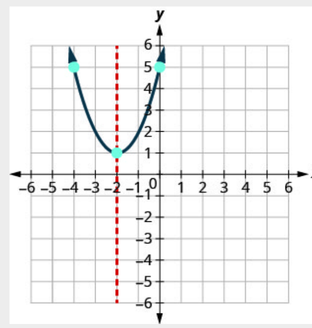
Example:
Exercise:

Problem: Graph $y = x^2 + 4x + 5$.

Solution:
Solution

The equation has *y* on one side.	$y = ax^2 + bx + c$ $y = x^2 + 4x + 5$
Since a is 1, the parabola opens upward.	
To find the axis of symmetry, find $x = -\frac{b}{2a}$.	$x = -\frac{b}{2a}$ $x = -\frac{4}{2(1)}$ $x = -2$ The axis of symmetry is $x = -2$. 
The vertex is on the line $x = -2$.	
Find *y* when $x = -2$.	$y = x^2 + 4x + 5$

	$y = (-2)^2 + 4 \cdot (-2) + 5$ $y = 4 - 8 + 5$ $y = 1$ The vertex is $(-2, 1)$. 
The *y*-intercept occurs when $x = 0$. Substitute $x = 0$. Simplify. The point $(0, 5)$ is two units to the right of the line of symmetry. The point two units to the left of the line of symmetry is $(-4, 5)$.	$y = x^2 + 4x + 5$ $y = (0)^2 + 4(0) + 5$ $y = 5$ The *y*-intercept is $(0, 5)$.  Point symmetric to the *y*-intercept is $(-4, 5)$.
The *x*- intercept occurs when $y = 0$.	

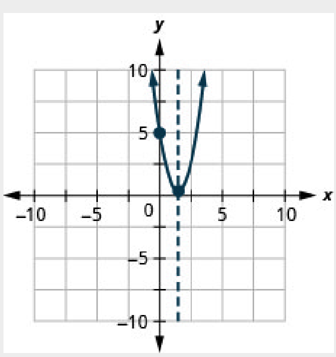
Substitute $y = 0$. Test the discriminant.		$y = x^2 + 4x + 5$ $0 = x^2 + 4x + 5$
		$b^2 - 4ac$ $4^2 - 4 \cdot 15$ $16 - 20$ -4
Since the value of the discriminant is negative, there is no solution and so no *x*- intercept. Connect the points to graph the parabola. You may want to choose two more points for greater accuracy.		

Note:
Exercise:

Problem: Graph the parabola $y = 2x^2 - 6x + 5$.

Solution:

y: $(0, 5)$; x: none;
axis: $x = \frac{3}{2}$; vertex: $\left(\frac{3}{2}, \frac{1}{2}\right)$;

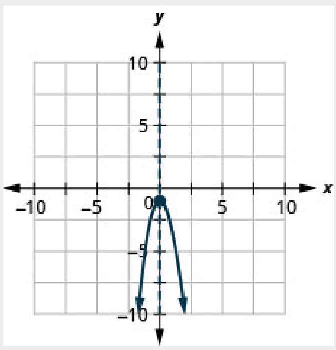


Note:
Exercise:

Problem: Graph the parabola $y=-2x^2-1$.

Solution:

y: $(0,-1)$; x: none;
axis: $x=0$; vertex: $(0,-1)$;



Finding the *y*-intercept by substituting $x=0$ into the equation is easy, isn't it? But we needed to use the Quadratic Formula to find the *x*-intercepts in [link]. We will use the Quadratic Formula again in the next example.

Example:
Exercise:

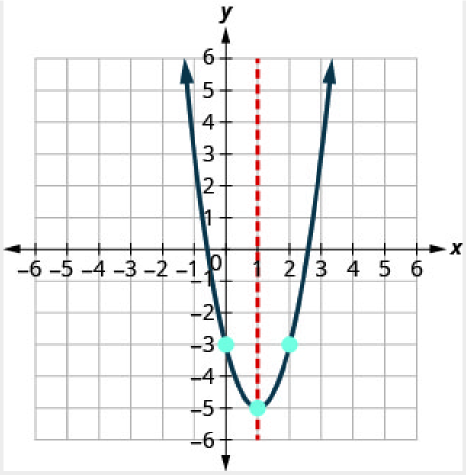
Problem: Graph $y=2x^2-4x-3$.

Solution:
Solution

	$y = ax^2 + bx + c$ $y = 2x^2 - 4x - 3$
The equation *y* has one side. Since *a* is 2, the parabola opens upward.	

To find the axis of symmetry, find $x = -\frac{b}{2a}$.	$x = -\frac{b}{2a}$ $x = -\frac{-4}{2 \cdot 2}$ $x = 1$ The axis of symmetry is $x = 1$.
The vertex on the line $x = 1$.	$y = 2x^2 - 4x - 3$
Find *y* when $x = 1$.	$y = 2(1)^2 - 4 \cdot (1) - 3$ $y = 2 - 4 - 3$ $y = -5$ The vertex is $(1, -5)$.
The *y*-intercept occurs when $x = 0$.	$y = 2x^2 - 4x - 3$
Substitute $x = 0$.	$y = 2 \cdot 0^2 - 4 \cdot 0 - 3$
Simplify.	$y = -3$ The *y*-intercept is $(0, -3)$.
The point $(0, -3)$ is one unit to the left of the line of symmetry. The point one unit to the right of the line of symmetry is $(2, -3)$	Point symmetric to the *y*-intercept is $(2, -3)$.
The *x*-intercept occurs when $y = 0$.	$y = 2x^2 - 4x - 3$

Substitute $y = 0$.	$0 = 2x^2 - 4x - 3$
Use the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Substitute in the values of *a*, *b*, *c*.	$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 2 \cdot (-3)}}{2 \cdot 2}$
Simplify.	$x = \frac{4 \pm \sqrt{16 + 24}}{4}$
Simplify inside the radical.	$x = \frac{4 \pm \sqrt{40}}{4}$
Simplify the radical.	$x = \frac{4 \pm 2\sqrt{10}}{4}$
Factor the GCF.	$x = \frac{2(2 \pm \sqrt{10})}{4}$
Remove common factors.	$x = \frac{2 \pm \sqrt{10}}{2}$
Write as two equations.	$x = \frac{2 + \sqrt{10}}{2}, x = \frac{2 - \sqrt{10}}{2}$
Approximate the values.	$x \approx 2.5, x \approx -0.6$
	The approximate values of the *x*-intercepts are $(2.5, 0)$ and $(-0.6, 0)$.
Graph the parabola using the points found.	

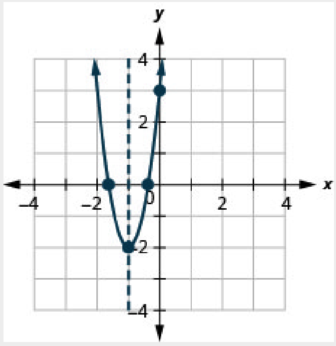


Note:
Exercise:

Problem: Graph the parabola $y=5x^2+10x+3$.

Solution:

y: $(0,3);$ x: $(-1.6,0),(-0.4,0);$
axis: $x=-1;$ vertex: $(-1,-2);$

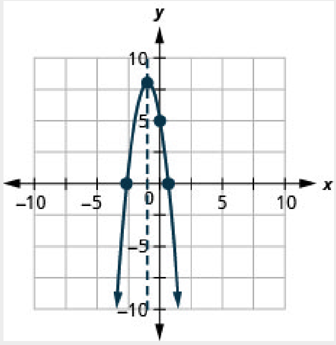


Note:
Exercise:

Problem: Graph the parabola $y=-3x^2-6x+5$.

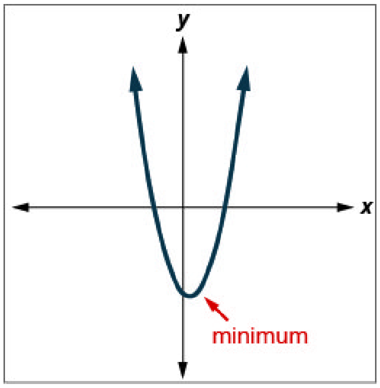
Solution:

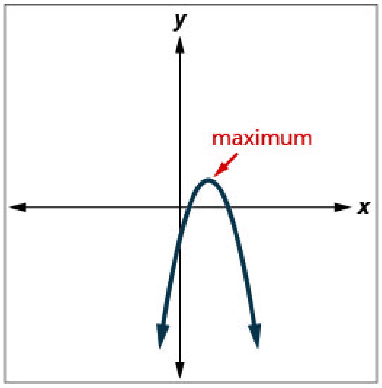
y: $(0,5)$; x: $(0.6,0),(-2.6,0)$;
axis: $x=-1$; vertex: $(-1,8)$;



Solve Maximum and Minimum Applications

Knowing that the vertex of a parabola is the lowest or highest point of the parabola gives us an easy way to determine the minimum or maximum value of a quadratic equation. The *y*-coordinate of the vertex is the minimum *y*-value of a parabola that opens upward. It is the maximum *y*-value of a parabola that opens downward. See [link].





Note:
Minimum or Maximum Values of a Quadratic Equation
The ***y*-coordinate of the vertex** of the graph of a quadratic equation is the

- minimum value of the quadratic equation if the parabola opens upward.
- maximum value of the quadratic equation if the parabola opens downward.

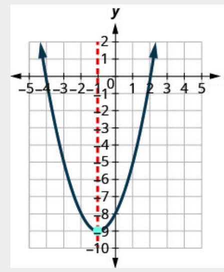
Example:

Exercise:

Problem: Find the minimum value of the quadratic equation $y = x^2 + 2x - 8$.

Solution:
Solution

	$y = x^2 + 2x - 8$
Since *a* is positive, the parabola opens upward.	
The quadratic equation has a minimum.	
Find the axis of symmetry.	$x = -\frac{b}{2a}$ $x = -\frac{2}{2 \cdot 1}$ $x = -1$ The axis of symmetry is $x = -1$.
The vertex is on the line $x = -1$.	$y = x^2 + 2x - 8$
Find *y* when $x = -1$.	$y = (-1)^2 + 2 \cdot (-1) - 8$ $y = 1 - 2 - 8$ $y = -9$ The vertex is $(-1, -9)$.

Since the parabola has a minimum, the *y*-coordinate of the vertex is the minimum *y*-value of the quadratic equation.	
The minimum value of the quadratic is -9 and it occurs when $x = -1$.	
Show the graph to verify the result.	

Note:
Exercise:

Problem: Find the maximum or minimum value of the quadratic equation $y = x^2 - 8x + 12$.

Solution:

The minimum value is -4 when $x = 4$.

Note:
Exercise:

Problem: Find the maximum or minimum value of the quadratic equation $y = -4x^2 + 16x - 11$.

Solution:

The maximum value is 5 when $x = 2$.

We have used the formula
Equation:

$$h = -16t^2 + v_0 t + h_0$$

to calculate the height in feet, h, of an object shot upwards into the air with initial velocity, v_0, after t seconds.

This formula is a quadratic equation in the variable t, so its graph is a parabola. By solving for the coordinates of the vertex, we can find how long it will take the object to reach its maximum height. Then, we can calculate the maximum height.

Example:
Exercise:

Problem:

The quadratic equation $h = -16t^2 + v_0 t + h_0$ models the height of a volleyball hit straight upwards with velocity 176 feet per second from a height of 4 feet.

ⓐ How many seconds will it take the volleyball to reach its maximum height?
ⓑ Find the maximum height of the volleyball.

Solution:
Solution

$h = -16t^2 + 176t + 4$

Since a is negative, the parabola opens downward.

The quadratic equation has a maximum.

ⓐ

Find the axis of symmetry.	$t = -\frac{b}{2a}$ $t = -\frac{176}{2(-16)}$ $t = 5.5$ The axis of symmetry is $t = 5.5$.
The vertex is on the line $t = 5.5$.	The maximum occurs when $t = 5.5$ seconds.

ⓑ

Find h when $t = 5.5$.	$h = -16t^2 + 176t + 4$ $h = -16(5.5)^2 + 176 \cdot (5.5) + 4$
Use a calculator to simplify.	$h = 488$
	The vertex is $(5.5, 488)$.

Since the parabola has a maximum, the *h*-coordinate of the vertex is the maximum *y*-value of the quadratic equation.	The maximum value of the quadratic is 488 feet and it occurs when $t = 5.5$ seconds.

Note:
Exercise:

Problem:

The quadratic equation $h = -16t^2 + 128t + 32$ is used to find the height of a stone thrown upward from a height of 32 feet at a rate of 128 ft/sec. How long will it take for the stone to reach its maximum height? What is the maximum height? Round answers to the nearest tenth.

Solution:

It will take 4 seconds to reach the maximum height of 288 feet.

Note:
Exercise:

Problem:

A toy rocket shot upward from the ground at a rate of 208 ft/sec has the quadratic equation of $h = -16t^2 + 208t$. When will the rocket reach its maximum height? What will be the maximum height? Round answers to the nearest tenth.

Solution:

It will take 6.5 seconds to reach the maximum height of 676 feet.

Note:
Access these online resources for additional instruction and practice graphing quadratic equations:

- Graphing Quadratic Functions
- How do you graph a quadratic function?
- Graphing Quadratic Equations

Key Concepts

- **The graph of every quadratic equation is a parabola.**
- **Parabola Orientation** For the quadratic equation $y = ax^2 + bx + c$, if

- $a > 0$, the parabola opens upward.
- $a < 0$, the parabola opens downward.

- **Axis of Symmetry and Vertex of a Parabola** For a parabola with equation $y = ax^2 + bx + c$:
  - The axis of symmetry of a parabola is the line $x = -\frac{b}{2a}$.
  - The vertex is on the axis of symmetry, so its *x*-coordinate is $-\frac{b}{2a}$.
  - To find the *y*-coordinate of the vertex we substitute $x = -\frac{b}{2a}$ into the quadratic equation.

- **Find the Intercepts of a Parabola** To find the intercepts of a parabola with equation $y = ax^2 + bx + c$:

y-intercept	x-intercepts
Let $x = 0$ and solve for y.	Let $y = 0$ and solve for x.

- **To Graph a Quadratic Equation in Two Variables**

  Write the quadratic equation with y on one side.
  Determine whether the parabola opens upward or downward.
  Find the axis of symmetry.
  Find the vertex.
  Find the *y*-intercept. Find the point symmetric to the *y*-intercept across the axis of symmetry.
  Find the *x*-intercepts.
  Graph the parabola.

- **Minimum or Maximum Values of a Quadratic Equation**
  - The *y*-**coordinate of the vertex** of the graph of a quadratic equation is the
  - **minimum** value of the quadratic equation if the parabola opens upward.
  - **maximum** value of the quadratic equation if the parabola opens downward.

Section Exercises

Practice Makes Perfect

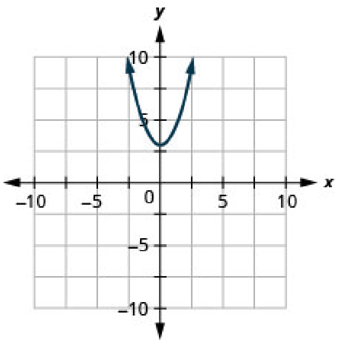
Recognize the Graph of a Quadratic Equation in Two Variables

In the following exercises, graph:

Exercise:

Problem: $y = x^2 + 3$

Solution:



Exercise:

Problem: $y = -x^2 + 1$

In the following exercises, determine if the parabola opens up or down.

Exercise:

Problem: $y = -2x^2 - 6x - 7$

Solution:

down

Exercise:

Problem: $y = 6x^2 + 2x + 3$

Exercise:

Problem: $y = 4x^2 + x - 4$

Solution:

up

Exercise:

Problem: $y = -9x^2 - 24x - 16$

Find the Axis of Symmetry and Vertex of a Parabola

In the following exercises, find ⓐ the axis of symmetry and ⓑ the vertex.

Exercise:

Problem: $y = x^2 + 8x - 1$

Solution:

ⓐ $x = -4$ ⓑ $(-4, -17)$

Exercise:

Problem: $y = x^2 + 10x + 25$

Exercise:

Problem: $y = -x^2 + 2x + 5$

Solution:

ⓐ $x = 1$ ⓑ $(1, 6)$

Exercise:

Problem: $y = -2x^2 - 8x - 3$

Find the Intercepts of a Parabola

In the following exercises, find the *x*- and *y*-intercepts.

Exercise:

Problem: $y = x^2 + 7x + 6$

Solution:

y: $(0, 6)$; x: $(-1, 0), (-6, 0)$

Exercise:

Problem: $y = x^2 + 10x - 11$

Exercise:

Problem: $y = -x^2 + 8x - 19$

Solution:

y: $(0, 19)$; x: none

Exercise:

Problem: $y = x^2 + 6x + 13$

Exercise:

Problem: $y = 4x^2 - 20x + 25$

Solution:

y: $(0, 25)$; x: $\left(\frac{5}{2}, 0\right)$

Exercise:

Problem: $y=-x^2-14x-49$

Graph Quadratic Equations in Two Variables

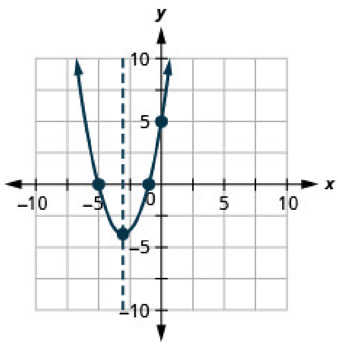
In the following exercises, graph by using intercepts, the vertex, and the axis of symmetry.

Exercise:

Problem: $y=x^2+6x+5$

Solution:

y: $(0,5);$ x: $(-1,0),(-5,0);$
axis: $x=-3;$ vertex: $(-3,-4)$



Exercise:

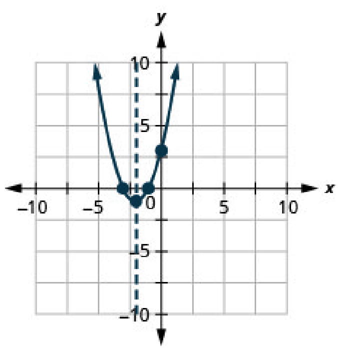
Problem: $y=x^2+4x-12$

Exercise:

Problem: $y=x^2+4x+3$

Solution:

y: $(0,3);$ x: $(-1,0),(-3,0);$
axis: $x=-2;$ vertex: $(-2,-1)$



Exercise:

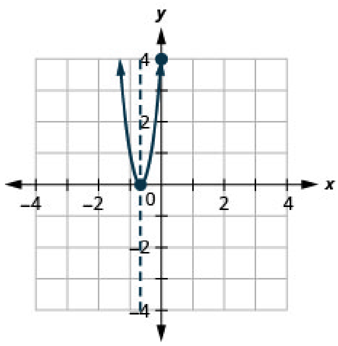
Problem: $y = x^2 - 6x + 8$

Exercise:

Problem: $y = 9x^2 + 12x + 4$

Solution:

y: $(0, 4)$ x: $(-\frac{2}{3}, 0)$;
axis: $x = -\frac{2}{3}$; vertex: $(-\frac{2}{3}, 0)$



Exercise:

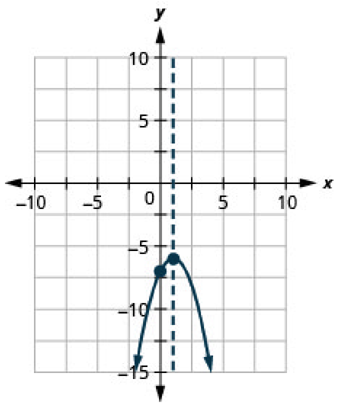
Problem: $y = -x^2 + 8x - 16$

Exercise:

Problem: $y = -x^2 + 2x - 7$

Solution:

y: $(0, -7)$; x: none;
axis: $x = 1$; vertex: $(1, -6)$



Exercise:

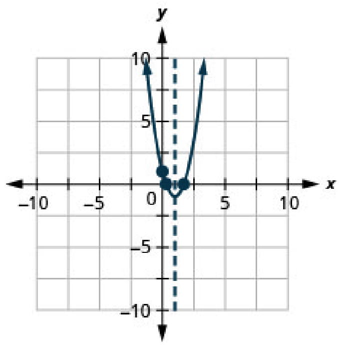
Problem: $y=5x^2+2$

Exercise:

Problem: $y=2x^2-4x+1$

Solution:

y: $(0,1)$; x: $(1.7,0)$, $(0.3,0)$;
axis: $x=1$; vertex: $(1,-1)$



Exercise:

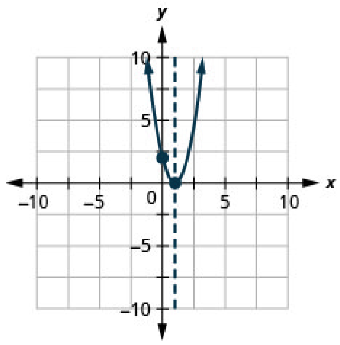
Problem: $y=3x^2-6x-1$

Exercise:

Problem: $y=2x^2-4x+2$

Solution:

y: $(0,2)$ x: $(1,0)$;
axis: $x=1$; vertex: $(1,0)$



Exercise:

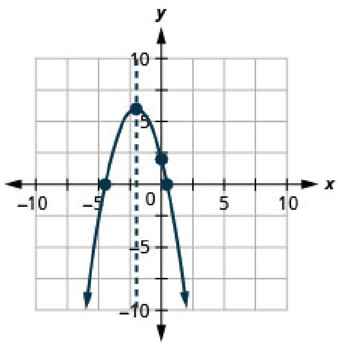
Problem: $y = -4x^2 - 6x - 2$

Exercise:

Problem: $y = -x^2 - 4x + 2$

Solution:

y: $(0, 2)$ x: $(-4.4, 0), (0.4, 0)$;
axis: $x = -2$; vertex: $(-2, 6)$



Exercise:

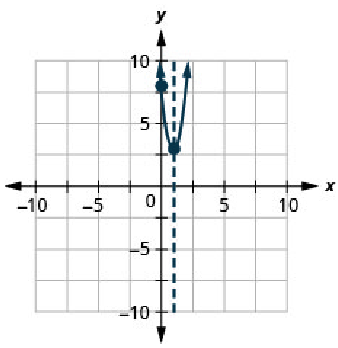
Problem: $y = x^2 + 6x + 8$

Exercise:

Problem: $y = 5x^2 - 10x + 8$

Solution:

y: $(0, 8)$; x: none;
axis: $x = 1$; vertex: $(1, 3)$



Exercise:

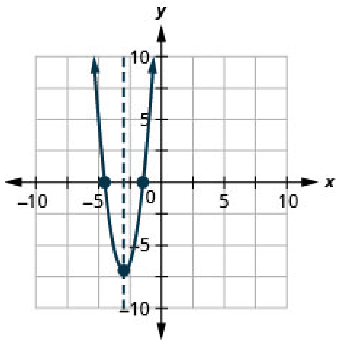
Problem: $y = -16x^2 + 24x - 9$

Exercise:

Problem: $y = 3x^2 + 18x + 20$

Solution:

y: $(0, 20)$ x: $(-4.5, 0), (-1.5, 0)$;
axis: $x = -3$; vertex: $(-3, -7)$



Exercise:

Problem: $y = -2x^2 + 8x - 10$

Solve Maximum and Minimum Applications

In the following exercises, find the maximum or minimum value.

Exercise:

Problem: $y = 2x^2 + x - 1$

Solution:

The minimum value is $-\frac{9}{8}$ when $x = -\frac{1}{4}$.

Exercise:

Problem: $y = -4x^2 + 12x - 5$

Exercise:

Problem: $y = x^2 - 6x + 15$

Solution:

The minimum value is 6 when $x = 3$.

Exercise:

Problem: $y = -x^2 + 4x - 5$

Exercise:

Problem: $y = -9x^2 + 16$

Solution:

The maximum value is 16 when $x = 0$.

Exercise:

Problem: $y = 4x^2 - 49$

In the following exercises, solve. Round answers to the nearest tenth.

Exercise:

Problem:

An arrow is shot vertically upward from a platform 45 feet high at a rate of 168 ft/sec. Use the quadratic equation $h = -16t^2 + 168t + 45$ to find how long it will take the arrow to reach its maximum height, and then find the maximum height.

Solution:

In 5.3 sec the arrow will reach maximum height of 486 ft.

Exercise:

Problem:

A stone is thrown vertically upward from a platform that is 20 feet high at a rate of 160 ft/sec. Use the quadratic equation $h = -16t^2 + 160t + 20$ to find how long it will take the stone to reach its maximum height, and then find the maximum height.

Exercise:

Problem:

A computer store owner estimates that by charging x dollars each for a certain computer, he can sell $40 - x$ computers each week. The quadratic equation $R = -x^2 + 40x$ is used to find the revenue, R, received when the selling price of a computer is x. Find the selling price that will give him the maximum revenue, and then find the amount of the maximum revenue.

Solution:

20 computers will give the maximum of $400 in receipts.

Exercise:

Problem:

A retailer who sells backpacks estimates that, by selling them for x dollars each, he will be able to sell $100 - x$ backpacks a month. The quadratic equation $R = -x^2 + 100x$ is used to find the R received when the selling price of a backpack is x. Find the selling price that will give him the maximum revenue, and then find the amount of the maximum revenue.

Exercise:

Problem:

A rancher is going to fence three sides of a corral next to a river. He needs to maximize the corral area using 240 feet of fencing. The quadratic equation $A = x\left(240 - 2x\right)$ gives the area of the corral, A, for the length, x, of the corral along the river. Find the length of the corral along the river that will give the maximum area, and then find the maximum area of the corral.

Solution:

The length of the side along the river of the corral is 120 feet and the maximum area is 7,200 sq ft.

Exercise:

Problem:

A veterinarian is enclosing a rectangular outdoor running area against his building for the dogs he cares for. He needs to maximize the area using 100 feet of fencing. The quadratic equation $A = x\left(100 - 2x\right)$ gives the area, A, of the dog run for the length, x, of the building that will border the dog run. Find the length of the building that should border the dog run to give the maximum area, and then find the maximum area of the dog run.

Everyday Math

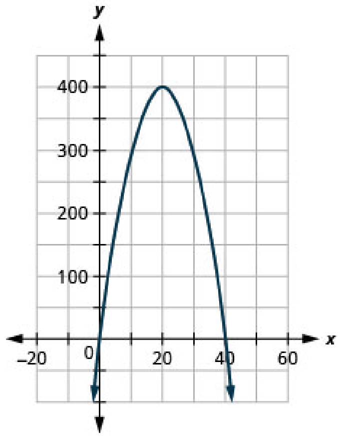
Exercise:

Problem:

In the previous set of exercises, you worked with the quadratic equation $R = -x^2 + 40x$ that modeled the revenue received from selling computers at a price of x dollars. You found the selling price that would give the maximum revenue and calculated the maximum revenue. Now you will look at more characteristics of this model.
ⓐ Graph the equation $R = -x^2 + 40x$. ⓑ Find the values of the *x*-intercepts.

Solution:

ⓐ



ⓑ $(0, 0), (40, 0)$

Exercise:

Problem:

In the previous set of exercises, you worked with the quadratic equation $R = -x^2 + 100x$ that modeled the revenue received from selling backpacks at a price of x dollars. You found the selling price that would give the maximum revenue and calculated the maximum revenue. Now you will look at more characteristics of this model.
ⓐ Graph the equation $R = -x^2 + 100x$. ⓑ Find the values of the *x*-intercepts.

Writing Exercises

Exercise:

Problem:

For the revenue model in [link] and [link], explain what the *x*-intercepts mean to the computer store owner.

Solution:

Answers will vary.

Exercise:

Problem:

For the revenue model in [link] and [link], explain what the *x*-intercepts mean to the backpack retailer.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
recognize the graph of a quadratic equation in two variables.			
find the axis of symmetry and vertex of a parabola.			
find the intercepts of a parabola.			
graph quadratic equations in two variables.			
solve maximum and minimum applications.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Chapter Review Exercises

10.1 Solve Quadratic Equations Using the Square Root Property

In the following exercises, solve using the Square Root Property.

Exercise:

Problem: $x^2 = 100$

Solution:

$x = \pm 10$

Exercise:

Problem: $y^2 = 144$

Exercise:

Problem: $m^2 - 40 = 0$

Solution:

$m = \pm 2\sqrt{10}$

Exercise:

Problem: $n^2 - 80 = 0$

Exercise:

Problem: $4a^2 = 100$

Solution:

$a = \pm 5$

Exercise:

Problem: $2b^2 = 72$

Exercise:

Problem: $r^2 + 32 = 0$

Solution:

no solution

Exercise:

Problem: $t^2 + 18 = 0$

Exercise:

Problem: $\frac{4}{3}v^2 + 4 = 28$

Solution:

$v = \pm\, 3\sqrt{2}$

Exercise:

Problem: $\frac{2}{3}w^2 - 20 = 30$

Exercise:

Problem: $5c^2 + 3 = 19$

Solution:

$c = \pm\, \frac{4\sqrt{5}}{5}$

Exercise:

Problem: $3d^2 - 6 = 43$

In the following exercises, solve using the Square Root Property.

Exercise:

Problem: $(p-5)^2 + 3 = 19$

Solution:

$p = 1, 9$

Exercise:

Problem: $(q+4)^2 = 9$

Exercise:

Problem: $(u+1)^2 = 45$

Solution:

$u = -1 \pm 3\sqrt{5}$

Exercise:

Problem: $(z-5)^2 = 50$

Exercise:

Problem: $\left(x-\frac{1}{4}\right)^2=\frac{3}{16}$

Solution:

$x=\frac{1}{4}\pm\frac{\sqrt{3}}{4}$

Exercise:

Problem: $\left(y-\frac{2}{3}\right)^2=\frac{2}{9}$

Exercise:

Problem: $(m-7)^2+6=30$

Solution:

$m=7\pm 2\sqrt{6}$

Exercise:

Problem: $(n-4)^2-50=150$

Exercise:

Problem: $(5c+3)^2=-20$

Solution:

no solution

Exercise:

Problem: $(4c-1)^2=-18$

Exercise:

Problem: $m^2-6m+9=48$

Solution:

$m=3\pm 4\sqrt{3}$

Exercise:

Problem: $n^2+10n+25=12$

Exercise:

Problem: $64a^2+48a+9=81$

Solution:

$a = -\frac{3}{2}, \frac{3}{4}$

Exercise:

Problem: $4b^2 - 28b + 49 = 25$

10.2 Solve Quadratic Equations Using Completing the Square

In the following exercises, complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

Exercise:

Problem: $x^2 + 22x$

Solution:

$(x + 11)^2$

Exercise:

Problem: $y^2 + 6y$

Exercise:

Problem: $m^2 - 8m$

Solution:

$(m - 4)^2$

Exercise:

Problem: $n^2 - 10n$

Exercise:

Problem: $a^2 - 3a$

Solution:

$\left(a - \frac{3}{2}\right)^2$

Exercise:

Problem: $b^2 + 13b$

Exercise:

Problem: $p^2 + \frac{4}{5}p$

Solution:

$\left(p + \frac{2}{5}\right)^2$

Exercise:

Problem: $q^2 - \frac{1}{3}q$

In the following exercises, solve by completing the square.

Exercise:

Problem: $c^2 + 20c = 21$

Solution:

$c = 1, -21$

Exercise:

Problem: $d^2 + 14d = -13$

Exercise:

Problem: $x^2 - 4x = 32$

Solution:

$x = -4, 8$

Exercise:

Problem: $y^2 - 16y = 36$

Exercise:

Problem: $r^2 + 6r = -100$

Solution:

no solution

Exercise:

Problem: $t^2 - 12t = -40$

Exercise:

Problem: $v^2 - 14v = -31$

Solution:

$v = 7 \pm 3\sqrt{2}$

Exercise:

Problem: $w^2 - 20w = 100$

Exercise:

Problem: $m^2 + 10m - 4 = -13$

Solution:

$m = -9, -1$

Exercise:

Problem: $n^2 - 6n + 11 = 34$

Exercise:

Problem: $a^2 = 3a + 8$

Solution:

$a = \frac{3}{2} \pm \frac{\sqrt{41}}{2}$

Exercise:

Problem: $b^2 = 11b - 5$

Exercise:

Problem: $(u + 8)\,(u + 4) = 14$

Solution:

$u = -6 \pm 2\sqrt{2}$

Exercise:

Problem: $(z - 10)\,(z + 2) = 28$

Exercise:

Problem: $3p^2 - 18p + 15 = 15$

Solution:

$p = 0, 6$

Exercise:

Problem: $5q^2 + 70q + 20 = 0$

Exercise:

Problem: $4y^2 - 6y = 4$

Solution:

$y = -\frac{1}{2}, 2$

Exercise:

Problem: $2x^2 + 2x = 4$

Exercise:

Problem: $3c^2 + 2c = 9$

Solution:

$c = -\frac{1}{3} \pm \frac{2\sqrt{7}}{3}$

Exercise:

Problem: $4d^2 - 2d = 8$

10.3 Solve Quadratic Equations Using the Quadratic Formula

In the following exercises, solve by using the Quadratic Formula.

Exercise:

Problem: $4x^2 - 5x + 1 = 0$

Solution:

$x = \frac{1}{4}, 1$

Exercise:

Problem: $7y^2 + 4y - 3 = 0$

Exercise:

Problem: $r^2 - r - 42 = 0$

Solution:

$r = -6, 7$

Exercise:

Problem: $t^2 + 13t + 22 = 0$

Exercise:

Problem: $4v^2 + v - 5 = 0$

Solution:

$v = -\frac{5}{4}, 1$

Exercise:

Problem: $2w^2 + 9w + 2 = 0$

Exercise:

Problem: $3m^2 + 8m + 2 = 0$

Solution:

$m = \frac{-4 \pm \sqrt{10}}{3}$

Exercise:

Problem: $5n^2 + 2n - 1 = 0$

Exercise:

Problem: $6a^2 - 5a + 2 = 0$

Solution:

no real solution

Exercise:

Problem: $4b^2 - b + 8 = 0$

Exercise:

Problem: $u\left(u - 10\right) + 3 = 0$

Solution:

$u = 5 \pm \sqrt{22}$

Exercise:

Problem: $5z(z - 2) = 3$

Exercise:

Problem: $\frac{1}{8}p^2 - \frac{1}{5}p = -\frac{1}{20}$

Solution:

$p = \frac{4 \pm \sqrt{6}}{5}$

Exercise:

Problem: $\frac{2}{5}q^2 + \frac{3}{10}q = \frac{1}{10}$

Exercise:

Problem: $4c^2 + 4c + 1 = 0$

Solution:

$c = -\frac{1}{2}$

Exercise:

Problem: $9d^2 - 12d = -4$

In the following exercises, determine the number of solutions to each quadratic equation.

Exercise:

Problem:

ⓐ $9x^2 - 6x + 1 = 0$
ⓑ $3y^2 - 8y + 1 = 0$
ⓒ $7m^2 + 12m + 4 = 0$
ⓓ $5n^2 - n + 1 = 0$

Solution:

ⓐ 1 ⓑ 2 ⓒ 2 ⓓ none

Exercise:

Problem:

ⓐ $5x^2 - 7x - 8 = 0$
ⓑ $7x^2 - 10x + 5 = 0$
ⓒ $25x^2 - 90x + 81 = 0$
ⓓ $15x^2 - 8x + 4 = 0$

In the following exercises, identify the most appropriate method (Factoring, Square Root, or Quadratic Formula) to use to solve each quadratic equation.

Exercise:

Problem:

ⓐ $16r^2 - 8r + 1 = 0$
ⓑ $5t^2 - 8t + 3 = 9$
ⓒ $3(c + 2)^2 = 15$

Solution:

ⓐ factor ⓑ Quadratic Formula ⓒ square root

Exercise:

Problem:

ⓐ $4d^2 + 10d - 5 = 21$
ⓑ $25x^2 - 60x + 36 = 0$
ⓒ $6(5v - 7)^2 = 150$

10.4 Solve Applications Modeled by Quadratic Equations

In the following exercises, solve by using methods of factoring, the square root principle, or the quadratic formula.

Exercise:

Problem: Find two consecutive odd numbers whose product is 323.

Solution:

Two consecutive odd numbers whose product is 323 are 17 and 19, and -17 and -19.

Exercise:

Problem: Find two consecutive even numbers whose product is 624.

Exercise:

Problem:

A triangular banner has an area of 351 square centimeters. The length of the base is two centimeters longer than four times the height. Find the height and length of the base.

Solution:

The height of the banner is 13 cm and the length of the side is 54 cm.

Exercise:

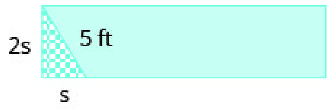
Problem:

Julius built a triangular display case for his coin collection. The height of the display case is six inches less than twice the width of the base. The area of the of the back of the case is 70 square inches. Find the height and width of the case.

Exercise:

Problem:

A tile mosaic in the shape of a right triangle is used as the corner of a rectangular pathway. The hypotenuse of the mosaic is 5 feet. One side of the mosaic is twice as long as the other side. What are the lengths of the sides? Round to the nearest tenth.



Solution:

The lengths of the sides of the mosaic are 2.2 and 4.4 feet.

Exercise:

Problem:

A rectangular piece of plywood has a diagonal which measures two feet more than the width. The length of the plywood is twice the width. What is the length of the plywood's diagonal? Round to the nearest tenth.

Exercise:

Problem:

The front walk from the street to Pam's house has an area of 250 square feet. Its length is two less than four times its width. Find the length and width of the sidewalk. Round to the nearest tenth.

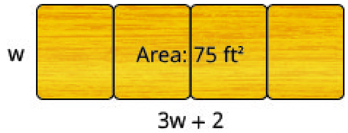
Solution:

The width of the front walk is 8.1 feet and its length is 30.8 feet.

Exercise:

Problem:

For Sophia's graduation party, several tables of the same width will be arranged end to end to give a serving table with a total area of 75 square feet. The total length of the tables will be two more than three times the width. Find the length and width of the serving table so Sophia can purchase the correct size tablecloth. Round answer to the nearest tenth.



Exercise:

Problem:

A ball is thrown vertically in the air with a velocity of 160 ft/sec. Use the formula $h = -16t^2 + v_0t$ to determine when the ball will be 384 feet from the ground. Round to the nearest tenth.

Solution:

The ball will reach 384 feet on its way up in 4 seconds and on the way down in 6 seconds.

Exercise:

Problem:

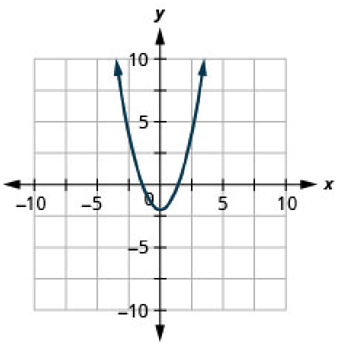
A bullet is fired straight up from the ground at a velocity of 320 ft/sec. Use the formula $h = -16t^2 + v_0t$ to determine when the bullet will reach 800 feet. Round to the nearest tenth.

10.5 Graphing Quadratic Equations in Two Variables

In the following exercises, graph by plotting point.
Exercise:

Problem: Graph $y = x^2 - 2$

Solution:



Exercise:

Problem: Graph $y = -x^2 + 3$

In the following exercises, determine if the following parabolas open up or down.
Exercise:

Problem: $y = -3x^2 + 3x - 1$

Solution:

down

Exercise:

Problem: $y = 5x^2 + 6x + 3$

Exercise:

Problem: $y = x^2 + 8x - 1$

Solution:

up

Exercise:

Problem: $y = -4x^2 - 7x + 1$

In the following exercises, find ⓐ the axis of symmetry and ⓑ the vertex.

Exercise:

Problem: $y = -x^2 + 6x + 8$

Solution:

ⓐ $x = 3$ ⓑ $(3, 17)$

Exercise:

Problem: $y = 2x^2 - 8x + 1$

In the following exercises, find the *x*- and *y*-intercepts.

Exercise:

Problem: $y = x^2 - 4x + 5$

Solution:

y: $(0, 5)$; x: $(5, 0), (-1, 0)$

Exercise:

Problem: $y = x^2 - 8x + 15$

Exercise:

Problem: $y = x^2 - 4x + 10$

Solution:

y: $(0, 10)$; x: none

Exercise:

Problem: $y = -5x^2 - 30x - 46$

Exercise:

Problem: $y = 16x^2 - 8x + 1$

Solution:

y: $(0,1)$; x: $\left(\frac{1}{4},0\right)$

Exercise:

Problem: $y=x^2+16x+64$

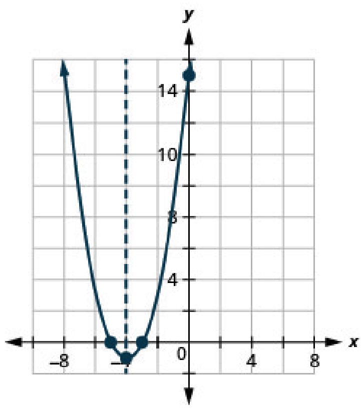
In the following exercises, graph by using intercepts, the vertex, and the axis of symmetry.

Exercise:

Problem: $y=x^2+8x+15$

Solution:

y: $(0,15)$; x: $(-3,0),(-5,0)$;
axis: $x=-4$; vertex: $(-4,-1)$



Exercise:

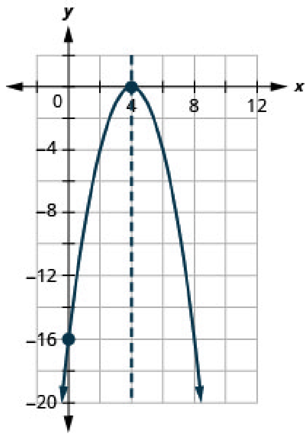
Problem: $y=x^2-2x-3$

Exercise:

Problem: $y=-x^2+8x-16$

Solution:

y: $(0,-16)$; x: $(4,0)$;
axis: $x=4$; vertex: $(4,0)$



Exercise:

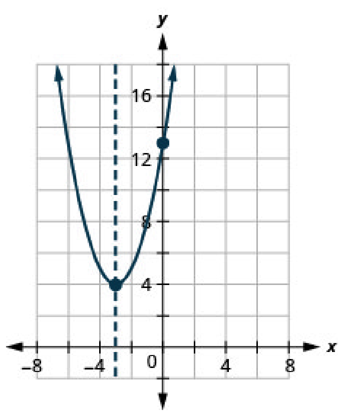
Problem: $y = 4x^2 - 4x + 1$

Exercise:

Problem: $y = x^2 + 6x + 13$

Solution:

y: $(0, 13)$; x: none;
axis: $x = -3$; vertex: $(-3, 4)$



Exercise:

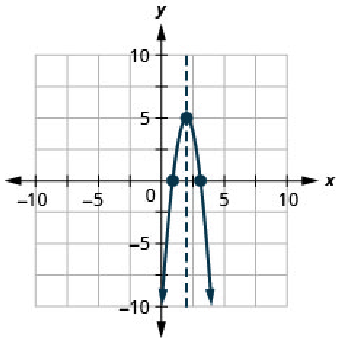
Problem: $y = -2x^2 - 8x - 12$

Exercise:

Problem: $y = -4x^2 + 16x - 11$

Solution:

y: $(0, -11)$ x: $(3.1, 0), (0.9, 0)$;
axis: $x = 2$; vertex: $(2, 5)$



Exercise:

Problem: $y = x^2 + 8x + 10$

In the following exercises, find the minimum or maximum value.

Exercise:

Problem: $y = 7x^2 + 14x + 6$

Solution:

The minimum value is -1 when $x = -1$.

Exercise:

Problem: $y = -3x^2 + 12x - 10$

In the following exercises, solve. Rounding answers to the nearest tenth.

Exercise:

Problem:

A ball is thrown upward from the ground with an initial velocity of 112 ft/sec. Use the quadratic equation $h = -16t^2 + 112t$ to find how long it will take the ball to reach maximum height, and then find the maximum height.

Solution:

In 3.5 seconds the ball is at its maximum height of 196 feet.

Exercise:

Problem:

A daycare facility is enclosing a rectangular area along the side of their building for the children to play outdoors. They need to maximize the area using 180 feet of fencing on three sides of the yard. The quadratic equation $A = -2x^2 + 180x$ gives the area, A, of the yard for the length, x, of the building that will border the yard. Find the length of the building that should border the yard to maximize the area, and then find the maximum area.

Practice Test

Exercise:

Problem: Use the Square Root Property to solve the quadratic equation: $3(w+5)^2 = 27$.

Solution:

$w = -2, -8$

Exercise:

Problem: Use Completing the Square to solve the quadratic equation: $a^2 - 8a + 7 = 23$.

Exercise:

Problem: Use the Quadratic Formula to solve the quadratic equation: $2m^2 - 5m + 3 = 0$.

Solution:

$m = 1, \frac{3}{2}$

Solve the following quadratic equations. Use any method.

Exercise:

Problem: $8v^2 + 3 = 35$

Exercise:

Problem: $3n^2 + 8n + 3 = 0$

Solution:

$n = \frac{-4 \pm \sqrt{7}}{3}$

Exercise:

Problem: $2b^2 + 6b - 8 = 0$

Exercise:

Problem: $x\,(x+3) + 12 = 0$

Solution:

no real solution

Exercise:

Problem: $\frac{4}{3}y^2 - 4y + 3 = 0$

Use the discriminant to determine the number of solutions of each quadratic equation.

Exercise:

Problem: $6p^2 - 13p + 7 = 0$

Solution:

2

Exercise:

Problem: $3q^2 - 10q + 12 = 0$

Solve by factoring, the Square Root Property, or the Quadratic Formula.

Exercise:

Problem: Find two consecutive even numbers whose product is 360.

Solution:

Two consecutive even number are -20 and -18 and 18 and 20.

Exercise:

Problem:

The length of a diagonal of a rectangle is three more than the width. The length of the rectangle is three times the width. Find the length of the diagonal. (Round to the nearest tenth.)

For each parabola, find ⓐ which ways it opens, ⓑ the axis of symmetry, ⓒ the vertex, ⓓ the *x*- and *y*-intercepts, and ⓔ the maximum or minimum value.

Exercise:

Problem: $y = 3x^2 + 6x + 8$

Solution:

ⓐ up ⓑ $x = -1$ ⓒ $(-1, 5)$ ⓓ y: $(0, 8)$; x: none ⓔ minimum value of 5 when $x = -1$

Exercise:

Problem: $y = x^2 - 4$

Exercise:

Problem: $y = x^2 + 10x + 24$

Solution:

ⓐ up ⓑ $x = -5$ ⓒ $(-5, -1)$ ⓓ y; $(0, 24)$; x: $(-6, 0), (-4, 0)$ ⓔ minimum value of -5 when $x = -1$

Exercise:

Problem: $y = -3x^2 + 12x - 8$

Exercise:

Problem: $y = -x^2 - 8x + 16$

Solution:

ⓐ down ⓑ $x = -4$
ⓒ $(-4, 32)$ ⓓ y; $(0, 16)$; x: $(-9.7, 0), (1.7, 0)$
ⓔ maximum value of 32 when $x = -4$

Graph the following parabolas by using intercepts, the vertex, and the axis of symmetry.

Exercise:

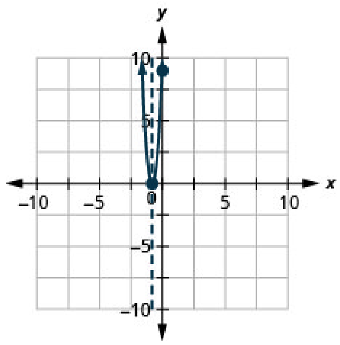
Problem: $y = 2x^2 + 6x + 2$

Exercise:

Problem: $y = 16x^2 + 24x + 9$

Solution:

y: $(0, 9)$; x: $\left(-\frac{3}{4}, 0\right)$;
axis: $x = -\frac{3}{4}$; vertex: $\left(-\frac{3}{4}, 0\right)$



Solve.

Exercise:

Problem:

A water balloon is launched upward at the rate of 86 ft/sec. Using the formula $h = -16t^2 + 86t$, find how long it will take the balloon to reach the maximum height and then find the maximum height. Round to the nearest tenth.

Glossary

axis of symmetry
: The axis of symmetry is the vertical line passing through the middle of the parabola $y = ax^2 + bx + c$.

parabola
: The graph of a quadratic equation in two variables is a parabola.

quadratic equation in two variables
: A quadratic equation in two variables, where $a, b, \text{ and } c$ are real numbers and $a \neq 0$ is an equation of the form $y = ax^2 + bx + c$.

vertex
: The point on the parabola that is on the axis of symmetry is called the *vertex* of the parabola; it is the lowest or highest point on the parabola, depending on whether the parabola opens upwards or downwards.

x-intercepts of a parabola
: The *x*-intercepts are the points on the parabola where $y = 0$.

y-intercept of a parabola
: The *y*-intercept is the point on the parabola where $x = 0$.

Introduction
class="introduction"

Hydroponic systems allow botanists to grow crops without land. (credit: “Izhamwong”/Wikimedia Commons)

As the world population continues to grow, food supplies are becoming less able to meet the increasing demand. At the same time, available resources of fertile soil for growing plants is dwindling. One possible solution—grow plants without soil. Botanists around the world are expanding the potential of hydroponics, which is the process of growing plants without soil. To provide the plants with the nutrients they need, the botanists keep careful growth records. Some growth is described by the types of functions you will explore in this chapter—exponential and logarithmic. You will evaluate and graph these functions, and solve equations using them.

Finding Composite and Inverse Functions: ASE
By the end of this section, you will be able to:

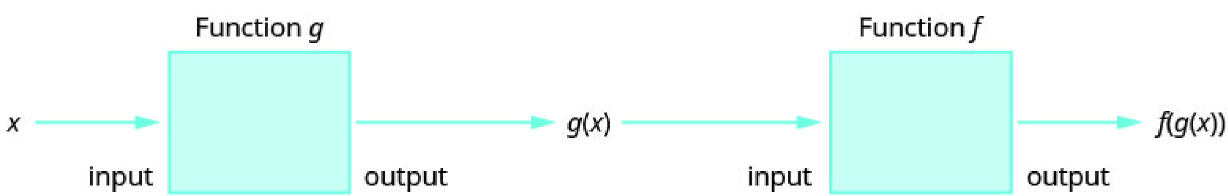
- Find and evaluate composite functions
- Determine whether a function is one-to-one
- Find the inverse of a function

In this chapter, we will introduce two new types of functions, exponential functions and logarithmic functions. These functions are used extensively in business and the sciences as we will see.

Find and Evaluate Composite Functions

Before we introduce the functions, we need to look at another operation on functions called composition. In composition, the output of one function is the input of a second function. For functions f and g, the composition is written $f \circ g$ and is defined by $\left(f \circ g\right)\left(x\right) = f\left(g\left(x\right)\right)$.

We read $f\left(g\left(x\right)\right)$ as "f of g of x."



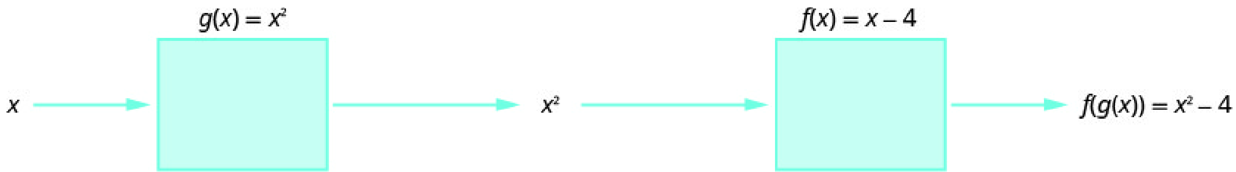
To do a composition, the output of the first function, $g\left(x\right)$, becomes the input of the second function, *f*, and so we must be sure that it is part of the domain of *f*.

Note:
Composition of Functions
The composition of functions *f* and *g* is written $f \cdot g$ and is defined by
Equation:

$$\left(f \circ g\right)\left(x\right) = f\left(g\left(x\right)\right)$$

We read $f\left(g\left(x\right)\right)$ as f of g of *x*.

We have actually used composition without using the notation many times before. When we graphed quadratic functions using translations, we were composing functions. For example, if we first graphed $g\left(x\right) = x^2$ as a parabola and then shifted it down vertically four units, we were using the composition defined by $\left(f \circ g\right)\left(x\right) = f\left(g\left(x\right)\right)$ where $f\left(x\right) = x - 4$.



The next example will demonstrate that $(f \circ g)(x)$, $(g \circ f)(x)$ and $(f \cdot g)(x)$ usually result in different outputs.

Example:
Exercise:

Problem:

For functions $f(x) = 4x - 5$ and $g(x) = 2x + 3$, find: ⓐ $(f \circ g)(x)$, ⓑ $(g \circ f)(x)$, and ⓒ $(f \cdot g)(x)$.

Solution:

ⓐ

Use the definition of $(f \circ g)(x)$.	$(f \circ g)(x) = f(g(x))$
Substitute $2x + 3$ for $g(x)$.	$(f \circ g)(x) = f(2x + 3)$
Find $f(2x + 3)$ where $f(x) = 4x - 5$.	$(f \circ g)(x) = 4(2x + 3) - 5$
Distribute.	$(f \circ g)(x) = 8x + 12 - 5$
Simplify.	$(f \circ g)(x) = 8x + 7$

ⓑ

Use the definition of $(f \circ g)(x)$.	$(g \circ f)(x) = g(f(x))$
Substitute $4x - 5$ for $f(x)$.	$(g \circ f)(x) = g(4x - 5)$
Find $g(4x - 5)$ where $g(x) = 2x + 3$.	$(g \circ f)(x) = 2(4x - 5) + 3$
Distribute.	$(g \circ f)(x) = 8x - 10 + 3$
Simplify.	$(g \circ f)(x) = 8x - 7$

Notice the difference in the result in part ⓐ and part ⓑ.

ⓒ Notice that $(f \cdot g)(x)$ is different than $(f \circ g)(x)$. In part ⓐ we did the composition of the functions. Now in part ⓒ we are not composing them, we are multiplying them.

Use the definition of $(f \cdot g)(x)$.	$(f \cdot g)(x) = f(x) \cdot g(x)$
Substitute $f(x) = 4x - 5$ and $g(x) = 2x + 3$.	$(f \cdot g)(x) = (4x - 5) \cdot (2x + 3)$
Multiply.	$(f \cdot g)(x) = 8x^2 + 2x - 15$

Note:
Exercise:

Problem:

For functions $f(x) = 3x - 2$ and $g(x) = 5x + 1$, find ⓐ $(f \circ g)(x)$ ⓑ $(g \circ f)(x)$ ⓒ $(f \cdot g)(x)$.

Solution:

ⓐ $15x + 1$ ⓑ $15x - 9$
ⓒ $15x^2 - 7x - 2$

Note:
Exercise:

Problem:

For functions $f(x) = 4x - 3$, and $g(x) = 6x - 5$, find ⓐ $(f \circ g)(x)$, ⓑ $(g \circ f)(x)$, and ⓒ $(f \cdot g)(x)$.

Solution:

ⓐ $24x - 23$ ⓑ $24x - 23$
ⓒ $24x^2 - 38x + 15$

In the next example we will evaluate a composition for a specific value.

Example:
Exercise:

Problem:

For functions $f(x) = x^2 - 4$, and $g(x) = 3x + 2$, find: ⓐ $(f \circ g)(-3)$, ⓑ $(g \circ f)(-1)$, and ⓒ $(f \circ f)(2)$.

Solution:

ⓐ

Use the definition of $(f \circ g)(-3)$.	$(f \circ g)(-3) = f(g(-3))$
Find $g(-3)$ where $g(x) = 3x + 2$.	$(f \circ g)(-3) = f(3 \cdot (-3) + 2)$
Simplify.	$(f \circ g)(-3) = f(-7)$
Find $f(-7)$ where $f(x) = x^2 - 4$.	$(f \circ g)(-3) = (-7)^2 - 4$
Simplify.	

	$(f \circ g)(-3) = 45$

ⓑ

Use the definition of $(g \circ f)(-1)$.	$(g \circ f)(-1) = g(f(-1))$
Find $f(-1)$ where $f(x) = x^2 - 4$.	$(g \circ f)(-1) = g((-1)^2 - 4)$
Simplify.	$(g \circ f)(-1) = g(-3)$
Find $g(-3)$ where $g(x) = 3x + 2$.	$(g \circ f)(-1) = 3(-3) + 2$
Simplify.	$(g \circ f)(-1) = -7$

ⓒ

Use the definition of $(f \circ f)(2)$.	$(f \circ f)(2) = f(f(2))$
Find $f(2)$ where $f(x) = x^2 - 4$.	$(f \circ f)(2) = f(2^2 - 4)$
Simplify.	$(f \circ f)(2) = f(0)$

Find $f(0)$ where $f(x) = x^2 - 4$.	$(f \circ f)(2) = 0^2 - 4$
Simplify.	$(f \circ f)(2) = -4$

Note:
Exercise:

Problem:

For functions $f(x) = x^2 - 9$, and $g(x) = 2x + 5$, find ⓐ $(f \circ g)(-2)$, ⓑ $(g \circ f)(-3)$, and ⓒ $(f \circ f)(4)$.

Solution:

ⓐ –8 ⓑ 5 ⓒ 40

Note:
Exercise:

Problem:

For functions $f(x) = x^2 + 1$, and $g(x) = 3x - 5$, find ⓐ $(f \circ g)(-1)$, ⓑ $(g \circ f)(2)$, and ⓒ $(f \circ f)(-1)$.

Solution:

ⓐ 65 ⓑ 10 ⓒ 5

Determine Whether a Function is One-to-One

When we first introduced functions, we said a function is a relation that assigns to each element in its domain exactly one element in the range. For each ordered pair in the relation, each *x*-value is matched with only one *y*-value.

We used the birthday example to help us understand the definition. Every person has a birthday, but no one has two birthdays and it is okay for two people to share a birthday. Since each person

has exactly one birthday, that relation is a function.

A function is **one-to-one** if each value in the range has exactly one element in the domain. For each ordered pair in the function, each *y*-value is matched with only one *x*-value.

Our example of the birthday relation is not a one-to-one function. Two people can share the same birthday. The range value August 2 is the birthday of Liz and June, and so one range value has two domain values. Therefore, the function is not one-to-one.

Note:
One-to-One Function
A function is **one-to-one** if each value in the range corresponds to one element in the domain. For each ordered pair in the function, each *y*-value is matched with only one *x*-value. There are no repeated *y*-values.

Example:
Exercise:

Problem:

For each set of ordered pairs, determine if it represents a function and, if so, if the function is one-to-one.

ⓐ $\{(-3,27),(-2,8),(-1,1),(0,0),(1,1),(2,8),(3,27)\}$ and ⓑ $\{(0,0),(1,1),(4,2),(9,3),(16,4)\}$.

Solution:

ⓐ

$$\{(-3,27),(-2,8),(-1,1),(0,0),(1,1),(2,8),(3,27)\}$$

Each *x*-value is matched with only one *y*-value. So this relation is a function.

But each *y*-value is not paired with only one *x*-value, $(-3,27)$ and $(3,27)$, for example. So this function is not one-to-one.

ⓑ

$$\{(0,0),(1,1),(4,2),(9,3),(16,4)\}$$

Each *x*-value is matched with only one *y*-value. So this relation is a function.

Since each *y*-value is paired with only one *x*-value, this function is one-to-one.

Note:
Exercise:

Problem:

For each set of ordered pairs, determine if it represents a function and if so, is the function one-to-one.

ⓐ $\{(-3,-6),(-2,-4),(-1,-2),(0,0),(1,2),(2,4),(3,6)\}$
ⓑ $\{(-4,8),(-2,4),(-1,2),(0,0),(1,2),(2,4),(4,8)\}$

Solution:

ⓐ One-to-one function
ⓑ Function; not one-to-one

Note:
Exercise:

Problem:

For each set of ordered pairs, determine if it represents a function and if so, is the function one-to-one.

ⓐ $\{(27,-3),(8,-2),(1,-1),(0,0),(1,1),(8,2),(27,3)\}$
ⓑ $\{(7,-3),(-5,-4),(8,0),(0,0),(-6,4),(-2,2),(-1,3)\}$

Solution:

ⓐ Not a function
ⓑ Function; not one-to-one

To help us determine whether a relation is a function, we use the vertical line test. A set of points in a rectangular coordinate system is the graph of a function if every vertical line intersects the graph in at most one point. Also, if any vertical line intersects the graph in more than one point, the graph does not represent a function.

The vertical line is representing an x-value and we check that it intersects the graph in only one y-value. Then it is a function.

To check if a function is one-to-one, we use a similar process. We use a horizontal line and check that each horizontal line intersects the graph in only one point. The horizontal line is representing a y-value and we check that it intersects the graph in only one x-value. If every horizontal line intersects the graph of a function in at most one point, it is a one-to-one function. This is the **horizontal line test**.

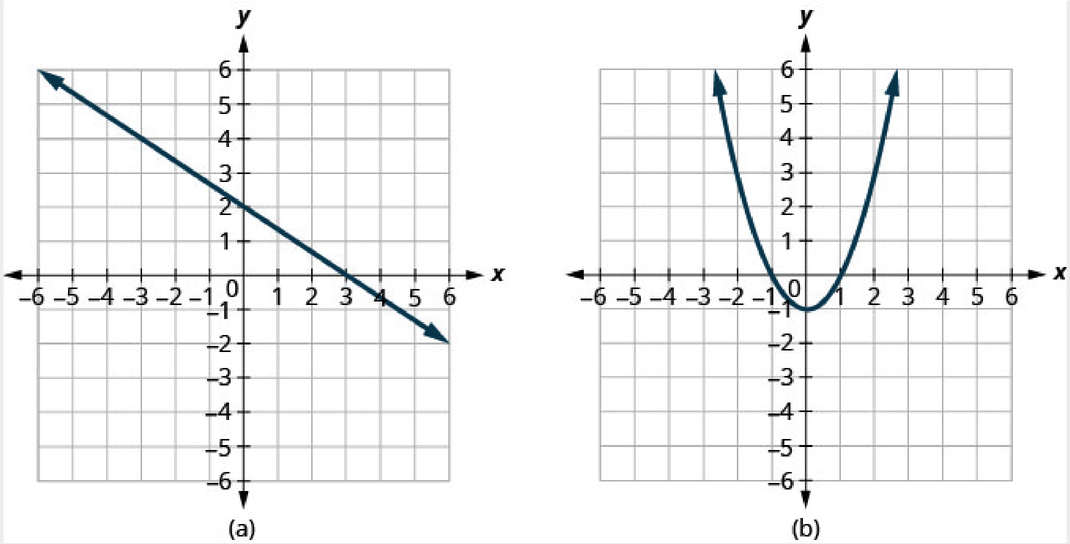
Note:
Horizontal Line Test
If every horizontal line intersects the graph of a function in at most one point, it is a one-to-one function.

We can test whether a graph of a relation is a function by using the vertical line test. We can then tell if the function is one-to-one by applying the horizontal line test.

Example:
Exercise:

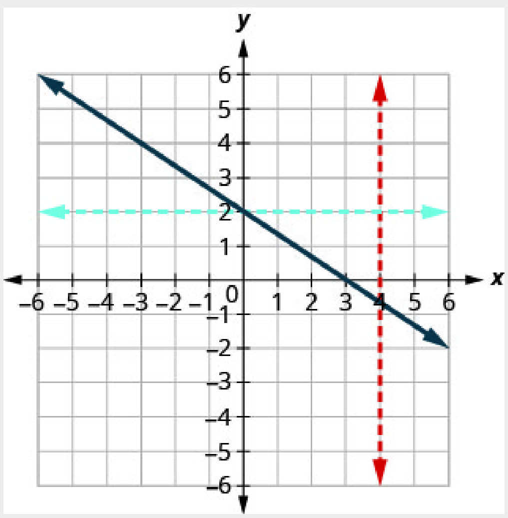
Problem:

Determine ⓐ whether each graph is the graph of a function and, if so, ⓑ whether it is one-to-one.



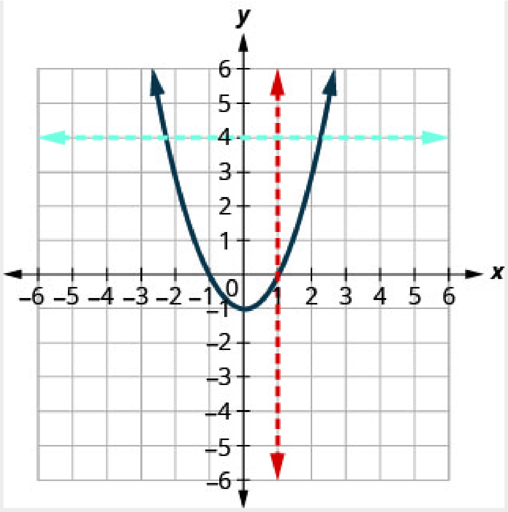
Solution:

ⓐ



Since any vertical line intersects the graph in at most one point, the graph is the graph of a function. Since any horizontal line intersects the graph in at most one point, the graph is the graph of a one-to-one function.

ⓑ

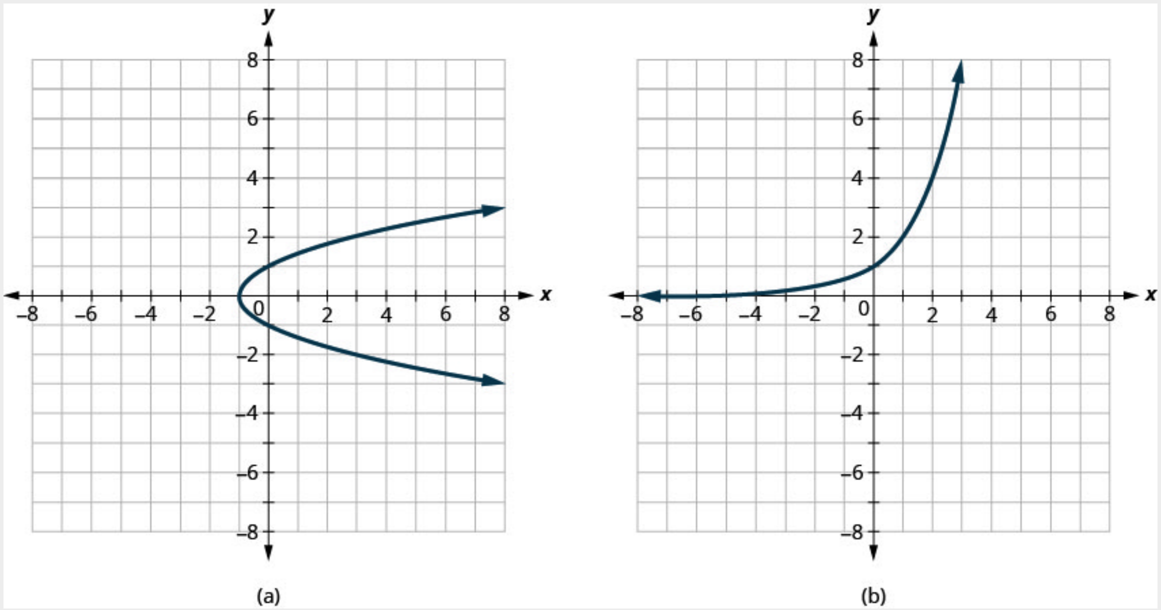


Since any vertical line intersects the graph in at most one point, the graph is the graph of a function. The horizontal line shown on the graph intersects it in two points. This graph does not represent a one-to-one function.

Note:
Exercise:

Problem:

Determine ⓐ whether each graph is the graph of a function and, if so, ⓑ whether it is one-to-one.



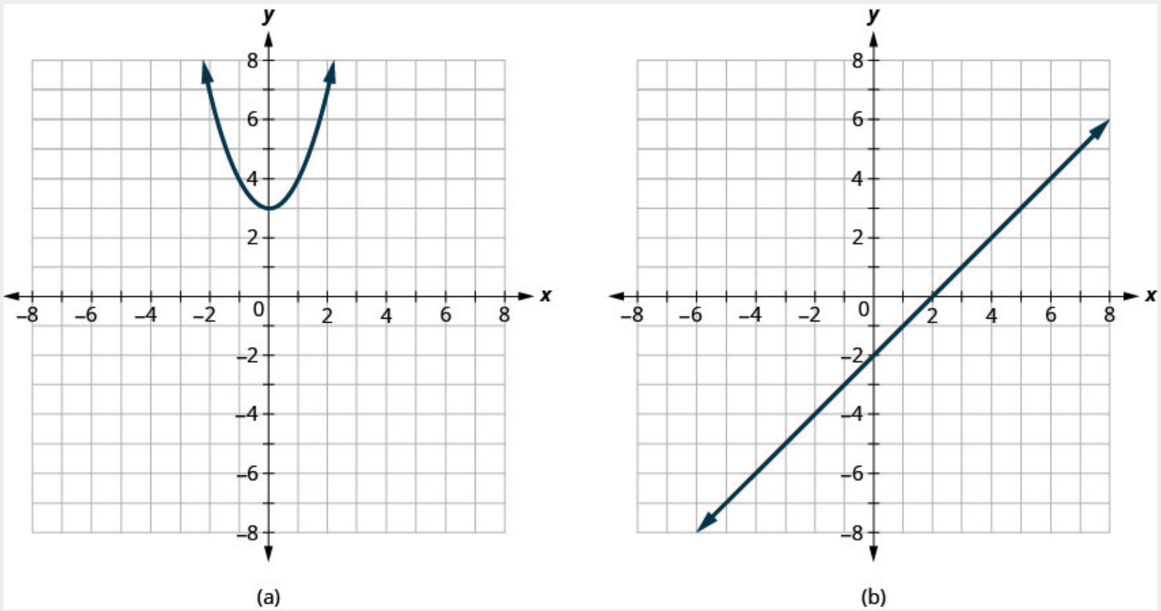
Solution:

ⓐ Not a function ⓑ One-to-one function

Note:
Exercise:

Problem:

Determine ⓐ whether each graph is the graph of a function and, if so, ⓑ whether it is one-to-one.

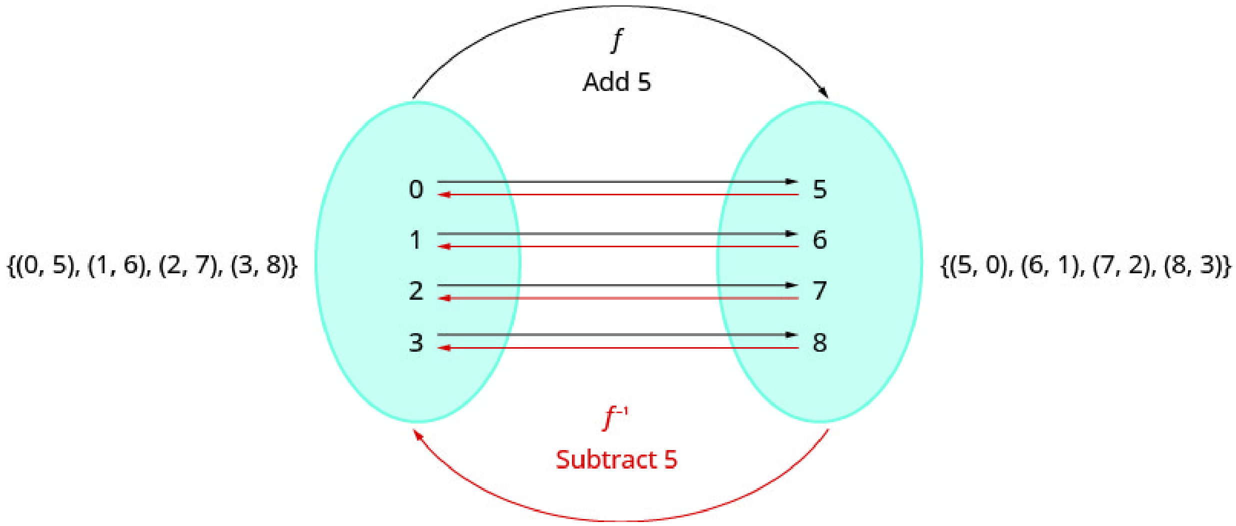


Solution:

ⓐ Function; not one-to-one ⓑ One-to-one function

Find the Inverse of a Function

Let’s look at a one-to one function, f, represented by the ordered pairs $\{(0,5),(1,6),(2,7),(3,8)\}$. For each x-value, f adds 5 to get the y-value. To ‘undo’ the addition of 5, we subtract 5 from each y-value and get back to the original x-value. We can call this “taking the inverse of f” and name the function f^{-1}.
Be careful. The -1 is NOT an exponent because f is not a variable but rather the name of a function.



Notice that that the ordered pairs of f and f^{-1} have their x-values and y-values reversed. The domain of f is the range of f^{-1} and the domain of f^{-1} is the range of f.

Note:
Inverse of a Function Defined by Ordered Pairs
If $f\left(x\right)$ is a one-to-one function whose ordered pairs are of the form $\left(x,y\right)$, then its inverse function $f^{-1}\left(x\right)$ is the set of ordered pairs $\left(y,x\right)$.

In the next example we will find the inverse of a function defined by ordered pairs.

Example:
Exercise:

Problem:

Find the inverse of the function $\{\left(0,3\right),\left(1,5\right),\left(2,7\right),\left(3,9\right)\}$. Determine the domain and range of the inverse function.

Solution:

This function is one-to-one since every x-value is paired with exactly one y-value.

To find the inverse we reverse the x-values and y-values in the ordered pairs of the function.

Function	$\{(0,3),(1,5),(2,7),(3,9)\}$
Inverse Function	$\{(3,0),(5,1),(7,2),(9,3)\}$
Domain of Inverse Function	$\{3,5,7,9\}$
Range of Inverse Function	$\{0,1,2,3\}$

Note:
Exercise:

Problem:

Find the inverse of $\{(0,4),(1,7),(2,10),(3,13)\}$. Determine the domain and range of the inverse function.

Solution:

Inverse function: $\{(4,0),(7,1),(10,2),(13,3)\}$. Domain: $\{4,7,10,13\}$. Range: $\{0,1,2,3\}$.

Note:
Exercise:

Problem:

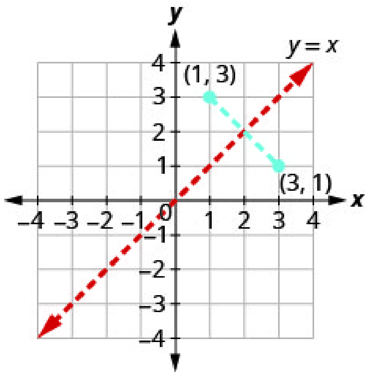
Find the inverse of $\{(-1,4),(-2,1),(-3,0),(-4,2)\}$. Determine the domain and range of the inverse function.

Solution:

Inverse function: $\{(4,-1),(1,-2),(0,-3),(2,-4)\}$. Domain: $\{0,1,2,4\}$. Range: $\{-4,-3,-2,-1\}$.

We just noted that if $f(x)$ is a one-to-one function whose ordered pairs are of the form (x,y), then its inverse function $f^{-1}(x)$ is the set of ordered pairs (y,x).

So if a point (a,b) is on the graph of a function $f(x)$, then the ordered pair (b,a) is on the graph of $f^{-1}(x)$. See [link].



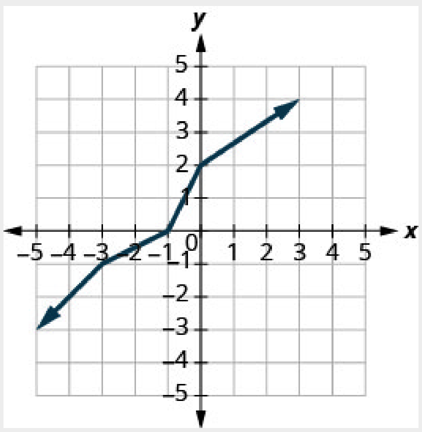
The distance between any two pairs (a, b) and (b, a) is cut in half by the line $y = x$. So we say the points are mirror images of each other through the line $y = x$.

Since every point on the graph of a function $f(x)$ is a mirror image of a point on the graph of $f^{-1}(x)$, we say the graphs are mirror images of each other through the line $y = x$. We will use this concept to graph the inverse of a function in the next example.

Example:
Exercise:

Problem:

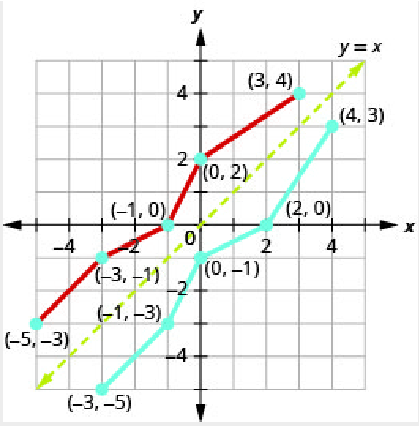
Graph, on the same coordinate system, the inverse of the one-to one function shown.



Solution:

We can use points on the graph to find points on the inverse graph. Some points on the graph are: $(-5, -3)$, $(-3, -1)$, $(-1, 0)$, $(0, 2)$, $(3, 4)$.

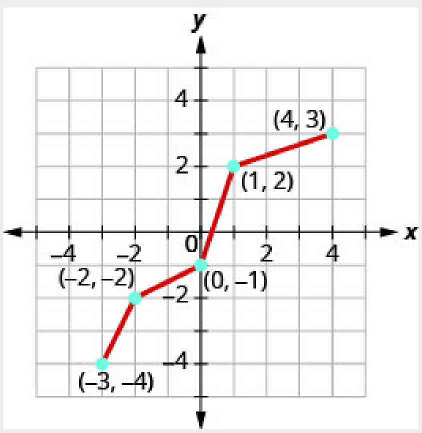
So, the inverse function will contain the points: $(-3, -5)$, $(-1, -3)$, $(0, -1)$, $(2, 0)$, $(4, 3)$.



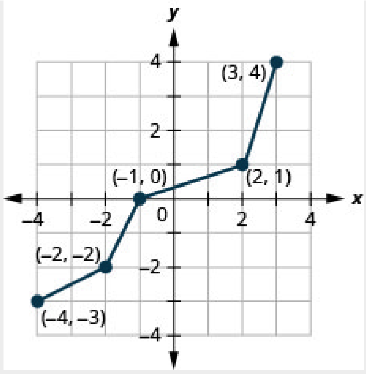
Notice how the graph of the original function and the graph of the inverse functions are mirror images through the line $y = x$.

Note:
Exercise:

Problem: Graph, on the same coordinate system, the inverse of the one-to one function.

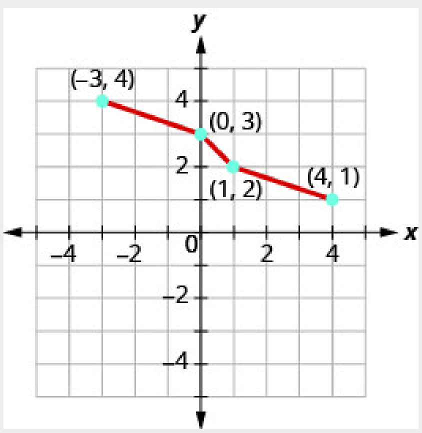


Solution:

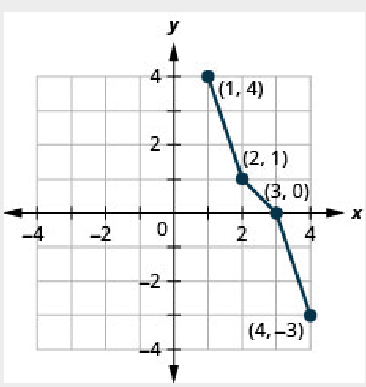


Note:
Exercise:

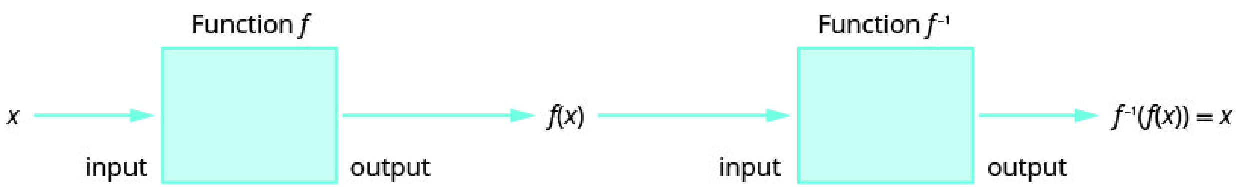
Problem: Graph, on the same coordinate system, the inverse of the one-to one function.



Solution:



When we began our discussion of an inverse function, we talked about how the inverse function ‘undoes’ what the original function did to a value in its domain in order to get back to the original *x*-value.



Note:
Inverse Functions
Equation:

$$\begin{aligned} f^{-1}\left(f\left(x\right)\right) &= x,\ \text{for all } x \text{ in the domain of } f \\ f\left(f^{-1}\left(x\right)\right) &= x,\ \text{for all } x \text{ in the domain of } f^{-1} \end{aligned}$$

We can use this property to verify that two functions are inverses of each other.

Example:
Exercise:

Problem: Verify that $f\left(x\right)=5x-1$ and $g\left(x\right)=\frac{x+1}{5}$ are inverse functions.

Solution:

The functions are inverses of each other if $g\left(f\left(x\right)\right)=x$ and $f\left(g\left(x\right)\right)=x$.

	$g(f(x))\stackrel{?}{=}x$
Substitute $5x-1$ for $f\left(x\right)$.	$g(5x-1)\stackrel{?}{=}x$

Find $g(5x-1)$ where $g(x)=\frac{x+1}{5}$.	$\frac{(5x-1)+1}{5}\stackrel{?}{=}x$
Simplify.	$\frac{5x}{5}\stackrel{?}{=}x$
Simplify.	$x=x$ ✓
	$f(g(x))\stackrel{?}{=}x$
Substitute $\frac{x+1}{5}$ for $g\left(x\right)$.	$f\left(\frac{x+1}{5}\right)\stackrel{?}{=}x$
Find $f\left(\frac{x+1}{5}\right)$ where $f(x)=5x-1$.	$5\left(\frac{x+1}{5}\right)-1\stackrel{?}{=}x$
Simplify.	$x+1-1\stackrel{?}{=}x$
Simplify.	$x=x$ ✓

Since both $g\left(f\left(x\right)\right)=x$ and $f\left(g\left(x\right)\right)=x$ are true, the functions $f\left(x\right)=5x-1$ and $g\left(x\right)=\frac{x+1}{5}$ are inverse functions. That is, they are inverses of each other.

Note:
Exercise:

Problem: Verify that the functions are inverse functions.

$f\left(x\right)=4x-3$ and $g\left(x\right)=\frac{x+3}{4}$.

Solution:

$g(f(x))=x$, and $f(g(x))=x$, so they are inverses.

Note:
Exercise:

Problem: Verify that the functions are inverse functions.

$f\left(x\right)=2x+6$ and $g\left(x\right)=\frac{x-6}{2}$.

Solution:

$g(f(x))=x$, and $f(g(x))=x$, so they are inverses.

We have found inverses of function defined by ordered pairs and from a graph. We will now look at how to find an inverse using an algebraic equation. The method uses the idea that if $f\left(x\right)$ is a one-to-one function with ordered pairs $\left(x,y\right)$, then its inverse function $f^{-1}\left(x\right)$ is the set of ordered pairs $\left(y,x\right)$.

If we reverse the *x* and *y* in the function and then solve for *y*, we get our inverse function.

Example:
How to Find the inverse of a One-to-One Function
Exercise:

Problem: Find the inverse of $f\left(x\right)=4x+7$.

Solution:

Step 1. Substitute *y* for *f*(*x*).	Replace *f*(*x*) with *y*.	$f(x)=4x+7$ $y=4x+7$
Step 2. Interchange the variables *x* and *y*.	Replace *x* with *y* and then *y* with *x*.	$x=4y+7$
Step 3. Solve for *y*.	Subtract 7 from each side. Divide by 4.	$x-7=4y$ $\frac{x-7}{4}=y$

Step 4. Substitute $f^{-1}(x)$ for y.	Replace y with $f^{-1}(x)$.	$\frac{x-7}{4} = f^{-1}(x)$

Step 5. Verify that the functions are inverses.	Show $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$	$f^{-1}(f(x)) \stackrel{?}{=} x$ $f^{-1}(4x+7) \stackrel{?}{=} x$ $\frac{(4x+7)-7}{4} \stackrel{?}{=} x$ $\frac{4x}{4} \stackrel{?}{=} x$ $x = x$ ✓ $f(f^{-1}(x)) \stackrel{?}{=} x$ $f\left(\frac{x-7}{4}\right) \stackrel{?}{=} x$ $4\left(\frac{x-7}{4}\right) + 7 \stackrel{?}{=} x$ $x - 7 + 7 \stackrel{?}{=} x$ $x = x$ ✓

Note:
Exercise:

Problem: Find the inverse of the function $f(x) = 5x - 3$.

Solution:

$f^{-1}(x) = \frac{x+3}{5}$

Note:
Exercise:

Problem: Find the inverse of the function $f(x) = 8x + 5$.

Solution:

$f^{-1}(x) = \frac{x-5}{8}$

We summarize the steps below.

Note:
How to Find the inverse of a One-to-One Function

Substitute y for $f\left(x\right)$.
Interchange the variables x and y.
Solve for y.
Substitute $f^{-1}\left(x\right)$ for y.
Verify that the functions are inverses.

Example:
How to Find the Inverse of a One-to-One Function
Exercise:

Problem: Find the inverse of $f\left(x\right)=\sqrt[5]{2x-3}$.

Solution:

$$\begin{array}{lrcl} & f\left(x\right) & = & \sqrt[5]{2x-3} \\ \text{Substitute } y \text{ for } f\left(x\right). & y & = & \sqrt[5]{2x-3} \\ \text{Interchange the variables } x \text{ and } y. & x & = & \sqrt[5]{2y-3} \\ \text{Solve for } y. & \left(x\right)^{5} & = & \left(\sqrt[5]{2y-3}\right)^{5} \\ & x^{5} & = & 2y-3 \\ & x^{5}+3 & = & 2y \\ & \frac{x^{5}+3}{2} & = & y \\ \text{Substitute } f^{-1}\left(x\right) \text{ for } y. & f^{-1}\left(x\right) & = & \frac{x^{5}+3}{2} \end{array}$$

Verify that the functions are inverses.

$$\begin{aligned} f^{-1}(f(x)) &\stackrel{?}{=} x \\ f^{-1}\left(\sqrt[5]{2x-3}\right) &\stackrel{?}{=} x \\ \frac{\left(\sqrt[5]{2x-3}\right)^5+3}{2} &\stackrel{?}{=} x \\ \frac{2x-3+3}{2} &\stackrel{?}{=} x \\ \frac{2x}{2} &\stackrel{?}{=} x \\ x &= x\checkmark \end{aligned} \qquad \begin{aligned} f\left(f^{-1}(x)\right) &\stackrel{?}{=} x \\ f\left(\frac{x^5+3}{2}\right) &\stackrel{?}{=} x \\ \sqrt[5]{2\left(\frac{x^5+3}{2}\right)-3} &\stackrel{?}{=} x \\ \sqrt[5]{x^5+3-3} &\stackrel{?}{=} x \\ \sqrt[5]{x^5} &\stackrel{?}{=} x \\ x &= x\checkmark \end{aligned}$$

Note:
Exercise:

Problem: Find the inverse of the function $f(x) = \sqrt[5]{3x-2}$.

Solution:

$f^{-1}(x) = \frac{x^5+2}{3}$

Note:
Exercise:

Problem: Find the inverse of the function $f(x) = \sqrt[4]{6x-7}$.

Solution:

$f^{-1}(x) = \frac{x^4+7}{6}$

Key Concepts

- **Composition of Functions:** The composition of functions f and g, is written $f \circ g$ and is defined by
  Equation:

$$(f \circ g)(x) = f(g(x))$$

We read $f(g(x))$ as f of g of x.

- **Horizontal Line Test:** If every horizontal line, intersects the graph of a function in at most one point, it is a one-to-one function.
- **Inverse of a Function Defined by Ordered Pairs:** If $f(x)$ is a one-to-one function whose ordered pairs are of the form (x, y), then its inverse function $f^{-1}(x)$ is the set of ordered pairs (y, x).
- **Inverse Functions:** For every x in the domain of one-to-one function f and f^{-1},
Equation:

$$\begin{aligned} f^{-1}(f(x)) &= x \\ f\left(f^{-1}(x)\right) &= x \end{aligned}$$

- **How to Find the Inverse of a One-to-One Function:**

Substitute*y*for$f(x)$.
Interchange the variables*x*and*y*.
Solve for*y*.
Substitute$f^{-1}(x)$for$y$.
Verify that the functions are inverses.

Practice Makes Perfect

Find and Evaluate Composite Functions

In the following exercises, find ⓐ $(f \circ g)(x)$, ⓑ $(g \circ f)(x)$, and ⓒ $(f \cdot g)(x)$.
Exercise:

Problem: $f(x) = 4x + 3$ and $g(x) = 2x + 5$

Solution:

ⓐ $8x + 23$ ⓑ $8x + 11$ ⓒ
$8x^2 + 26x + 15$

Exercise:

Problem: $f(x) = 3x - 1$ and $g(x) = 5x - 3$

Exercise:

Problem: $f(x) = 6x - 5$ and $g(x) = 4x + 1$

Solution:

ⓐ $24x+1$ ⓑ $24x-19$
ⓒ $24x^2+14x-5$

Exercise:

Problem: $f(x)=2x+7$ and $g(x)=3x-4$

Exercise:

Problem: $f(x)=3x$ and $g(x)=2x^2-3x$

Solution:

ⓐ $6x^2-9x$ ⓑ $18x^2-9x$
ⓒ $6x^3-9x^2$

Exercise:

Problem: $f(x)=2x$ and $g(x)=3x^2-1$

Exercise:

Problem: $f(x)=2x-1$ and $g(x)=x^2+2$

Solution:

ⓐ $2x^2+3$ ⓑ $4x^2-4x+3$
ⓒ $2x^3-x^2+4x-2$

Exercise:

Problem: $f(x)=4x+3$ and $g(x)=x^2-4$

In the following exercises, find the values described.

Exercise:

Problem: For functions $f(x)=2x^2+3$ and $g(x)=5x-1$, find
ⓐ $(f\circ g)(-2)$
ⓑ $(g\circ f)(-3)$
ⓒ $(f\circ f)(-1)$

Solution:

ⓐ 245 ⓑ 104 ⓒ 53

Exercise:

For functions $f\left(x\right)=5x^2-1$ and $g\left(x\right)=4x-1$, find
ⓐ $\left(f\circ g\right)\left(1\right)$
ⓑ $\left(g\circ f\right)\left(-1\right)$
Problem: ⓒ $\left(f\circ f\right)\left(2\right)$

Exercise:

For functions $f\left(x\right)=2x^3$ and $g\left(x\right)=3x^2+2$, find
ⓐ $\left(f\circ g\right)\left(-1\right)$
ⓑ $\left(g\circ f\right)\left(1\right)$
Problem: ⓒ $\left(g\circ g\right)\left(1\right)$

Solution:

ⓐ 250 ⓑ 14 ⓒ 77

Exercise:

For functions $f\left(x\right)=3x^3+1$ and $g\left(x\right)=2x^2-3$, find
ⓐ $\left(f\circ g\right)\left(-2\right)$
ⓑ $\left(g\circ f\right)\left(-1\right)$
Problem: ⓒ $\left(g\circ g\right)\left(1\right)$

Determine Whether a Function is One-to-One

In the following exercises, determine if the set of ordered pairs represents a function and if so, is the function one-to-one.

Exercise:

Problem: $\{(-3,9),(-2,4),(-1,1),(0,0),(1,1),(2,4),(3,9)\}$

Solution:

Function; not one-to-one

Exercise:

Problem: $\{(9,-3),(4,-2),(1,-1),(0,0),(1,1),(4,2),(9,3)\}$

Exercise:

Problem: $\{(-3,-5),(-2,-3),(-1,-1),(0,1),(1,3),(2,5),(3,7)\}$

Solution:

One-to-one function

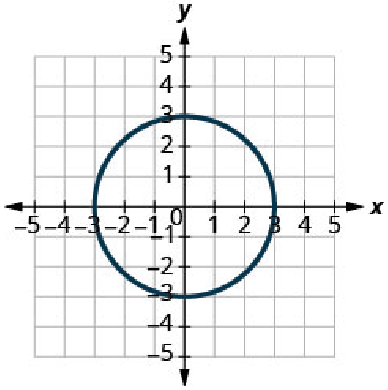
Exercise:

Problem: $\{(5,3),(4,2),(3,1),(2,0),$
$(1,-1),(0,-2),(-1,-3)\}$

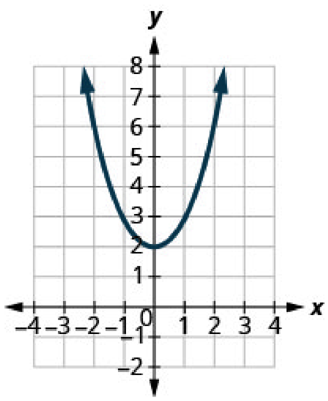
In the following exercises, determine whether each graph is the graph of a function and if so, is it one-to-one.

Exercise:

Problem: ⓐ



ⓑ

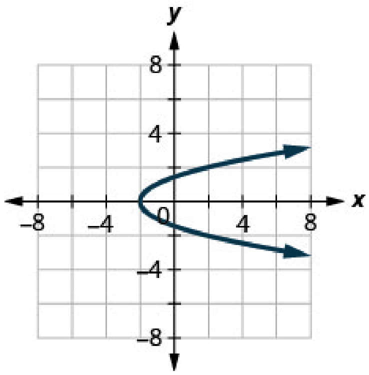


Solution:

ⓐ Not a function ⓑ Function; not one-to-one

Exercise:

Problem: ⓐ



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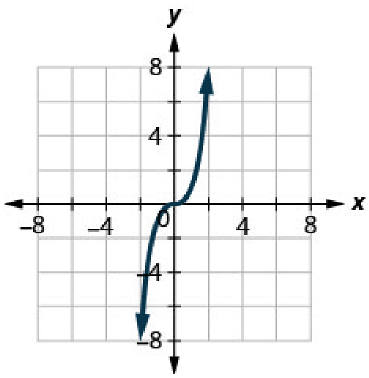
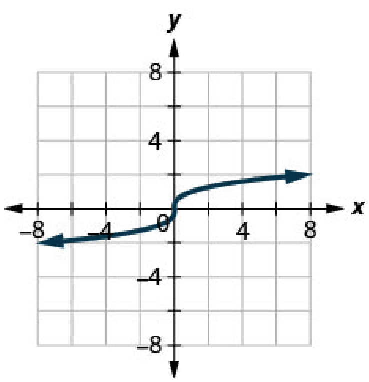
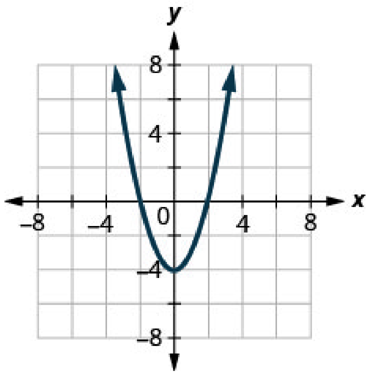


Exercise:

Problem: ⓐ



ⓑ

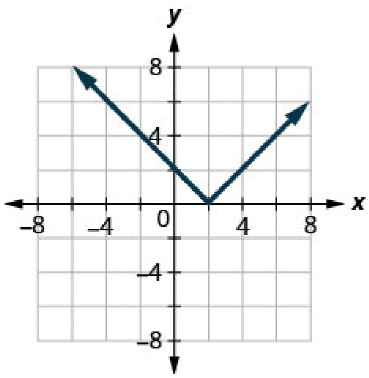


Solution:

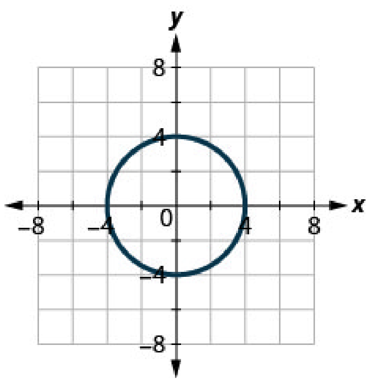
ⓐ One-to-one function
ⓑ Function; not one-to-one

Exercise:

Problem: ⓐ



ⓑ



In the following exercises, find the inverse of each function. Determine the domain and range of the inverse function.

Exercise:

Problem: $\{(2,1),(4,2),(6,3),(8,4)\}$

Solution:

Inverse function: $\{(1,2),(2,4),(3,6),(4,8)\}$. Domain: $\{1,2,3,4\}$. Range: $\{2,4,6,8\}$.

Exercise:

Problem: $\{(6,2),(9,5),(12,8),(15,11)\}$

Exercise:

Problem: $\{(0,-2),(1,3),(2,7),(3,12)\}$

Solution:

Inverse function: $\{(-2,0),(3,1),(7,2),(12,3)\}$. Domain: $\{-2,3,7,12\}$. Range: $\{0,1,2,3\}$.

Exercise:

Problem: $\{(0,0),(1,1),(2,4),(3,9)\}$

Exercise:

Problem: $\{(-2,-3),(-1,-1),(0,1),(1,3)\}$

Solution:

Inverse function: $\{(-3,-2),(-1,-1),(1,0),(3,1)\}$. Domain: $\{-3,-1,1,3\}$. Range: $\{-2,-1,0,1\}$.

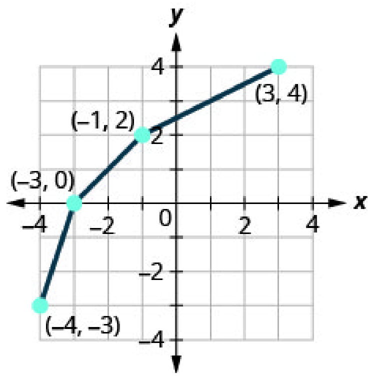
Exercise:

Problem: $\{(5,3),(4,2),(3,1),(2,0)\}$

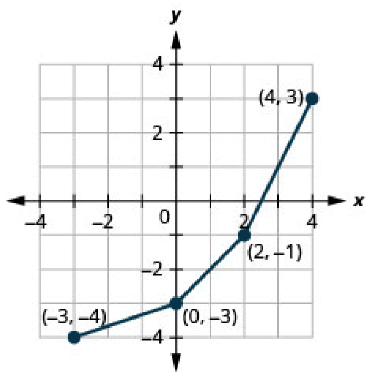
In the following exercises, graph, on the same coordinate system, the inverse of the one-to-one function shown.

Exercise:

Problem:

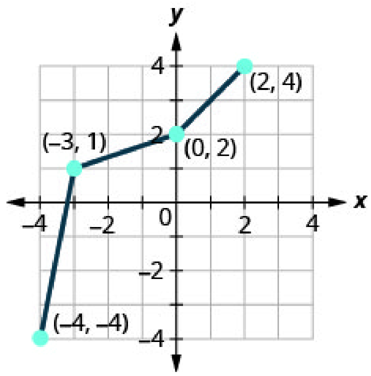


Solution:



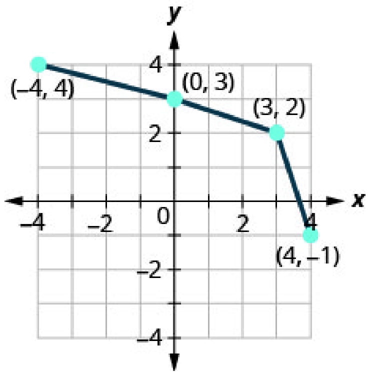
Exercise:

Problem:

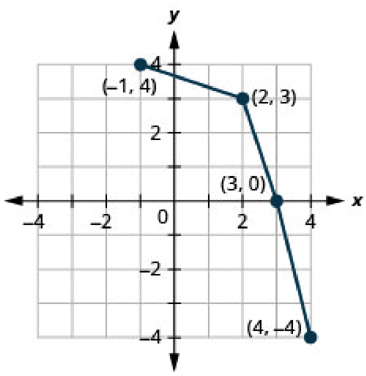


Exercise:

Problem:

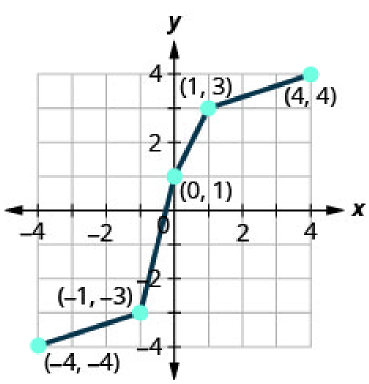


Solution:



Exercise:

Problem:



In the following exercises, determine whether or not the given functions are inverses.

Exercise:

Problem: $f\left(x\right)=x+8$ and $g\left(x\right)=x-8$

Solution:

$g(f(x)) = x$, and $f(g(x)) = x$, so they are inverses.

Exercise:

Problem: $f\left(x\right) = x - 9$ and $g\left(x\right) = x + 9$

Exercise:

Problem: $f\left(x\right) = 7x$ and $g\left(x\right) = \frac{x}{7}$

Solution:

$g(f(x)) = x$, and $f(g(x)) = x$, so they are inverses.

Exercise:

Problem: $f\left(x\right) = \frac{x}{11}$ and $g\left(x\right) = 11x$

Exercise:

Problem: $f\left(x\right) = 7x + 3$ and $g\left(x\right) = \frac{x-3}{7}$

Solution:

$g(f(x)) = x$, and $f(g(x)) = x$, so they are inverses.

Exercise:

Problem: $f\left(x\right) = 5x - 4$ and $g\left(x\right) = \frac{x-4}{5}$

Exercise:

Problem: $f\left(x\right) = \sqrt{x+2}$ and $g\left(x\right) = x^2 - 2$

Solution:

$g(f(x)) = x$, and $f(g(x)) = x$, so they are inverses (for nonnegative x).

Exercise:

Problem: $f\left(x\right) = \sqrt[3]{x-4}$ and $g\left(x\right) = x^3 + 4$

In the following exercises, find the inverse of each function.

Exercise:

Problem: $f\left(x\right)=x-12$

Solution:

$f^{-1}(x)=x+12$

Exercise:

Problem: $f\left(x\right)=x+17$

Exercise:

Problem: $f\left(x\right)=9x$

Solution:

$f^{-1}(x)=\frac{x}{9}$

Exercise:

Problem: $f\left(x\right)=8x$

Exercise:

Problem: $f\left(x\right)=\frac{x}{6}$

Solution:

$f^{-1}(x)=6x$

Exercise:

Problem: $f\left(x\right)=\frac{x}{4}$

Exercise:

Problem: $f\left(x\right)=6x-7$

Solution:

$f^{-1}(x)=\frac{x+7}{6}$

Exercise:

Problem: $f\left(x\right)=7x-1$

Exercise:

Problem: $f\left(x\right)=-2x+5$

Solution:

$f^{-1}(x)=\frac{x-5}{-2}$

Exercise:

Problem: $f\left(x\right)=-5x-4$

Exercise:

Problem: $f\left(x\right)=x^2+6,\,x\geq 0$

Solution:

$f^{-1}(x)=\sqrt{x-6}$

Exercise:

Problem: $f\left(x\right)=x^2-9,\,x\geq 0$

Exercise:

Problem: $f\left(x\right)=x^3-4$

Solution:

$f^{-1}(x)=\sqrt[3]{x+4}$

Exercise:

Problem: $f\left(x\right)=x^3+6$

Exercise:

Problem: $f\left(x\right)=\frac{1}{x+2}$

Solution:

$f^{-1}(x)=\frac{1}{x}-2$

Exercise:

Problem: $f\left(x\right)=\frac{1}{x-6}$

Exercise:

Problem: $f\left(x\right)=\sqrt{x-2}, x\ge 2$

Solution:

$f^{-1}(x)=x^2+2, x\ge 0$

Exercise:

Problem: $f\left(x\right)=\sqrt{x+8}, x\ge -8$

Exercise:

Problem: $f\left(x\right)=\sqrt[3]{x-3}$

Solution:

$f^{-1}(x)=x^3+3$

Exercise:

Problem: $f\left(x\right)=\sqrt[3]{x+5}$

Exercise:

Problem: $f\left(x\right)=\sqrt[4]{9x-5}, x\ge \frac{5}{9}$

Solution:

$f^{-1}(x)=\frac{x^4+5}{9}, x\ge 0$

Exercise:

Problem: $f\left(x\right)=\sqrt[4]{8x-3}, x\ge \frac{3}{8}$

Exercise:

Problem: $f\left(x\right)=\sqrt[5]{-3x+5}$

Solution:

$f^{-1}(x)=\frac{x^5-5}{-3}$

Exercise:

Problem: $f\left(x\right)=\sqrt[5]{-4x-3}$

Writing Exercises

Exercise:

Problem:

Explain how the graph of the inverse of a function is related to the graph of the function.

Solution:

Answers will vary.

Exercise:

Problem:

Explain how to find the inverse of a function from its equation. Use an example to demonstrate the steps.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
find and evaluate composite functions.			
determine whether a function is one-to-one.			
find the inverse of a function.			

ⓑ If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no—I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

Glossary

one-to-one function

A function is one-to-one if each value in the range has exactly one element in the domain. For each ordered pair in the function, each y-value is matched with only one x-value.

Evaluate and Graph Exponential Functions: ASE
By the end of this section, you will be able to:

- Graph exponential functions
- Solve Exponential equations
- Use exponential models in applications

Graph Exponential Functions

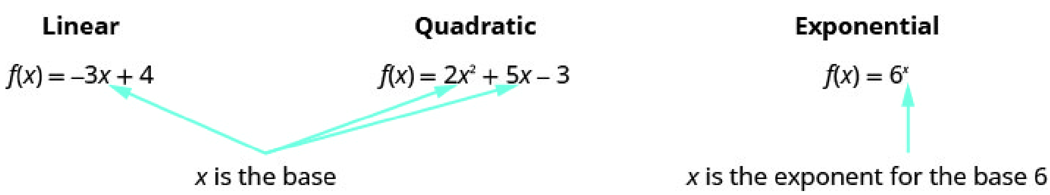
The functions we have studied so far do not give us a model for many naturally occurring phenomena. From the growth of populations and the spread of viruses to radioactive decay and compounding interest, the models are very different from what we have studied so far. These models involve exponential functions.

An **exponential function** is a function of the form $f\left(x\right)=a^{x}$ where $a>0$ and $a\ne 1$.

Note:
Exponential Function
An exponential function, where $a>0$ and $a\ne 1$, is a function of the form
Equation:

$$f\left(x\right)=a^{x}$$

Notice that in this function, the variable is the exponent. In our functions so far, the variables were the base.



Our definition says $a\ne 1$. If we let $a=1$, then $f\left(x\right)=a^{x}$ becomes $f\left(x\right)=1^{x}$. Since $1^{x}=1$ for all real numbers, $f\left(x\right)=1$. This is the constant function.

Our definition also says $a>0$. If we let a base be negative, say -4, then $f\left(x\right)=\left(-4\right)^{x}$ is not a real number when $x=\frac{1}{2}$.
Equation:

$$\begin{array}{rcl} f\left(x\right) & = & \left(-4\right)^{x} \\ f\left(\frac{1}{2}\right) & = & \left(-4\right)^{\frac{1}{2}} \\ f\left(\frac{1}{2}\right) & = & \sqrt{-4} \quad \text{not a real number} \end{array}$$

In fact, $f(x) = (-4)^x$ would not be a real number any time x is a fraction with an even denominator. So our definition requires $a > 0$.

By graphing a few exponential functions, we will be able to see their unique properties.

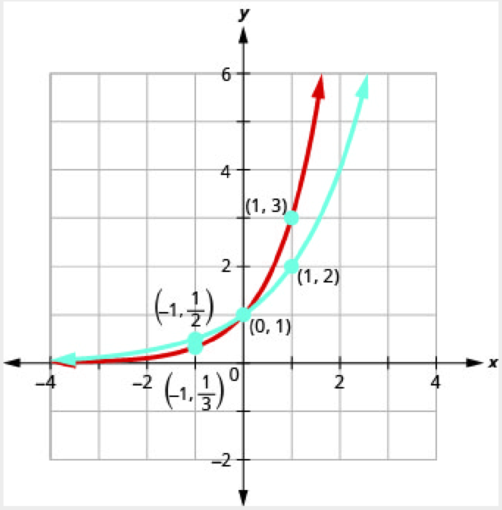
Example:
Exercise:

Problem: On the same coordinate system graph $f(x) = 2^x$ and $g(x) = 3^x$.

Solution:

We will use point plotting to graph the functions.

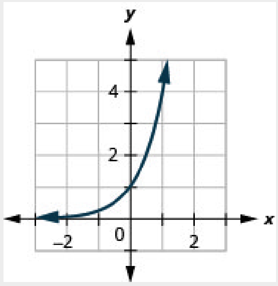
x	$f(x) = 2^x$	$(x, f(x))$		$g(x) = 3^x$	$(x, g(x))$
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$\left(-2, \frac{1}{4}\right)$		$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$	$\left(-2, \frac{1}{9}\right)$
-1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$		$3^{-1} = \frac{1}{3^1} = \frac{1}{3}$	$\left(-1, \frac{1}{3}\right)$
0	$2^0 = 1$	(0, 1)		$3^0 = 1$	(0, 1)
1	$2^1 = 2$	(1, 2)		$3^1 = 3$	(1, 3)
2	$2^2 = 4$	(2, 4)		$3^2 = 9$	(2, 9)
3	$2^3 = 8$	(3, 8)		$3^3 = 27$	(3, 27)



Note:
Exercise:

Problem: Graph: $f\left(x\right)=4^{x}$.

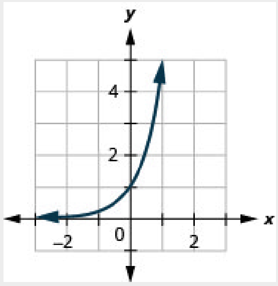
Solution:



Note:
Exercise:

Problem: Graph: $g\left(x\right)=5^{x}$.

Solution:



If we look at the graphs from the previous Example and Try Its, we can identify some of the properties of exponential functions.

The graphs of $f\left(x\right)=2^{x}$ and $g\left(x\right)=3^{x}$, as well as the graphs of $f\left(x\right)=4^{x}$ and $g\left(x\right)=5^{x}$, all have the same basic shape. This is the shape we expect from an exponential function where $a>1$.

We notice, that for each function, the graph contains the point $\left(0,1\right)$. This make sense because $a^{0}=1$ for any a.

The graph of each function, $f\left(x\right)={a}^{x}$ also contains the point $\left(1,a\right)$. The graph of $f\left(x\right)={2}^{x}$ contained $\left(1,2\right)$ and the graph of $g\left(x\right)={3}^{x}$ contained $\left(1,3\right)$. This makes sense as ${a}^{1}=a$.

Notice too, the graph of each function $f\left(x\right)={a}^{x}$ also contains the point $\left(-1,\frac{1}{a}\right)$. The graph of $f\left(x\right)={2}^{x}$ contained $\left(-1,\frac{1}{2}\right)$ and the graph of $g\left(x\right)={3}^{x}$ contained $\left(-1,\frac{1}{3}\right)$. This makes sense as ${a}^{-1}=\frac{1}{a}$.

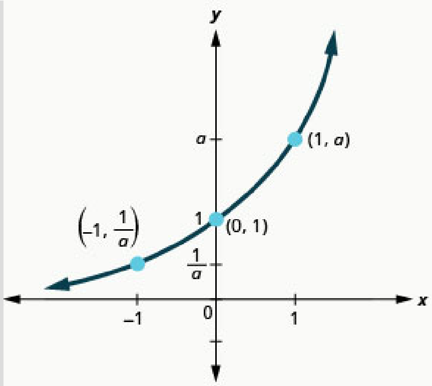
What is the domain for each function? From the graphs we can see that the domain is the set of all real numbers. There is no restriction on the domain. We write the domain in interval notation as $\left(-\infty ,\infty \right)$.

Look at each graph. What is the range of the function? The graph never hits the x-axis. The range is all positive numbers. We write the range in interval notation as $\left(0,\infty \right)$.

Whenever a graph of a function approaches a line but never touches it, we call that line an **asymptote**. For the exponential functions we are looking at, the graph approaches the x-axis very closely but will never cross it, we call the line $y=0$, the *x*-axis, a horizontal asymptote.

Note:
Properties of the Graph of $f\left(x\right)={a}^{x}$ when $a>1$

Domain	$\left(-\infty ,\infty \right)$
Range	$\left(0,\infty \right)$
x-intercept	None
y-intercept	$\left(0,1\right)$
Contains	$\left(1,a\right),\ \left(-1,\frac{1}{a}\right)$
Asymptote	x-axis, the line $y=0$



Our definition of an exponential function $f\left(x\right)={a}^{x}$ says $a>0$, but the examples and discussion so far has been about functions where $a>1$. What happens when $0<a<1$? The next example will explore this possibility.

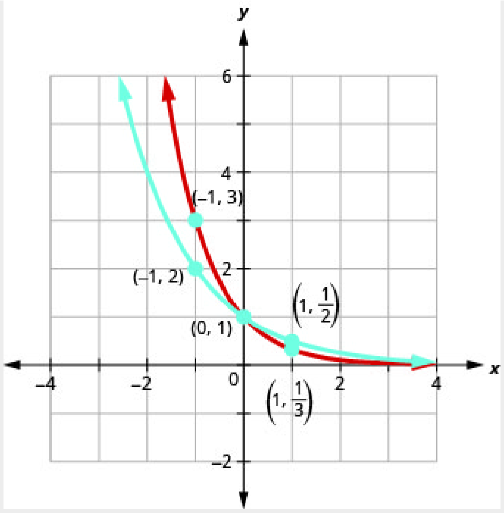
Example:
Exercise:

Problem: On the same coordinate system, graph $f\left(x\right)={\left(\frac{1}{2}\right)}^{x}$ and $g\left(x\right)={\left(\frac{1}{3}\right)}^{x}$.

Solution:

We will use point plotting to graph the functions.

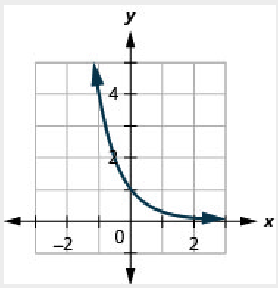
x	$f(x)=\left(\frac{1}{2}\right)^x$	$(x, f(x))$		$g(x)=\left(\frac{1}{3}\right)^x$	$(x, g(x))$
–2	$\left(\frac{1}{2}\right)^{-2}=2^2=4$	$(-2, 4)$		$\left(\frac{1}{3}\right)^{-2}=3^2=9$	$(-2, 9)$
–1	$\left(\frac{1}{2}\right)^{-1}=2^1=2$	$(-1, 2)$		$\left(\frac{1}{3}\right)^{-1}=3^1=3$	$(-1, 3)$
0	$\left(\frac{1}{2}\right)^{0}=1$	$(0, 1)$		$\left(\frac{1}{3}\right)^{0}=1$	$(0, 1)$
1	$\left(\frac{1}{2}\right)^{1}=\frac{1}{2}$	$\left(1, \frac{1}{2}\right)$		$\left(\frac{1}{3}\right)^{1}=\frac{1}{3}$	$\left(1, \frac{1}{3}\right)$
2	$\left(\frac{1}{2}\right)^{2}=\frac{1}{4}$	$\left(2, \frac{1}{4}\right)$		$\left(\frac{1}{3}\right)^{2}=\frac{1}{9}$	$\left(2, \frac{1}{9}\right)$
3	$\left(\frac{1}{2}\right)^{3}=\frac{1}{8}$	$\left(3, \frac{1}{8}\right)$		$\left(\frac{1}{3}\right)^{3}=\frac{1}{27}$	$\left(3, \frac{1}{27}\right)$



Note:
Exercise:

Problem: Graph: $f(x) = \left(\frac{1}{4}\right)^x$.

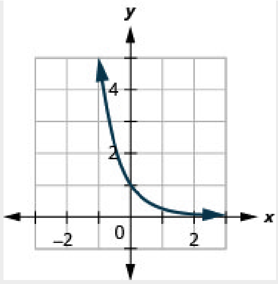
Solution:



Note:
Exercise:

Problem: Graph: $g(x) = \left(\frac{1}{5}\right)^x$.

Solution:



Now let’s look at the graphs from the previous Example and Try Its so we can now identify some of the properties of exponential functions where $0 < a < 1$.

The graphs of $f(x) = \left(\frac{1}{2}\right)^x$ and $g(x) = \left(\frac{1}{3}\right)^x$ as well as the graphs of $f(x) = \left(\frac{1}{4}\right)^x$ and $g(x) = \left(\frac{1}{5}\right)^x$ all have the same basic shape. While this is the shape we expect from an exponential function where $0 < a < 1$, the graphs go down from left to right while the previous graphs, when $a > 1$, went from up from left to right.

We notice that for each function, the graph still contains the point (0, 1). This make sense because $a^0 = 1$ for any a.

As before, the graph of each function, $f(x) = a^x$, also contains the point $(1, a)$. The graph of $f(x) = \left(\frac{1}{2}\right)^x$ contained $\left(1, \frac{1}{2}\right)$ and the graph of $g(x) = \left(\frac{1}{3}\right)^x$ contained $\left(1, \frac{1}{3}\right)$. This makes sense as $a^1 = a$.

Notice too that the graph of each function, $f(x) = a^x$, also contains the point $\left(-1, \frac{1}{a}\right)$. The graph of $f(x) = \left(\frac{1}{2}\right)^x$ contained $(-1, 2)$ and the graph of $g(x) = \left(\frac{1}{3}\right)^x$ contained $(-1, 3)$. This makes sense as $a^{-1} = \frac{1}{a}$.

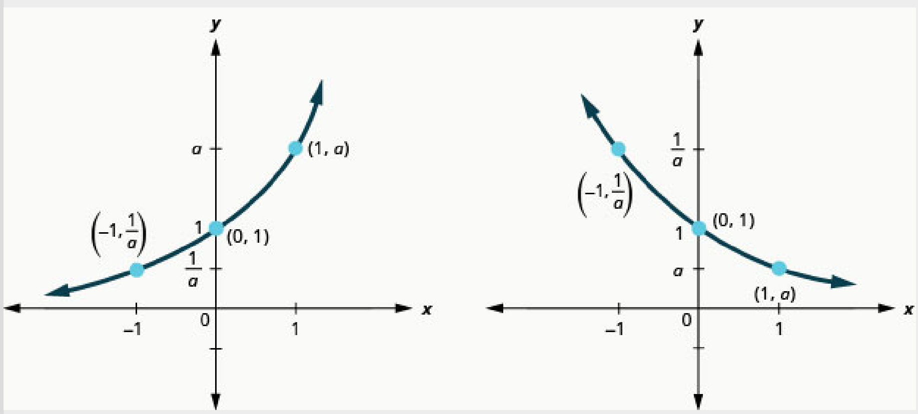
What is the domain and range for each function? From the graphs we can see that the domain is the set of all real numbers and we write the domain in interval notation as $(-\infty, \infty)$. Again, the graph never hits the x-axis. The range is all positive numbers. We write the range in interval notation as $(0, \infty)$.

We will summarize these properties in the chart below. Which also include when $a > 1$.

Note:
Properties of the Graph of $f(x) = a^x$

when $a > 1$	**when** $0 < a < 1$

when $a > 1$		**when** $0 < a < 1$	
Domain	$(-\infty, \infty)$	Domain	$(-\infty, \infty)$
Range	$(0, \infty)$	Range	$(0, \infty)$
x-intercept	none	x-intercept	none
y-intercept	$(0, 1)$	y-intercept	$(0, 1)$
Contains	$(1, a),\ \left(-1, \frac{1}{a}\right)$	Contains	$(1, a),\ \left(-1, \frac{1}{a}\right)$
Asymptote	x-axis, the line $y = 0$	Asymptote	x-axis, the line $y = 0$
Basic shape	increasing	Basic shape	decreasing



It is important for us to notice that both of these graphs are one-to-one, as they both pass the horizontal line test. This means the exponential function will have an inverse. We will look at this later.

When we graphed quadratic functions, we were able to graph using translation rather than just plotting points. Will that work in graphing exponential functions?

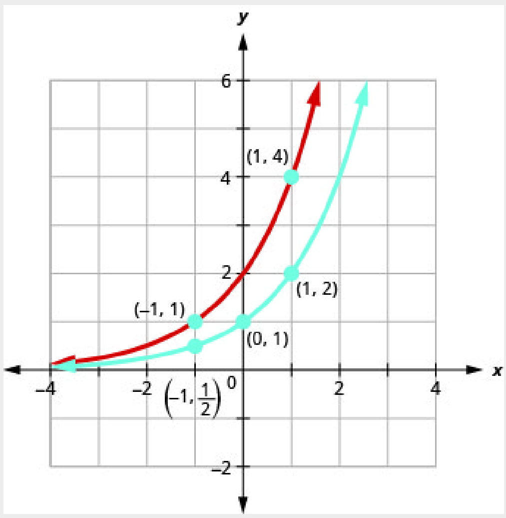
Example:
Exercise:

Problem: On the same coordinate system graph $f\left(x\right) = 2^x$ and $g\left(x\right) = 2^{x+1}$.

Solution:

We will use point plotting to graph the functions.

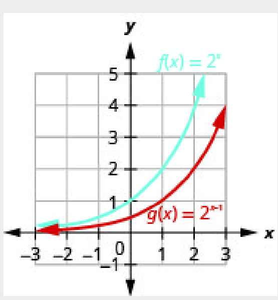
x	$f(x) = 2^x$	$(x, f(x))$		$g(x) = 2^{x+1}$	$(x, g(x))$
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$\left(-2, \frac{1}{4}\right)$		$2^{-2+1} = \frac{1}{2^1} = \frac{1}{2}$	$\left(-2, \frac{1}{2}\right)$
-1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$		$2^{-1+1} = 2^0 = 1$	$(-1, 1)$
0	$2^0 = 1$	$(0, 1)$		$2^{0+1} = 2^1 = 2$	$(0, 2)$
1	$2^1 = 2$	$(1, 2)$		$2^{1+1} = 2^2 = 4$	$(1, 4)$
2	$2^2 = 4$	$(2, 4)$		$2^{2+1} = 2^3 = 8$	$(2, 8)$
3	$2^3 = 8$	$(3, 8)$		$2^{3+1} = 2^4 = 16$	$(3, 16)$



Note:
Exercise:

Problem: On the same coordinate system, graph: $f(x) = 2^x$ and $g(x) = 2^{x-1}$.

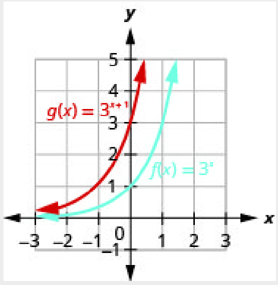
Solution:



Note:
Exercise:

Problem: On the same coordinate system, graph: $f(x) = 3^x$ and $g(x) = 3^{x+1}$.

Solution:



Looking at the graphs of the functions $f(x) = 2^x$ and $g(x) = 2^{x+1}$ in the last example, we see that adding one in the exponent caused a horizontal shift of one unit to the left. Recognizing this pattern allows us to graph other functions with the same pattern by translation.

Let’s now consider another situation that might be graphed more easily by translation, once we recognize the pattern.

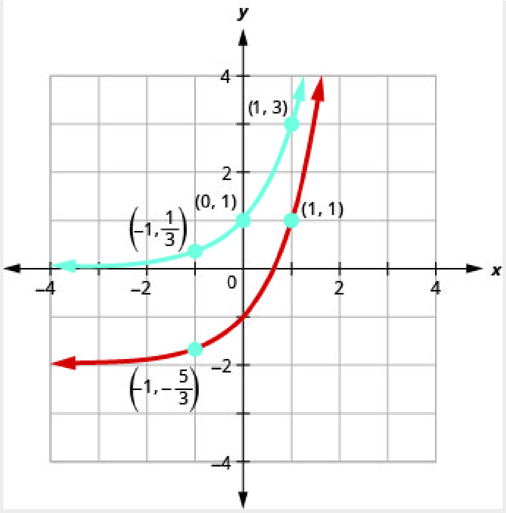
Example:
Exercise:

Problem: On the same coordinate system graph $f(x) = 3^x$ and $g(x) = 3^x - 2$.

Solution:

We will use point plotting to graph the functions.

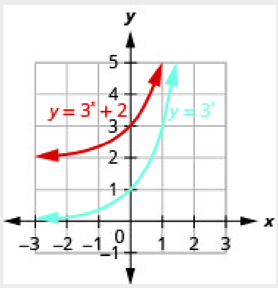
x	$f(x) = 3^x$	$(x, g(x))$		$g(x) = 3^x - 2$	$(x, g(x))$
-2	$3^{-2} = \frac{1}{9}$	$\left(-2, \frac{1}{9}\right)$		$3^{-2} - 2 = \frac{1}{9} - 2 = -\frac{17}{9}$	$\left(-2, -\frac{17}{9}\right)$
-1	$3^{-1} = \frac{1}{3}$	$\left(-1, \frac{1}{3}\right)$		$3^{-1} - 2 = \frac{1}{3} - 2 = -\frac{5}{3}$	$\left(-1, -\frac{5}{3}\right)$
0	$3^0 = 1$	$(0, 1)$		$3^0 - 2 = 1 - 2 = -1$	$(0, -1)$
1	$3^1 = 3$	$(1, 3)$		$3^1 - 2 = 3 - 2 = 1$	$(1, 1)$
2	$3^2 = 9$	$(2, 9)$		$3^2 - 2 = 9 - 2 = 7$	$(2, 8)$



Note:
Exercise:

Problem: On the same coordinate system, graph: $f\left(x\right)=3^{x}$ and $g\left(x\right)=3^{x}+2$.

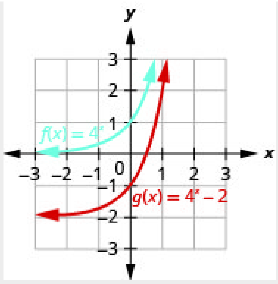
Solution:



Note:
Exercise:

Problem: On the same coordinate system, graph: $f\left(x\right)=4^{x}$ and $g\left(x\right)=4^{x}-2$.

Solution:



Looking at the graphs of the functions $f\left(x\right)=3^x$ and $g\left(x\right)=3^x-2$ in the last example, we see that subtracting 2 caused a vertical shift of down two units. Notice that the horizontal asymptote also shifted down 2 units. Recognizing this pattern allows us to graph other functions with the same pattern by translation.

All of our exponential functions have had either an integer or a rational number as the base. We will now look at an exponential function with an irrational number as the base.

Before we can look at this exponential function, we need to define the irrational number, *e*. This number is used as a base in many applications in the sciences and business that are modeled by exponential functions. The number is defined as the value of $\left(1+\frac{1}{n}\right)^n$ as *n* gets larger and larger. We say, as *n* approaches infinity, or increases without bound. The table shows the value of $\left(1+\frac{1}{n}\right)^n$ for several values of n.

n	$\left(1+\frac{1}{n}\right)^n$
1	2
2	2.25
5	2.48832
10	2.59374246
100	2.704813829…
1,000	2.716923932…
10,000	2.718145927…
100,000	2.718268237…

n	$\left(1+\frac{1}{n}\right)^n$
1,000,000	2.718280469…
1,000,000,000	2.718281827…

Equation:

$$e \approx 2.718281827$$

The number e is like the number π in that we use a symbol to represent it because its decimal representation never stops or repeats. The irrational number e is called the **natural base**.

Note:
Natural Base e
The number e is defined as the value of $\left(1+\frac{1}{n}\right)^n$, as n increases without bound. We say, as n approaches infinity,
Equation:

$$e \approx 2.718281827...$$

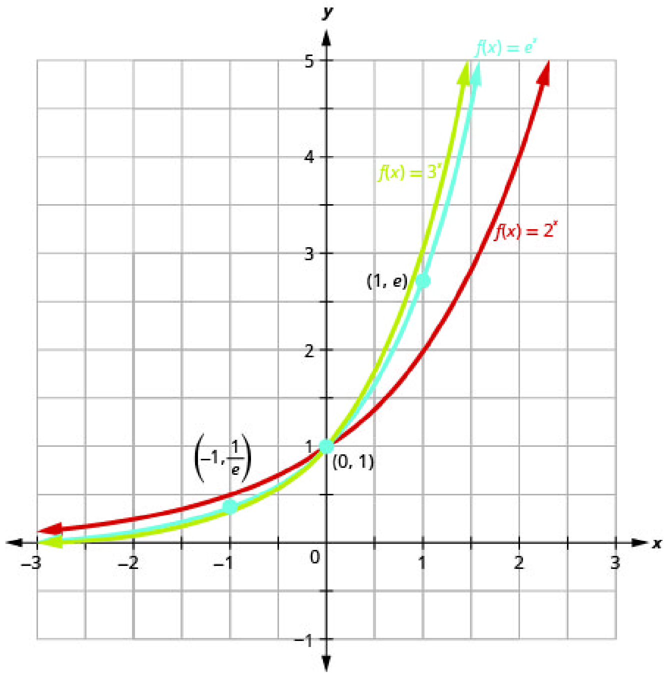
The exponential function whose base is e, $f(x) = e^x$ is called the **natural exponential function**.

Note:
Natural Exponential Function
The natural exponential function is an exponential function whose base is e
Equation:

$$f(x) = e^x$$

The domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.

Let's graph the function $f(x) = e^x$ on the same coordinate system as $g(x) = 2^x$ and $h(x) = 3^x$.



Notice that the graph of $f\left(x\right)={e}^{x}$ is “between” the graphs of $g\left(x\right)={2}^{x}$ and $h\left(x\right)={3}^{x}$. Does this make sense as $2<e<3$?

Solve Exponential Equations

Equations that include an exponential expression ${a}^{x}$ are called exponential equations. To solve them we use a property that says as long as $a>0$ and $a\ne 1$, if ${a}^{x}={a}^{y}$ then it is true that $x=y$. In other words, in an exponential equation, if the bases are equal then the exponents are equal.

Note:
One-to-One Property of Exponential Equations
For $a>0$ and $a\ne 1$,
Equation:

$$\text{If } {a}^{x}={a}^{y}, \text{ then } x=y.$$

To use this property, we must be certain that both sides of the equation are written with the same base.

Example:
How to Solve an Exponential Equation
Exercise:

Problem: Solve: ${3}^{2x-5}=27.$

Solution:

Step 1. Write both sides of the equation with the same base.	Since the left side has base 3, we write the right side with base 3. $27 = 3^3$	$3^{2x-5} = 27$ $3^{2x-5} = 3^3$
Step 2. Write a new equation by setting the exponents equal.	Since the bases are the same, the exponents must be equal.	$2x - 5 = 3$
Step 3. Solve the equation.	Add 5 to each side. Divide by 2.	$2x = 8$ $x = 4$
Step 4. Check the solution.	Substitute $x = 4$ into the original equation.	$3^{2x-5} = 27$ $3^{2 \cdot 4 - 5} \stackrel{?}{=} 27$ $3^3 \stackrel{?}{=} 27$ $27 = 27$ ✓

Note:
Exercise:

Problem: Solve: $3^{3x-2} = 81$.

Solution:

$x = 2$

Note:
Exercise:

Problem: Solve: $7^{x-3} = 7$.

Solution:

$x = 4$

The steps are summarized below.

Note:
How to Solve an Exponential Equation

Write both sides of the equation with the same base, if possible.
Write a new equation by setting the exponents equal.
Solve the equation.
Check the solution.

In the next example, we will use our properties on exponents.

Example:
Exercise:

Problem: Solve $\frac{e^{x^2}}{e^3} = e^{2x}$.

Solution:

	$\frac{e^{x^2}}{e^3} = e^{2x}$
Use the Property of Exponents: $\frac{a^m}{a^n} = a^{m-n}$.	$e^{x^2-3} = e^{2x}$
Write a new equation by setting the exponents equal.	$x^2 - 3 = 2x$
Solve the equation.	$x^2 - 2x - 3 = 0$
	$(x-3)(x+1) = 0$
	$x = 3,\ x = -1$
Check the solutions.	

$x = 3$	$x = -1$
$\frac{e^{x^2}}{e^3} \stackrel{?}{=} e^{2x}$	$\frac{e^{x^2}}{e^3} \stackrel{?}{=} e^{2x}$
$\frac{e^{3^2}}{e^3} \stackrel{?}{=} e^{2\cdot 3}$	$\frac{e^{(-1)^2}}{e^3} \stackrel{?}{=} e^{2\cdot(-1)}$
$\frac{e^9}{e^3} \stackrel{?}{=} e^6$	$\frac{e^1}{e^3} \stackrel{?}{=} e^{-2}$
$e^6 = e^6$ ✓	$e^{-2} = e^{-2}$ ✓

Note:
Exercise:

Problem: Solve: $\frac{e^{x^2}}{e^x} = e^2$.

Solution:

$x = -1, x = 2$

Note:
Exercise:

Problem: Solve: $\frac{e^{x^2}}{e^x} = e^6$.

Solution:

$x = -2, x = 3$

Use Exponential Models in Applications

Exponential functions model many situations. If you own a bank account, you have experienced the use of an exponential function. There are two formulas that are used to determine the balance in the account when interest is earned. If a principal, *P*, is invested at an interest rate, *r*, for *t* years, the new balance, *A*, will depend on how often the interest is compounded. If the interest is compounded *n* times a year we use the formula $A = P\left(1 + \frac{r}{n}\right)^{nt}$. If the interest is compounded continuously, we use the formula $A = Pe^{rt}$. These are the formulas for **compound interest**.

Note:

Compound Interest
For a principal, P, invested at an interest rate, r, for t years, the new balance, A, is:
Equation:

$$\begin{array}{ll} A = P\left(1 + \frac{r}{n}\right)^{nt} & \text{when compounded } n \text{ times a year.} \\ A = Pe^{rt} & \text{when compounded continuously.} \end{array}$$

As you work with the Interest formulas, it is often helpful to identify the values of the variables first and then substitute them into the formula.

Example:
Exercise:

Problem:

A total of \$10,000 was invested in a college fund for a new grandchild. If the interest rate is 5%, how much will be in the account in 18 years by each method of compounding?

ⓐ compound quarterly

ⓑ compound monthly

ⓒ compound continuously

Solution:

Identify the values of each variable in the formulas. Remember to express the percent as a decimal.

$$\begin{aligned} A &= ? \\ P &= \$10{,}000 \\ r &= 0.05 \\ t &= 18 \text{ years} \end{aligned}$$

ⓐ

For quarterly compounding, $n = 4$. There are 4 quarters in a year.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Substitute the values in the formula.

$$A = 10{,}000\left(1 + \frac{0.05}{4}\right)^{4 \cdot 18}$$

Compute the amount. Be careful to consider the order of operations as you enter the expression into your calculator.

$$A = \$24{,}459.20$$

ⓑ
For monthly compounding, $n = 12$. There are 12 months in a year. $A = P\left(1 + \frac{r}{n}\right)^{nt}$

Substitute the values in the formula. $A = 10{,}000\left(1 + \frac{0.05}{12}\right)^{12 \cdot 18}$

Compute the amount. $A = \$24{,}550.08$

ⓒ
For compounding continuously, $A = Pe^{rt}$

Substitute the values in the formula. $A = 10{,}000e^{0.05 \cdot 18}$

Compute the amount. $A = \$24{,}596.03$

Note:
Exercise:

Problem:

Angela invested $15,000 in a savings account. If the interest rate is 4%, how much will be in the account in 10 years by each method of compounding?

ⓐ compound quarterly

ⓑ compound monthly

ⓒ compound continuously

Solution:

ⓐ $22,332.96
ⓑ $22,362.49 ⓒ $22,377.37

Note:
Exercise:

Problem:

Allan invested $10,000 in a mutual fund. If the interest rate is 5%, how much will be in the account in 15 years by each method of compounding?

ⓐ compound quarterly

ⓑ compound monthly

ⓒ compound continuously

Solution:

ⓐ $21,071.81 ⓑ $21,137.04
ⓒ $21,170.00

Other topics that are modeled by exponential functions involve growth and decay. Both also use the formula $A = Pe^{rt}$ we used for the growth of money. For growth and decay, generally we use A_0, as the original amount instead of calling it P, the principal. We see that **exponential growth** has a positive rate of growth and **exponential decay** has a negative rate of growth.

Note:
Exponential Growth and Decay
For an original amount, A_0, that grows or decays at a rate, r, for a certain time, t, the final amount, A, is:
Equation:

$$A = A_0e^{rt}$$

Exponential growth is typically seen in the growth of populations of humans or animals or bacteria. Our next example looks at the growth of a virus.

Example:
Exercise:

Problem:

Chris is a researcher at the Center for Disease Control and Prevention and he is trying to understand the behavior of a new and dangerous virus. He starts his experiment with 100 of the virus that grows at a rate of 25% per hour. He will check on the virus in 24 hours. How many viruses will he find?

Solution:

Identify the values of each variable in the formulas. Be sure to put the percent in decimal form. Be sure the units match—the rate is per hour and the time is in hours.	$\begin{aligned} A &= ? \\ A_0 &= 100 \\ r &= 0.25/\text{hour} \\ t &= 24\text{ hours} \end{aligned}$
Substitute the values in the formula: $A = A_0e^{rt}$.	$A = 100e^{0.25 \cdot 24}$
Compute the amount.	$A = 40{,}342.88$
Round to the nearest whole virus.	$A = 40{,}343$
	The researcher will find 40,343 viruses.

Note:
Exercise:

Problem:

Another researcher at the Center for Disease Control and Prevention, Lisa, is studying the growth of a bacteria. She starts his experiment with 50 of the bacteria that grows at a rate of 15% per hour. He will check on the bacteria every 8 hours. How many bacteria will he find in 8 hours?

Solution:

She will find 166 bacteria.

Note:
Exercise:

Problem:

Maria, a biologist is observing the growth pattern of a virus. She starts with 100 of the virus that grows at a rate of 10% per hour. She will check on the virus in 24 hours. How many viruses will she find?

Solution:

She will find 1,102 viruses.

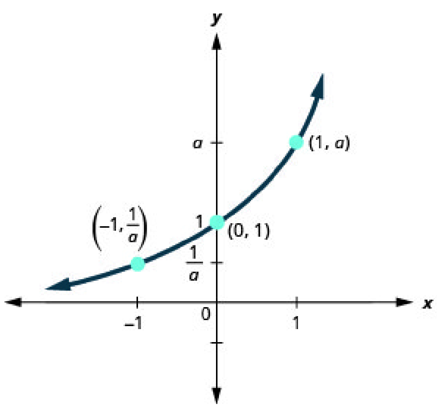
Note:
Access these online resources for additional instruction and practice with evaluating and graphing exponential functions.

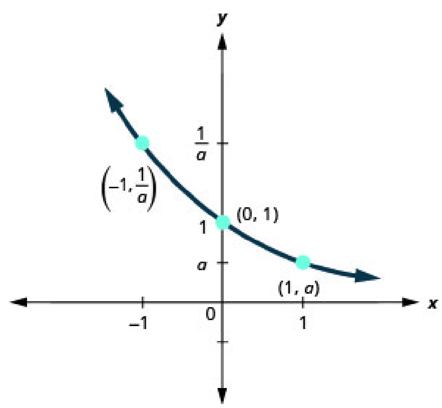
- Graphing Exponential Functions
- Solving Exponential Equations
- Applications of Exponential Functions
- Continuously Compound Interest
- Radioactive Decay and Exponential Growth

Key Concepts

- **Properties of the Graph of $f(x) = a^x$:**

when $a > 1$		when $0 < a < 1$	
Domain	$(-\infty, \infty)$	Domain	$(-\infty, \infty)$
Range	$(0, \infty)$	Range	$(0, \infty)$
x-intercept	none	x-intercept	none
y-intercept	$(0, 1)$	y-intercept	$(0, 1)$
Contains	$(1, a),\ \left(-1, \frac{1}{a}\right)$	Contains	$(1, a),\ \left(-1, \frac{1}{a}\right)$
Asymptote	x-axis, the line $y = 0$	Asymptote	x-axis, the line $y = 0$
Basic shape	increasing	Basic shape	decreasing





- **One-to-One Property of Exponential Equations:**
  For $a > 0$ and $a \neq 1$,
  Equation:

$$A = A_0 e^{rt}$$

- **How to Solve an Exponential Equation**

  Write both sides of the equation with the same base, if possible.
  Write a new equation by setting the exponents equal.
  Solve the equation.
  Check the solution.

- **Compound Interest:** For a principal, P, invested at an interest rate, r, for t years, the new balance, A, is

$$A = P\left(1 + \frac{r}{n}\right)^{nt} \qquad \text{when compounded } n \text{ times a year.}$$
$$A = Pe^{rt} \qquad \text{when compounded continuously.}$$

- **Exponential Growth and Decay:** For an original amount, A_0 that grows or decays at a rate, r, for a certain time t, the final amount,A, is $A = A_0 e^{rt}$.

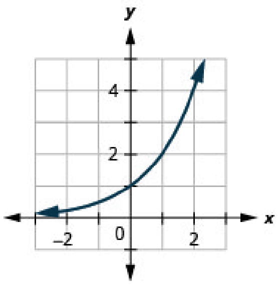
Practice Makes Perfect

Graph Exponential Functions

In the following exercises, graph each exponential function.
Exercise:

Problem: $f\left(x\right) = 2^x$

Solution:



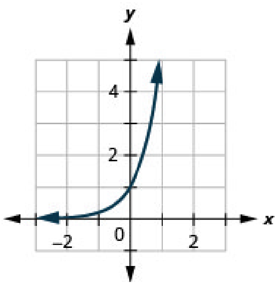
Exercise:

Problem: $g\left(x\right) = 3^x$

Exercise:

Problem: $f\left(x\right) = 6^x$

Solution:



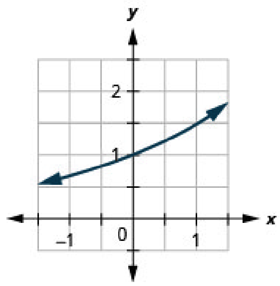
Exercise:

Problem: $g\left(x\right) = 7^x$

Exercise:

Problem: $f\left(x\right)=\left(1.5\right)^{x}$

Solution:



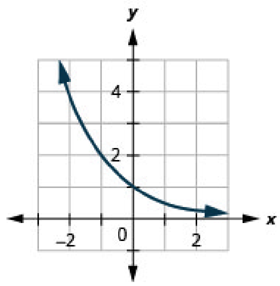
Exercise:

Problem: $g\left(x\right)=\left(2.5\right)^{x}$

Exercise:

Problem: $f\left(x\right)=\left(\frac{1}{2}\right)^{x}$

Solution:



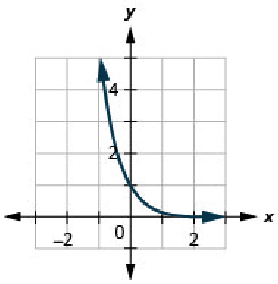
Exercise:

Problem: $g\left(x\right)=\left(\frac{1}{3}\right)^{x}$

Exercise:

Problem: $f\left(x\right)=\left(\frac{1}{6}\right)^{x}$

Solution:



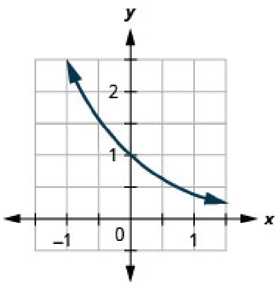
Exercise:

Problem: $g\left(x\right)={\left(\frac{1}{7}\right)}^{x}$

Exercise:

Problem: $f\left(x\right)={\left(0.4\right)}^{x}$

Solution:



Exercise:

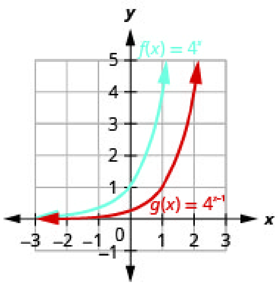
Problem: $g\left(x\right)={\left(0.6\right)}^{x}$

In the following exercises, graph each function in the same coordinate system.

Exercise:

Problem: $f\left(x\right)={4}^{x},\ g\left(x\right)={4}^{x-1}$

Solution:



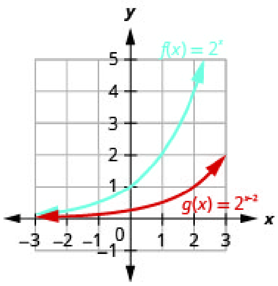
Exercise:

Problem: $f\left(x\right)=3^x,\ g\left(x\right)=3^{x-1}$

Exercise:

Problem: $f\left(x\right)=2^x,\ g\left(x\right)=2^{x-2}$

Solution:



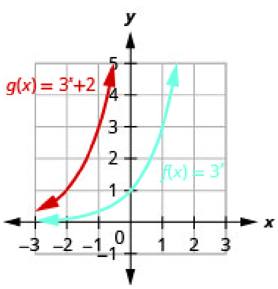
Exercise:

Problem: $f\left(x\right)=2^x,\ g\left(x\right)=2^{x+2}$

Exercise:

Problem: $f\left(x\right)=3^x,\ g\left(x\right)=3^x+2$

Solution:



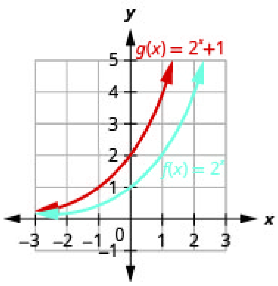
Exercise:

Problem: $f\left(x\right)=4^x,\ g\left(x\right)=4^x+2$

Exercise:

Problem: $f\left(x\right)=2^x,\ g\left(x\right)=2^x+1$

Solution:



Exercise:

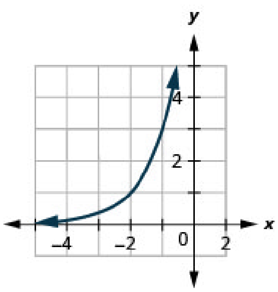
Problem: $f\left(x\right)=2^{x},\ g\left(x\right)=2^{x}-1$

In the following exercises, graph each exponential function.

Exercise:

Problem: $f\left(x\right)=3^{x+2}$

Solution:



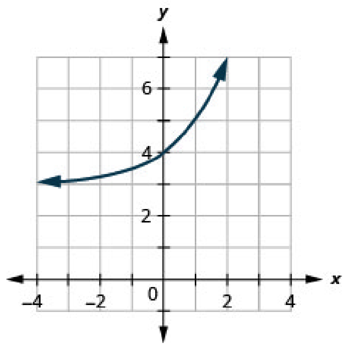
Exercise:

Problem: $f\left(x\right)=3^{x-2}$

Exercise:

Problem: $f\left(x\right)=2^{x}+3$

Solution:



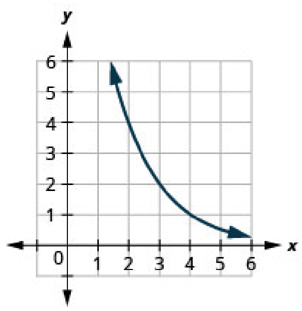
Exercise:

Problem: $f(x) = 2^x - 3$

Exercise:

Problem: $f(x) = \left(\frac{1}{2}\right)^{x-4}$

Solution:



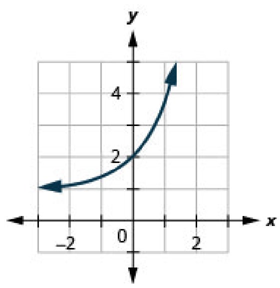
Exercise:

Problem: $f(x) = \left(\frac{1}{2}\right)^{x} - 3$

Exercise:

Problem: $f(x) = e^x + 1$

Solution:



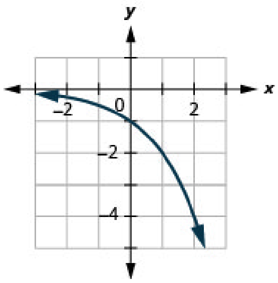
Exercise:

Problem: $f\left(x\right)={e}^{x-2}$

Exercise:

Problem: $f\left(x\right)=-{2}^{x}$

Solution:



Exercise:

Problem: $f\left(x\right)={2}^{-x-1}-1$

Solve Exponential Equations

In the following exercises, solve each equation.

Exercise:

Problem: ${2}^{3x-8}=16$

Solution:

$x=4$

Exercise:

Problem: ${2}^{2x-3}=32$

Exercise:

Problem: ${3}^{x+3}=9$

Solution:

$x=-1$

Exercise:

Problem: ${3}^{{x}^{2}}=81$

Exercise:

Problem: $4^{x^2} = 4$

Solution:

$x = -1, x = 1$

Exercise:

Problem: $4^x = 32$

Exercise:

Problem: $4^{x+2} = 64$

Solution:

$x = 1$

Exercise:

Problem: $4^{x+3} = 16$

Exercise:

Problem: $2^{x^2+2x} = \frac{1}{2}$

Solution:

$x = -1$

Exercise:

Problem: $3^{x^2-2x} = \frac{1}{3}$

Exercise:

Problem: $e^{3x} \cdot e^4 = e^{10}$

Solution:

$x = 2$

Exercise:

Problem: $e^{2x} \cdot e^3 = e^9$

Exercise:

Problem: $\frac{e^{x^2}}{e^2} = e^x$

Solution:

$x = -1, x = 2$

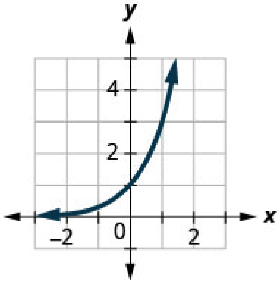
Exercise:

Problem: $\frac{e^{x^2}}{e^3} = e^{2x}$

In the following exercises, match the graphs to one of the following functions: ⓐ 2^x ⓑ 2^{x+1} ⓒ 2^{x-1} ⓓ $2^x + 2$ ⓔ $2^x - 2$ ⓕ 3^x

Exercise:

Problem:

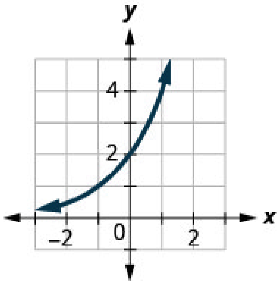


Solution:

ⓕ

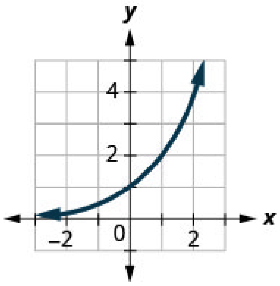
Exercise:

Problem:



Exercise:

Problem:

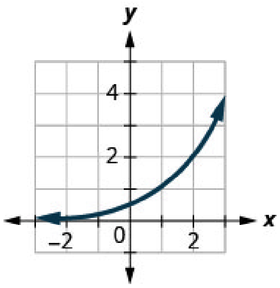


Solution:

ⓐ

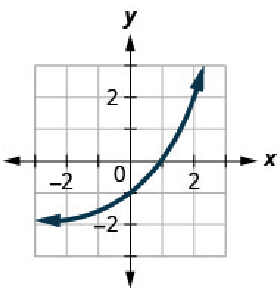
Exercise:

Problem:



Exercise:

Problem:

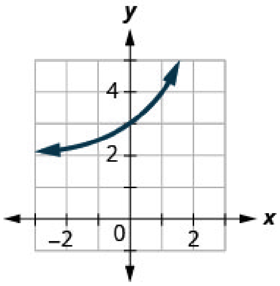


Solution:

ⓔ

Exercise:

Problem:



Use exponential models in applications

In the following exercises, use an exponential model to solve.

Exercise:

Problem:

Edgar accumulated $\$5,000$ in credit card debt. If the interest rate is 20% per year, and he does not make any payments for 2 years, how much will he owe on this debt in 2 years by each method of compounding?
ⓐ compound quarterly
ⓑ compound monthly
ⓒ compound continuously

Solution:

ⓐ $\$7,387.28$ ⓑ $\$7,434.57$ ⓒ $\$7,459.12$

Exercise:

Problem:

Cynthia invested $\$12,000$ in a savings account. If the interest rate is 6%, how much will be in the account in 10 years by each method of compounding?
ⓐ compound quarterly
ⓑ compound monthly
ⓒ compound continuously

Exercise:

Problem:

Rochelle deposits $\$5,000$ in an IRA. What will be the value of her investment in 25 years if the investment is earning 8% per year and is compounded continuously?

Solution:

$\$36,945.28$

Exercise:

Problem:

Nazerhy deposits $\$8,000$ in a certificate of deposit. The annual interest rate is 6% and the interest will be compounded quarterly. How much will the certificate be worth in 10 years?

Exercise:

Problem:

A researcher at the Center for Disease Control and Prevention is studying the growth of a bacteria. He starts his experiment with 100 of the bacteria that grows at a rate of 6% per hour. He will check on the bacteria every 8 hours. How many bacteria will he find in 8 hours?

Solution:

159 bacteria

Exercise:

Problem:

A biologist is observing the growth pattern of a virus. She starts with 50 of the virus that grows at a rate of 20% per hour. She will check on the virus in 24 hours. How many viruses will she find?

Exercise:

Problem:

In the last ten years the population of Indonesia has grown at a rate of 1.12% per year to 258,316,051. If this rate continues, what will be the population in 10 more years?

Solution:

288,929,825

Exercise:

Problem:

In the last ten years the population of Brazil has grown at a rate of 0.9% per year to 205,823,665. If this rate continues, what will be the population in 10 more years?

Writing Exercises

Exercise:

Problem:

Explain how you can distinguish between exponential functions and polynomial functions.

Solution:

Answers will vary.

Exercise:

Problem: Compare and contrast the graphs of $y = x^2$ and $y = 2^x$.

Exercise:

Problem:

What happens to an exponential function as the values of x decreases? Will the graph ever cross the
x-axis? Explain.

Solution:

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
graph exponential eunctions.			
solve exponential equations.			
use exponential models in applications.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

Glossary

asymptote
: A line which a graph of a function approaches closely but never touches.

exponential function
: An exponential function, where $a > 0$ and $a \neq 1$, is a function of the form $f(x) = a^x$.

natural base
: The number *e* is defined as the value of $\left(1 + \frac{1}{n}\right)^n$, as *n* gets larger and larger. We say, as *n* increases without bound, $e \approx 2.718281827...$

natural exponential function
: The natural exponential function is an exponential function whose base is *e*: $f(x) = e^x$. The domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.

Evaluate and Graph Logarithmic Functions: ASE
By the end of this section, you will be able to:

- Convert between exponential and logarithmic form
- Evaluate logarithmic functions
- Graph Logarithmic functions
- Solve logarithmic equations
- Use logarithmic models in applications

We have spent some time finding the inverse of many functions. It works well to 'undo' an operation with another operation. Subtracting 'undoes' addition, multiplication 'undoes' division, taking the square root 'undoes' squaring.

As we studied the exponential function, we saw that it is one-to-one as its graphs pass the horizontal line test. This means an exponential function does have an inverse. If we try our algebraic method for finding an inverse, we run into a problem.

	$f(x) = a^x$
Rewrite with $y = f(x)$.	$y = a^x$
Interchange the variables x and y.	$x = a^y$
Solve for y.	Oops! We have no way to solve for y!

To deal with this we define the logarithm function with base a to be the inverse of the exponential function $f(x) = a^x$. We use the notation $f^{-1}(x) = \log_a x$ and say the inverse function of the exponential function is the logarithmic function.

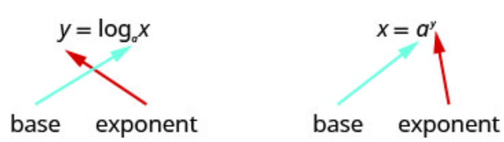
Note:
Logarithmic Function
The function $f(x) = \log_a x$ is the **logarithmic function** with base a, where $a > 0, x > 0$, and $a \neq 1$.
Equation:

$$y = \log_a x \text{ is equivalent to } x = a^y$$

Just as every different base for the exponential function is a different function, this is also true for the logarithm function. For any particular base there is an exponential function and it's corresponding inverse a logarithmic function.

Convert Between Exponential and Logarithmic Form

Since the equations $y = \log_a x$ and $x = a^y$ are equivalent, we can go back and forth between them. This will often be the method to solve some exponential and logarithmic equations. To help with converting back and forth let's take a close look at the equations. See [link]. Notice the positions of the exponent and base.



If we realize the logarithm is the exponent it makes the conversion easier. You may want to repeat, “base to the exponent give us the number.”

Example:
Exercise:

Problem: Convert to logarithmic form: ⓐ $2^3 = 8$, ⓑ $5^{\frac{1}{2}} = \sqrt{5}$, and ⓒ $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$.

Solution:

Identify the base and the exponent.

(a)

$2^3 = 8$

$y = \log_a x$

$3 = \log_2 8$

If $2^3 = 8$, then $3 = \log_2 8$.

(b)

$5^{\frac{1}{2}} = \sqrt{5}$

$y = \log_a x$

$\frac{1}{2} = \log_5 \sqrt{5}$

If $5^{\frac{1}{2}} = \sqrt{5}$, then $\frac{1}{2} = \log_5 \sqrt{5}$.

(c)

$\left(\frac{1}{2}\right)^4 = \frac{1}{16}$

$y = \log_a x$

$4 = \log_{\frac{1}{2}} \frac{1}{16}$

If $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$, then $4 = \log_{\frac{1}{2}} \frac{1}{16}$.

Note:
Exercise:

Problem: Convert to logarithmic form: ⓐ $3^2 = 9$ ⓑ $7^{\frac{1}{2}} = \sqrt{7}$ ⓒ $\left(\frac{1}{3}\right)^x = \frac{1}{27}$

Solution:

ⓐ $\log_3 9 = 2$
ⓑ $\log_7 \sqrt{7} = \frac{1}{2}$ ⓒ $\log_{\frac{1}{3}} \frac{1}{27} = x$

Note:
Exercise:

Problem: Convert to logarithmic form: ⓐ $4^3 = 64$ ⓑ $4^{\frac{1}{3}} = \sqrt[3]{4}$ ⓒ $\left(\frac{1}{2}\right)^x = \frac{1}{32}$

Solution:

ⓐ $\log_4 64 = 3$
ⓑ $\log_4 \sqrt[3]{4} = \frac{1}{3}$ ⓒ $\log_{\frac{1}{2}} \frac{1}{32} = x$

In the next example we do the reverse—convert logarithmic form to exponential form.

Example:
Exercise:

Problem: Convert to exponential form: ⓐ $2 = \log_8 64$, ⓑ $0 = \log_4 1$, and ⓒ $-3 = \log_{10} \frac{1}{1000}$.

Solution:

Identify the base and the exponent.

(a)	(b)	(c)
$2 = \log_8 64$	$0 = \log_4 1$	$-3 = \log_{10} \frac{1}{1000}$
$x = a^y$	$x = a^y$	$x = a^y$
$64 = 8^2$	$1 = 4^0$	$\frac{1}{1000} = \log^{-3}$
If $2 = \log_8 64$, then $64 = 8^2$.	If $0 = \log_4 1$, then $1 = 4^0$.	If $-3 = \log_{10} \frac{1}{1000}$, then $\frac{1}{1000} = 10^{-3}$.

Note:
Exercise:

Problem: Convert to exponential form: ⓐ $3 = \log_4 64$ ⓑ $0 = \log_x 1$ ⓒ $-2 = \log_{10} \frac{1}{100}$

Solution:

ⓐ $64 = 4^3$
ⓑ $1 = x^0$ ⓒ $\frac{1}{100} = 10^{-2}$

Note:
Exercise:

Problem: Convert to exponential form: ⓐ $3 = \log_3 27$ ⓑ $0 = \log_x 1$ ⓒ $-1 = \log_{10} \frac{1}{10}$

Solution:

ⓐ $27 = 3^3$ ⓑ $1 = x^0$
ⓒ $\frac{1}{10} = 10^{-1}$

Evaluate Logarithmic Functions

We can solve and evaluate logarithmic equations by using the technique of converting the equation to its equivalent exponential equation.

Example:
Exercise:

Problem: Find the value of x: ⓐ $\log_x 36 = 2$, ⓑ $\log_4 x = 3$, and ⓒ $\log_{\frac{1}{2}} \frac{1}{8} = x$.

Solution:

ⓐ

	$\log_x 36 = 2$	
Convert to exponential form.	$x^2 = 36$	
Solve the quadratic.	$x = 6, \quad \cancel{x = -6}$	
The base of a logarithmic function must be positive, so we eliminate $x = -6$.	$x = 6$	Therefore, $\log_6 36 = 2$.

ⓑ

	$\log_4 x = 3$	
Convert to exponential form.	$4^3 = x$	
Simplify.	$x = 64$	Therefore, $\log_4 64 = 3$.

ⓒ

	$\log_{\frac{1}{2}} \frac{1}{8} = x$	
Convert to exponential form.	$\left(\frac{1}{2}\right)^x = \frac{1}{8}$	
Rewrite $\frac{1}{8}$ as $\left(\frac{1}{2}\right)^3$.	$\left(\frac{1}{2}\right)^x = \left(\frac{1}{2}\right)^3$	
With the same base, the exponents must be equal.	$x = 3$	Therefore, $\log_{\frac{1}{2}} \frac{1}{8} = 3$

Note:
Exercise:

Problem: Find the value of x : ⓐ $\log_x 64 = 2$ ⓑ $\log_5 x = 3$ ⓒ $\log_{\frac{1}{2}} \frac{1}{4} = x$

Solution:

ⓐ $x = 8$ ⓑ $x = 125$ ⓒ $x = 2$

Note:
Exercise:

Problem: Find the value of x : ⓐ $\log_x 81 = 2$ ⓑ $\log_3 x = 5$ ⓒ $\log_{\frac{1}{3}} \frac{1}{27} = x$

Solution:

ⓐ
$x = 9$ ⓑ $x = 243$ ⓒ $x = 3$

When see an expression such as $\log_3 27$, we can find its exact value two ways. By inspection we realize it means \\3 to what power will be 27//? Since $3^3 = 27$, we know $\log_3 27 = 3$. An alternate way is to set the expression equal to x and then convert it into an exponential equation.

Example:
Exercise:

Problem: Find the exact value of each logarithm without using a calculator:
ⓐ $\log_5 25$,
ⓑ $\log_9 3$, and ⓒ $\log_2 \frac{1}{16}$.

Solution:

ⓐ

	$\log_5 25$	
5 to what power will be 25?	$\log_5 25 = 2$	
Or		
Set the expression equal to x.	$\log_5 25 = x$	
Change to exponential form.	$5^x = 25$	
Rewrite 25 as 5^2.	$5^x = 5^2$	
With the same base the exponents must be equal.	$x = 2$	Therefore, $\log_5 25 = 2$.

ⓑ

	$\log_9 3$	
Set the expression equal to x.	$\log_9 3 = x$	
Change to exponential form.	$9^x = 3$	
Rewrite 9 as 3^2.	$\left(3^2\right)^x = 3^1$	
Simplify the exponents.	$3^{2x} = 3^1$	
With the same base the exponents must be equal.	$2x = 1$	
Solve the equation.	$x = \frac{1}{2}$	Therefore, $\log_9 3 = \frac{1}{2}$.

ⓒ

$$\begin{array}{lrcl} & \log_2 \frac{1}{16} & & \\ \text{Set the expression equal to } x. & \log_2 \frac{1}{16} & = & x \\ \text{Change to exponential form.} & 2^x & = & \frac{1}{16} \\ \text{Rewrite 16 as } 2^4. & 2^x & = & \frac{1}{2^4} \\ & 2^x & = & 2^{-4} \\ \text{With the same base the exponents must be equal.} & x & = & -4 \quad \text{Therefore, } \log_2 \frac{1}{16} = -4. \end{array}$$

Note:
Exercise:

Problem: Find the exact value of each logarithm without using a calculator:
ⓐ $\log_{12}144$
ⓑ $\log_4 2$
ⓒ $\log_2 \frac{1}{32}$

Solution:

ⓐ
2 ⓑ $\frac{1}{2}$ ⓒ -5

Note:
Exercise:

Problem: Find the exact value of each logarithm without using a calculator:
ⓐ $\log_9 81$
ⓑ $\log_8 2$
ⓒ $\log_3 \frac{1}{9}$

Solution:

ⓐ 2 ⓑ $\frac{1}{3}$ ⓒ -2

Graph Logarithmic Functions

To graph a logarithmic function $y = \log_a x$, it is easiest to convert the equation to its exponential form, $x = a^y$. Generally, when we look for ordered pairs for the graph of a function, we usually choose an *x*-value and then determine its corresponding *y*-value. In this case you may find it easier to choose *y*-values and then determine its corresponding *x*-value.

Example:

Exercise:

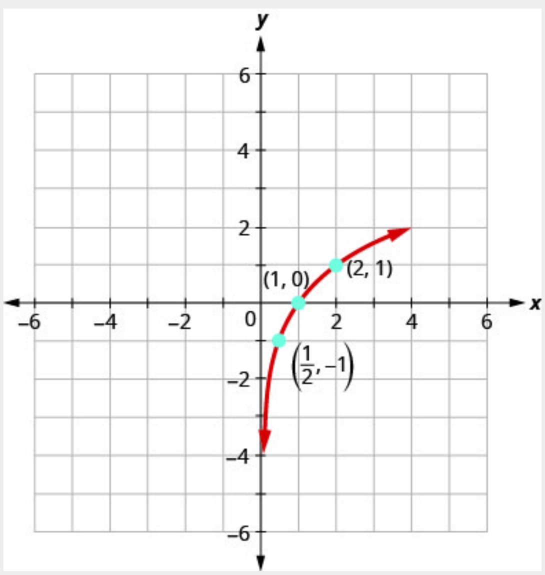
Problem: Graph $y = \log_2 x$.

Solution:

To graph the function, we will first rewrite the logarithmic equation, $y = \log_2 x$, in exponential form, $2^y = x$.

We will use point plotting to graph the function. It will be easier to start with values of *y* and then get *x*.

y	$2^y = x$	(x, y)
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$\left(\frac{1}{4}, 2\right)$
-1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$\left(\frac{1}{2}, -1\right)$
0	$2^0 = 1$	$(1, 0)$
1	$2^1 = 2$	$(2, 1)$
2	$2^2 = 4$	$(4, 2)$
3	$2^3 = 8$	$(8, 3)$

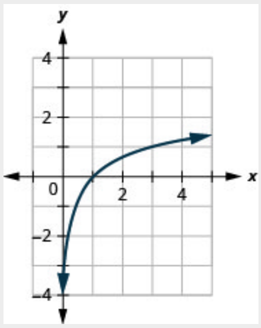


Note:

Exercise:

Problem: Graph: $y = \log_3 x$.

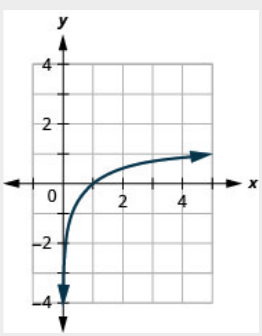
Solution:



Note:
Exercise:

Problem: Graph: $y = \log_5 x$.

Solution:



The graphs of $y = \log_2 x$,$y = \log_3 x$, and $y = \log_5 x$ are the shape we expect from a logarithmic function where $a > 1$.

We notice that for each function the graph contains the point $(1, 0)$. This make sense because $0 = \log_a 1$ means $a^0 = 1$ which is true for any a.

The graph of each function, also contains the point $(a, 1)$. This makes sense as $1 = \log_a a$ means $a^1 = a$. which is true for any a.

Notice too, the graph of each function $y = \log_a x$ also contains the point $\left(\frac{1}{a}, -1\right)$. This makes sense as $-1 = \log_a \frac{1}{a}$ means $a^{-1} = \frac{1}{a}$, which is true for any a.

Look at each graph again. Now we will see that many characteristics of the logarithm function are simply 'mirror images' of the characteristics of the corresponding exponential function.

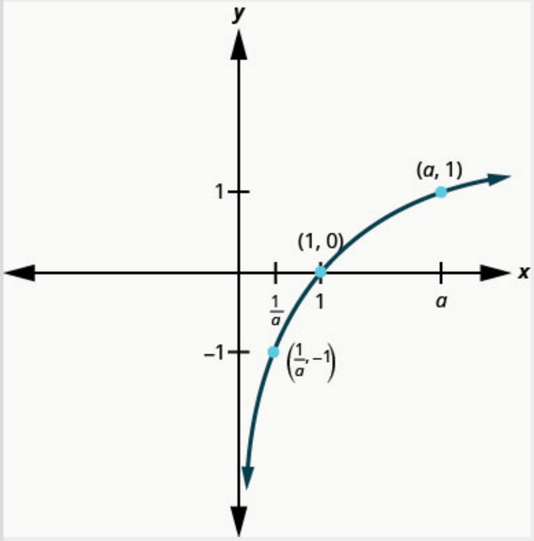
What is the domain of the function? The graph never hits the *y*-axis. The domain is all positive numbers. We write the domain in interval notation as $(0, \infty)$.

What is the range for each function? From the graphs we can see that the range is the set of all real numbers. There is no restriction on the range. We write the range in interval notation as $(-\infty, \infty)$.

When the graph approaches the *y*-axis so very closely but will never cross it, we call the line $x = 0$, the *y*-axis, a vertical asymptote.

Note:
Properties of the Graph of $y = \log_a x$ when $a > 1$

Domain	$(0, \infty)$
Range	$(-\infty, \infty)$
x-intercept	$(1, 0)$
y-intercept	None
Contains	$(a, 1), \left(\frac{1}{a}, -1\right)$
Asymptote	y-axis



Our next example looks at the graph of $y = \log_a x$ when $0 < a < 1$.

Example:
Exercise:

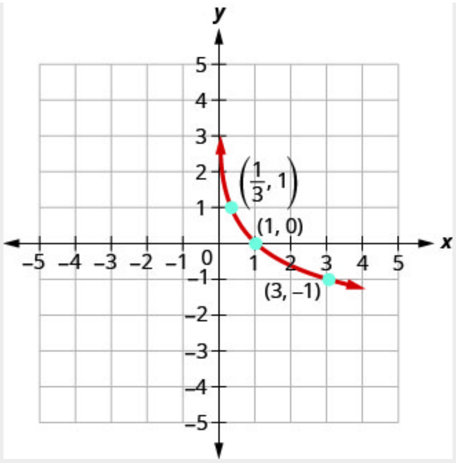
Problem: Graph $y = \log_{\frac{1}{3}} x$.

Solution:

To graph the function, we will first rewrite the logarithmic equation, $y = \log_{\frac{1}{3}} x$, in exponential form, $\left(\frac{1}{3}\right)^y = x$.

We will use point plotting to graph the function. It will be easier to start with values of *y* and then get *x*.

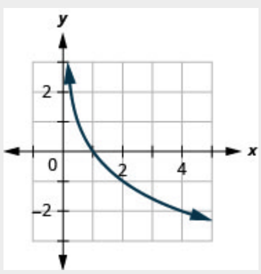
y	$\left(\frac{1}{3}\right)^y = x$	(x, y)
-2	$\left(\frac{1}{3}\right)^{-2} = 3^2 = 9$	$(9, -2)$
-1	$\left(\frac{1}{3}\right)^{-1} = 3^1 = 3$	$(3, -1)$
0	$\left(\frac{1}{3}\right)^0 = 1$	$(1, 0)$
1	$\left(\frac{1}{3}\right)^1 = \frac{1}{3}$	$\left(\frac{1}{3}, 1\right)$
2	$\left(\frac{1}{3}\right)^2 = \frac{1}{9}$	$\left(\frac{1}{9}, 2\right)$
3	$\left(\frac{1}{3}\right)^3 = \frac{1}{27}$	$\left(\frac{1}{27}, 3\right)$



Note:
Exercise:

Problem: Graph: $y = \log_{\frac{1}{2}} x$.

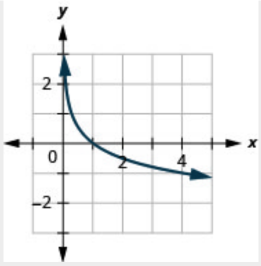
Solution:



Note:
Exercise:

Problem: Graph: $y = \log_{\frac{1}{4}} x$.

Solution:



Now, let's look at the graphs $y = \log_{\frac{1}{2}} x$, $y = \log_{\frac{1}{3}} x$ and $y = \log_{\frac{1}{4}} x$, so we can identify some of the properties of logarithmic functions where $0 < a < 1$.

The graphs of all have the same basic shape. While this is the shape we expect from a logarithmic function where $0 < a < 1$.

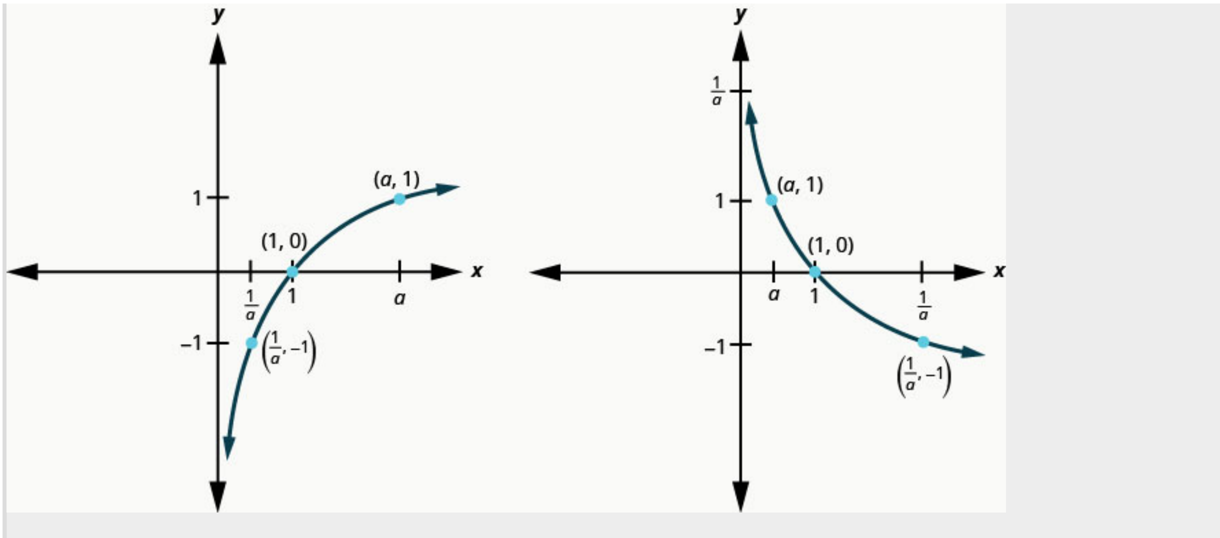
We notice, that for each function again, the graph contains the points,$(1, 0)$,$(a, 1)$,$\left(\frac{1}{a}, -1\right)$. This make sense for the same reasons we argued above.

We notice the domain and range are also the same—the domain is $(0, \infty)$ and the range is $(-\infty, \infty)$. The y-axis is again the vertical asymptote.

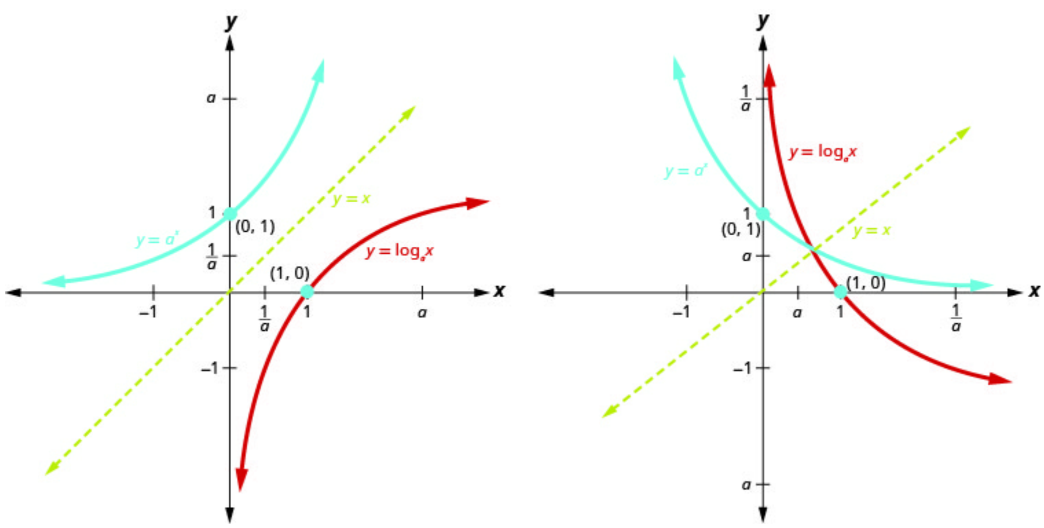
We will summarize these properties in the chart below. Which also include when $a > 1$.

Note:
Properties of the Graph of $y = \log_a x$

when $a > 1$		**when** $0 < a < 1$	
Domain	$(0, \infty)$	Domain	$(0, \infty)$
Range	$(-\infty, \infty)$	Range	$(-\infty, \infty)$
x-intercept	$(1, 0)$	x-intercept	$(1, 0)$
y-intercept	none	y-intercept	None
Contains	$(a, 1)$,$\left(\frac{1}{a}, -1\right)$	Contains	$(a, 1)$,$\left(\frac{1}{a}, -1\right)$
Asymptote	y-axis	Asymptote	y-axis
Basic shape	increasing	Basic shape	Decreasing



We talked earlier about how the logarithmic function $f^{-1}(x) = \log_a x$ is the inverse of the exponential function $f(x) = a^x$. The graphs in [link] show both the exponential (blue) and logarithmic (red) functions on the same graph for both $a > 1$ and $0 < a < 1$.



Notice how the graphs are reflections of each other through the line $y = x$. We know this is true of inverse functions. Keeping a visual in your mind of these graphs will help you remember the domain and range of each function. Notice the *x*-axis is the horizontal asymptote for the exponential functions and the *y*-axis is the vertical asymptote for the logarithmic functions.

Solve Logarithmic Equations

When we talked about exponential functions, we introduced the number e. Just as e was a base for an exponential function, it can be used a base for logarithmic functions too. The logarithmic function with base e is called the **natural logarithmic function**. The function $f(x) = \log_e x$ is generally written $f(x) = \ln x$ and we read it as “el en of x.”

Note:
Natural Logarithmic Function
The function $f(x) = \ln x$ is the **natural logarithmic function** with base e, where $x > 0$.
Equation:

$$y = \ln x \text{ is equivalent to } x = e^y$$

When the base of the logarithm function is 10, we call it the **common logarithmic function** and the base is not shown. If the base *a* of a logarithm is not shown, we assume it is 10.

Note:
Common Logarithmic Function
The function $f(x) = \log x$ is the **common logarithmic function** with base10, where $x > 0$.
Equation:

$$y = \log x \text{ is equivalent to } x = 10^y$$

It will be important for you to use your calculator to evaluate both common and natural logarithms.

Look for the log and ln keys on your calculator.

To solve logarithmic equations, one strategy is to change the equation to exponential form and then solve the exponential equation as we did before. As we solve logarithmic equations, $y = \log_a x$, we need to remember that for the base a, $a > 0$ and $a \neq 1$. Also, the domain is $x > 0$. Just as with radical equations, we must check our solutions to eliminate any extraneous solutions.

Example:
Exercise:

Problem: Solve: ⓐ $\log_a 49 = 2$ and ⓑ $\ln x = 3$.

Solution:

ⓐ

$$\log_a 49 = 2$$

Rewrite in exponential form. $a^2 = 49$

Solve the equation using the square root property. $a = \pm 7$

The base cannot be negative, so we eliminate $a = -7$. $a = 7,\quad \cancel{a = -7}$

Check.

$a = 7$

$$\begin{aligned} \log_a 49 &= 2 \\ \log_7 49 &\stackrel{?}{=} 2 \\ 7^2 &\stackrel{?}{=} 49 \\ 49 &= 49\checkmark \end{aligned}$$

ⓑ

$$\ln x = 3$$

Rewrite in exponential form. $e^3 = x$

Check.

$x = e^3$

$$\begin{aligned} \ln x &= 3 \\ \ln e^3 &\stackrel{?}{=} 3 \\ e^3 &= e^3\checkmark \end{aligned}$$

Note:
Exercise:

Problem: Solve: ⓐ $\log_a 121 = 2$ ⓑ $\ln x = 7$

Solution:

ⓐ
$a = 11$
ⓑ $x = e^7$

Note:
Exercise:

Problem: Solve: ⓐ $\log_a 64 = 3$ ⓑ $\ln x = 9$

Solution:

ⓐ
$a = 4$
ⓑ $x = e^9$

Example:
Exercise:

Problem: Solve: ⓐ $\log_2(3x-5)=4$ and ⓑ $\ln e^{2x}=4$.

Solution:

ⓐ

	$\log_2(3x-5) = 4$
Rewrite in exponential form.	$2^4 = 3x-5$
Simplify.	$16 = 3x-5$
Solve the equation.	$21 = 3x$
	$7 = x$

Check.

$x=7$

$$\begin{aligned} \log_2(3x-5) &= 4 \\ \log_2(3\cdot 7-5) &\stackrel{?}{=} 4 \\ \log_2(16) &\stackrel{?}{=} 4 \\ 2^4 &\stackrel{?}{=} 16 \\ 16 &= 16\checkmark \end{aligned}$$

ⓑ

	$\ln\ e^{2x} = 4$
Rewrite in exponential form.	$e^4 = e^{2x}$
Since the bases are the same the exponents are equal.	$4 = 2x$
Solve the equation.	$2 = x$

Check.

$x=2$

$$\begin{aligned} \ln\ e^{2x} &= 4 \\ \ln\ e^{2\cdot 2} &\stackrel{?}{=} 4 \\ \ln\ e^4 &\stackrel{?}{=} 4 \\ e^4 &= e^4\checkmark \end{aligned}$$

Note:
Exercise:

Problem: Solve: ⓐ $\log_2(5x-1)=6$ ⓑ $\ln e^{3x}=6$

Solution:

ⓐ
$x=13$
ⓑ $x=2$

Note:
Exercise:

Problem: Solve: ⓐ $\log_3(4x+3)=3$ ⓑ $\ln e^{4x}=4$

Solution:

ⓐ
$x=6$
ⓑ $x=1$

Use Logarithmic Models in Applications

There are many applications that are modeled by logarithmic equations. We will first look at the logarithmic equation that gives the decibel (dB) level of sound. Decibels range from 0, which is barely audible to 160, which can rupture an eardrum. The 10^{-12} in the formula represents the intensity of sound that is barely audible.

Note:
Decibel Level of Sound
The loudness level, D, measured in decibels, of a sound of intensity, I, measured in watts per square inch is
Equation:

$$D=10\log\left(\frac{I}{10^{-12}}\right)$$

Example:
Exercise:

Problem:

Extended exposure to noise that measures 85 dB can cause permanent damage to the inner ear which will result in hearing loss. What is the decibel level of music coming through ear phones with intensity 10^{-2} watts per square inch?

Solution:

	$D=10\log\left(\frac{I}{10^{-12}}\right)$
Substitute in the intensity level, I.	

	$D = 10 \log\left(\frac{10^{-2}}{10^{-12}}\right)$
Simplify.	$D = 10 \log(10^{10})$
Since $\log 10^{10} = 10$.	$D = 10 \cdot 10$
Multiply.	$D = 100$
	The decibel level of music coming through earphones is 100 dB.

Note:
Exercise:

Problem:

What is the decibel level of one of the new quiet dishwashers with intensity 10^{-7} watts per square inch?

Solution:

The quiet dishwashers have a decibel level of 50 dB.

Note:
Exercise:

Problem: What is the decibel level heavy city traffic with intensity 10^{-3} watts per square inch?

Solution:

The decibel level of heavy traffic is 90 dB.

The magnitude R of an earthquake is measured by a logarithmic scale called the Richter scale. The model is $R = \log I$, where I is the intensity of the shock wave. This model provides a way to measure earthquake intensity.

Note:
Earthquake Intensity
The magnitude R of an earthquake is measured by $R = \log I$, where I is the intensity of its shock wave.

Example:
Exercise:

Problem:

In 1906, San Francisco experienced an intense earthquake with a magnitude of 7.8 on the Richter scale. Over 80% of the city was destroyed by the resulting fires. In 2014, Los Angeles experienced a moderate earthquake that measured 5.1 on the Richter scale and caused $108 million dollars of damage. Compare the intensities of the two earthquakes.

Solution:

To compare the intensities, we first need to convert the magnitudes to intensities using the log formula. Then we will set up a ratio to compare the intensities.

Convert the magnitudes to intensities.	$R = \log I$
1906 earthquake	$7.8 = \log I$
Convert to exponential form.	$I = 10^{7.8}$
2014 earthquake	$5.1 = \log I$
Convert to exponential form.	$I = 10^{5.1}$
Form a ratio of the intensities.	$\frac{\text{Intensity for 1906}}{\text{Intensity for 2014}}$
Substitute in the values.	$\frac{10^{7.8}}{10^{5.1}}$
Divide by subtracting the exponents.	$10^{2.7}$
Evaluate.	501
	The intensity of the 1906 earthquake was about 501 times the intensity of the 2014 earthquake.

Note:
Exercise:

Problem:

In 1906, San Francisco experienced an intense earthquake with a magnitude of 7.8 on the Richter scale. In 1989, the Loma Prieta earthquake also affected the San Francisco area, and measured 6.9 on the Richter scale. Compare the intensities of the two earthquakes.

Solution:

The intensity of the 1906 earthquake was about 8 times the intensity of the 1989 earthquake.

Note:
Exercise:

Problem:

In 2014, Chile experienced an intense earthquake with a magnitude of 8.2 on the Richter scale. In 2014, Los Angeles also experienced an earthquake which measured 5.1 on the Richter scale. Compare the intensities of the two earthquakes.

Solution:

The intensity of the earthquake in Chile was about 1,259 times the intensity of the earthquake in Los Angeles.

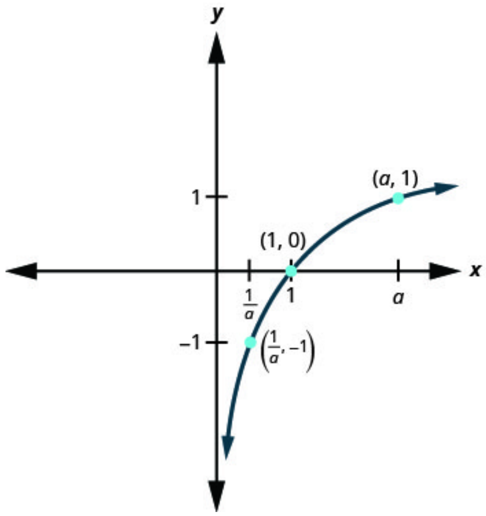
Note:
Access these online resources for additional instruction and practice with evaluating and graphing logarithmic functions.

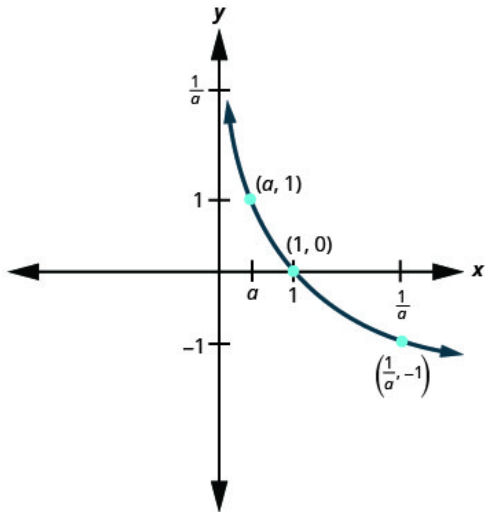
- Re-writing logarithmic equations in exponential form
- Simplifying Logarithmic Expressions
- Graphing logarithmic functions
- Using logarithms to calculate decibel levels

Key Concepts

- **Properties of the Graph of** $y = \log_a x$:

when $a > 1$		**when** $0 < a < 1$	
Domain	$(0, \infty)$	Domain	$(0, \infty)$
Range	$(-\infty, \infty)$	Range	$(-\infty, \infty)$
x-intercept	$(1, 0)$	*x*-intercept	$(1, 0)$
y-intercept	none	*y*-intercept	none
Contains	$(a, 1), \left(\frac{1}{a}, -1\right)$	Contains	$(a, 1), \left(\frac{1}{a}, -1\right)$
Asymptote	*y*-axis	Asymptote	*y*-axis
Basic shape	increasing	Basic shape	decreasing





- **Decibel Level of Sound:** The loudness level, D, measured in decibels, of a sound of intensity, I, measured in watts per square inch is $D = 10\log\left(\frac{I}{10^{-12}}\right)$.
- **Earthquake Intensity:** The magnitude R of an earthquake is measured by $R = \log I$, where I is the intensity of its shock wave.

Practice Makes Perfect

Convert Between Exponential and Logarithmic Form

In the following exercises, convert from exponential to logarithmic form.

Exercise:

Problem: $4^2 = 16$

Exercise:

Problem: $2^5 = 32$

Solution:

$\log_2 32 = 5$

Exercise:

Problem: $3^3 = 27$

Exercise:

Problem: $5^3 = 125$

Solution:

$\log_5 125 = 3$

Exercise:

Problem: $10^3 = 1000$

Exercise:

Problem: $10^{-2} = \frac{1}{100}$

Solution:

$\log \frac{1}{100} = -2$

Exercise:

Problem: $x^{\frac{1}{2}} = \sqrt{3}$

Exercise:

Problem: $x^{\frac{1}{3}} = \sqrt[3]{6}$

Solution:

$\log_x \sqrt[3]{6} = \frac{1}{3}$

Exercise:

Problem: $32^x = \sqrt[4]{32}$

Exercise:

Problem: $17^x = \sqrt[5]{17}$

Solution:

$\log_{17} \sqrt[5]{17} = x$

Exercise:

Problem: $\left(\frac{1}{4}\right)^2 = \frac{1}{16}$

Exercise:

Problem: $\left(\frac{1}{3}\right)^4 = \frac{1}{81}$

Solution:

$\log_{\frac{1}{3}} \frac{1}{81} = 4$

Exercise:

Problem: $3^{-2} = \frac{1}{9}$

Exercise:

Problem: $4^{-3} = \frac{1}{64}$

Solution:

$\log_4 \frac{1}{64} = -3$

Exercise:

Problem: $e^x = 6$

Exercise:

Problem: $e^3 = x$

Solution:

$\ln x = 3$

In the following exercises, convert each logarithmic equation to exponential form.

Exercise:

Problem: $3 = \log_4 64$

Exercise:

Problem: $6 = \log_2 64$

Solution:

$64 = 2^6$

Exercise:

Problem: $4 = \log_x 81$

Exercise:

Problem: $5 = \log_x 32$

Solution:

$32 = x^5$

Exercise:

Problem: $0 = \log_{12} 1$

Exercise:

Problem: $0 = \log_7 1$

Solution:

$1 = 7^0$

Exercise:

Problem: $1 = \log_3 3$

Exercise:

Problem: $1 = \log_9 9$

Solution:

$9 = 9^1$

Exercise:

Problem: $-4 = \log_{10} \frac{1}{10{,}000}$

Exercise:

Problem: $3 = \log_{10} 1{,}000$

Solution:

$1{,}000 = 10^3$

Exercise:

Problem: $5 = \log_e x$

Exercise:

Problem: $x = \log_e 43$

Solution:

$43 = e^x$

Evaluate Logarithmic Functions

In the following exercises, find the value of x in each logarithmic equation.

Exercise:

Problem: $\log_x 49 = 2$

Exercise:

Problem: $\log_x 121 = 2$

Solution:

$x = 11$

Exercise:

Problem: $\log_x 27 = 3$

Exercise:

Problem: $\log_x 64 = 3$

Solution:

$x = 4$

Exercise:

Problem: $\log_3 x = 4$

Exercise:

Problem: $\log_5 x = 3$

Solution:

$x = 125$

Exercise:

Problem: $\log_2 x = -6$

Exercise:

Problem: $\log_3 x = -5$

Solution:

$x = \frac{1}{243}$

Exercise:

Problem: $\log_{\frac{1}{4}} \frac{1}{16} = x$

Exercise:

Problem: $\log_{\frac{1}{3}} \frac{1}{9} = x$

Solution:

$x = 2$

Exercise:

Problem: $\log_{\frac{1}{4}} 64 = x$

Exercise:

Problem: $\log_{\frac{1}{9}} 81 = x$

Solution:

$x = -2$

In the following exercises, find the exact value of each logarithm without using a calculator.

Exercise:

Problem: $\log_7 49$

Exercise:

Problem: $\log_6 36$

Solution:

2

Exercise:

Problem: $\log_4 1$

Exercise:

Problem: $\log_5 1$

Solution:

0

Exercise:

Problem: $\log_{16} 4$

Exercise:

Problem: $\log_{27} 3$

Solution:

$\frac{1}{3}$

Exercise:

Problem: $\log_{\frac{1}{2}} 2$

Exercise:

Problem: $\log_{\frac{1}{2}} 4$

Solution:

-2

Exercise:

Problem: $\log_2 \frac{1}{16}$

Exercise:

Problem: $\log_3 \frac{1}{27}$

Solution:

-3

Exercise:

Problem: $\log_4 \frac{1}{16}$

Exercise:

Problem: $\log_9 \frac{1}{81}$

Solution:

-2

Graph Logarithmic Functions

In the following exercises, graph each logarithmic function.

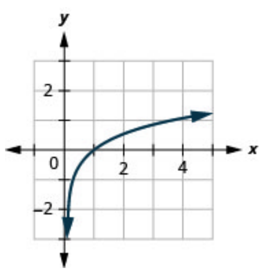
Exercise:

Problem: $y = \log_2 x$

Exercise:

Problem: $y = \log_4 x$

Solution:



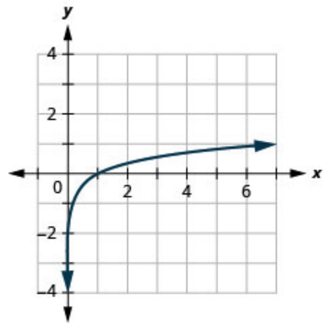
Exercise:

Problem: $y = \log_6 x$

Exercise:

Problem: $y = \log_7 x$

Solution:



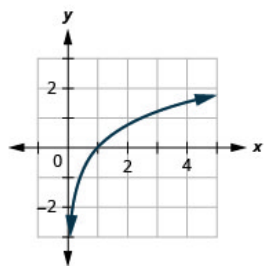
Exercise:

Problem: $y = \log_{1.5} x$

Exercise:

Problem: $y = \log_{2.5} x$

Solution:



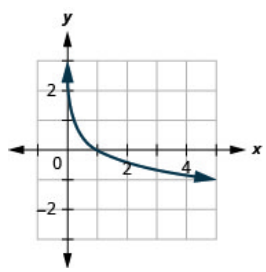
Exercise:

Problem: $y = \log_{\frac{1}{3}} x$

Exercise:

Problem: $y = \log_{\frac{1}{5}} x$

Solution:



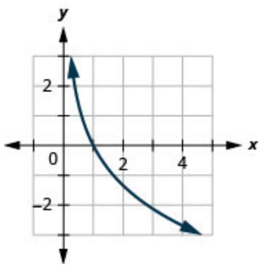
Exercise:

Problem: $y = \log_{0.4} x$

Exercise:

Problem: $y = \log_{0.6}x$

Solution:



Solve Logarithmic Equations

In the following exercises, solve each logarithmic equation.

Exercise:

Problem: $\log_{a}16 = 2$

Exercise:

Problem: $\log_{a}81 = 2$

Solution:

$a = 9$

Exercise:

Problem: $\log_{a}8 = 3$

Exercise:

Problem: $\log_{a}27 = 3$

Solution:

$a = 3$

Exercise:

Problem: $\log_{a}32 = 2$

Exercise:

Problem: $\log_{a}24 = 3$

Solution:

$a = \sqrt[3]{24}$

Exercise:

Problem: $\ln x = 5$

Exercise:

Problem: $\ln x = 4$

Solution:

$x = e^4$

Exercise:

Problem: $\log_2 (5x + 1) = 4$

Exercise:

Problem: $\log_2 (6x + 2) = 5$

Solution:

$x = 5$

Exercise:

Problem: $\log_3 (4x - 3) = 2$

Exercise:

Problem: $\log_3 (5x - 4) = 4$

Solution:

$x = 17$

Exercise:

Problem: $\log_4 (5x + 6) = 3$

Exercise:

Problem: $\log_4 (3x - 2) = 2$

Solution:

$x = 6$

Exercise:

Problem: $\ln e^{4x} = 8$

Exercise:

Problem: $\ln e^{2x} = 6$

Solution:

$x = 3$

Exercise:

Problem: $\log x^2 = 2$

Exercise:

Problem: $\log(x^2 - 25) = 2$

Solution:

$x = -5\sqrt{5}, x = 5\sqrt{5}$

Exercise:

Problem: $\log_2 \left(x^2 - 4\right) = 5$

Exercise:

Problem: $\log_3 \left(x^2 + 2\right) = 3$

Solution:

$x = -5, x = 5$

Use Logarithmic Models in Applications

In the following exercises, use a logarithmic model to solve.

Exercise:

Problem: What is the decibel level of normal conversation with intensity 10^{-6} watts per square inch?

Exercise:

Problem: What is the decibel level of a whisper with intensity 10^{-10} watts per square inch?

Solution:

A whisper has a decibel level of 20 dB.

Exercise:

Problem:

What is the decibel level of the noise from a motorcycle with intensity 10^{-2} watts per square inch?

Exercise:

Problem:

What is the decibel level of the sound of a garbage disposal with intensity 10^{-2} watts per square inch?

Solution:

The sound of a garbage disposal has a decibel level of 100 dB.

Exercise:

Problem:

In 2014, Chile experienced an intense earthquake with a magnitude of 8.2 on the Richter scale. In 2010, Haiti also experienced an intense earthquake which measured 7.0 on the Richter scale. Compare the intensities of the two earthquakes.

Exercise:

Problem:

The Los Angeles area experiences many earthquakes. In 1994, the Northridge earthquake measured magnitude of 6.7 on the Richter scale. In 2014, Los Angeles also experienced an earthquake which measured 5.1 on the Richter scale. Compare the intensities of the two earthquakes.

Solution:

The intensity of the 1994 Northridge earthquake in the Los Angeles area was about 40 times the intensity of the 2014 earthquake.

Writing Exercises

Exercise:

Problem: Explain how to change an equation from logarithmic form to exponential form.

Exercise:

Problem: Explain the difference between common logarithms and natural logarithms.

Solution:

Answers will vary.

Exercise:

Problem: Explain why $\log_a a^x = x$.

Exercise:

Problem: Explain how to find the $\log_7 32$ on your calculator.

Solution:

Answers will vary.

Self Check

ⓐ
After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
convert between exponential and logarithmic form.			
evaluate logarithmic functions.			
graph logarithmic functions.			
solve logarithmic functions.			
use logarithmic models in applications.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

Glossary

common logarithmic function
The function $f\left(x\right)=\log x$ is the common logarithmic function with base10, where $x>0$.
Equation:

$$y=\log x\text{ is equivalent to }x={10}^{y}$$

logarithmic function
The function $f\left(x\right)={\log}_{a}x$ is the logarithmic function with base a, where $a>0$,$x>0$, and $a\ne 1$.
Equation:

$$y={\log}_{a}x\text{ is equivalent to }x={a}^{y}$$

natural logarithmic function
The function $f\left(x\right)=\ln x$ is the natural logarithmic function with base e, where $x>0$.
Equation:

$$y=\ln x\text{ is equivalent to }x={e}^{y}$$

HiSET Formula Sheet
HiSET Formula Sheet

HiSET Formula Sheet

Formula Sheet

Perimeter / Circumference

Rectangle

$Perimeter = 2(length) + 2(width)$

Circle

$Circumference = 2\pi(radius)$

Area

Circle

$Area = \pi(radius)^2$

Triangle

$Area = \frac{1}{2}(base)(height)$

Parallelogram

$Area = (base)(height)$

Trapezoid

$Area = \frac{1}{2}(base_1 + base_2)(height)$

Volume

Prism/Cylinder

$Volume = (area\ of\ the\ base)(height)$

Pyramid/Cone

$Volume = \frac{1}{3}(area\ of\ the\ base)(height)$

Sphere

$Volume = \frac{4}{3}\pi(radius)^3$

Length

1 foot = 12 inches

1 yard = 3 feet

1 mile = 5,280 feet

1 meter = 1,000 millimeters

1 meter = 100 centimeters

1 kilometer = 1,000 meters

1 mile ≈ 1.6 kilometers

1 inch = 2.54 centimeters

1 foot ≈ 0.3 meter

Capacity / Volume

1 cup = 8 fluid ounces

1 pint = 2 cups

1 quart = 2 pints

1 gallon = 4 quarts

1 gallon = 231 cubic inches

1 liter = 1,000 milliliters

1 liter ≈ 0.264 gallon

Weight

1 pound = 16 ounces

1 ton = 2,000 pounds

1 gram = 1,000 milligrams

1 kilogram = 1,000 grams

1 kilogram ≈ 2.2 pounds

1 ounce ≈ 28.3 grams

GED Formula Sheet
GED Formula Sheet

Unfortunately, the GED Formula Sheet is copyrighted so the best I can do is provide links to copy and paste into your browser:
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